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Determining the domain of attraction of hybrid non–linear systems using maximal Lyapunov functions

DETERMINING THE DOMAIN OF ATTRACTION OF HYBRID NON–LINEAR SYSTEMS USING MAXIMAL LYAPUNOV FUNCTIONS

Szabolcs Rozgonyi, Katalin M. Hangos and Gábor Szederkényi

In this article a method is presented to find systematically the domain of attraction (DOA) of hybrid non-linear systems. It has already been shown that there exists a sequence of special kind of Lyapunov functions V_n in a rational functional form approximating a maximal Lyapunov function V_M that can be used to find an estimation for the DOA. Based on this idea, an improved method has been developed and implemented in a *Mathematica*-package to find such Lyapunov functions V_n for a class of hybrid (piecewise non-linear) systems, where the dynamics is continuous on the boundary of the different regimes in the state space.

In addition, a computationally feasible method is proposed to estimate the DOA using a maximal fitting hypersphere.

1. INTRODUCTION

The stability and the stability region of physical (and industrial) systems are important properties to be determined. In practice, it is often not enough to prove the local asymptotic stability of an equilibrium point, but one may also need to know the extent of the stability region, too.

A substantial part of methods on finding the region of stability described in the literature is based upon the classical results of LaSalle and Lefschetz [15] using a suitably chosen Lyapunov function in [28] and [12]. One of the first results concerning the estimation of stability domains is described by Zubov [30] using a constructed Lyapunov function of which preimage (contained by an open sphere) over an interval gives an approximation for the domain (or region) of attraction (DOA). This construction, unfortunately, requires the solutions of the given system (and so the knowledge of the DOA itself) [12, 30]. The same problem of applicability is true for the method given by Knobloch and Kappel in [13]. A linear matrix inequality (LMI) technique of polynomial Lyapunov functions for a class of non-polynomial systems is presented in [5] where the non-polynomial part of the Taylor expansion of the Lyapunov function is dropped. A multidimensional gridding approach based upon

the use of Chebyshev points is presented in [10]. [4] describes an LMI based method for estimating the so-called Robust Domain of Attraction for uncertain polynomial systems where the multiplicative uncertainties belong to a polytope. In [6], the domain of attraction of polynomial system is estimated using the union of a continuous family of Lyapunov function estimates. The computation of the minimal distance between a point and a surface in finite dimensional spaces is a fundamental subproblem in DOA estimation. [7] presents a general framework in which certain classes of minimum distance problems are solved via LMI computations. The above mentioned methods, however, are applicable for smooth non-linear ordinary differential equation (ODE) systems.

Computing the DOA for hybrid systems containing continuous and discrete components is far less advanced [1]. Here again, Lyapunov function based methods are usually applied. One possible approach that is followed e. g. in [19] is to construct the overall Lyapunov function of the system from known nonstrict Lyapunov functions for the dynamics and finite sums of persistence of excitation parameters.

The subject of this paper is to use a special Lyapunov function, the so called maximal Lyapunov function proposed by Vanelli and Vidyasagar in [25], to develop an algorithm to estimate the DOA of a class of hybrid non-linear systems. The maximal Lyapunov function can be constructed by using an iterative procedure and the DOA is estimated by its appropriately chosen level-set. The algorithm requires to perform a combination of symbolic and numerical calculations, therefore the *Mathematica* system has been chosen for its implementation. A similar iterative approximation algorithm for the approximate analytical solution to a non-linear undamped Duffing equation is reported in [27], where the authors have also implemented their method in *Mathematica*. The approximation of the stability region of a special BDF–Runge–Kutta type formulas implemented in *Mathematica* has also recently been reported in [26].

The structure of the paper is the following. First the fundamental notions and definitions are given including the stability of ODE systems, maximal Lyapunov-functions and hybrid systems. Then the numerical method is shown to find the DOA of a non-linear hybrid system which is then demonstrated via examples.

2. BASIC NOTIONS

In the following we will consider the ordinary autonomous differential system in the form of

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and moreover we assume that for each $x \in \mathbb{R}^n$ a unique solution $\phi(t, x)$ exists which is defined on $\mathbb{R}$ such that $\phi(0, x) = x$. Then all necessary conditions are satisfied [8] to allow $\phi(t, x)$ to define a dynamical system on $\mathbb{R}^n$. Note that if f, for instance, is of global Lipschitz-type then the mentioned conditions are met.

Throughout the paper (X, ρ) denotes *locally compact metric space* with metric ρ. Sets $\mathcal{I}(A)$, $\bar{A}$, $\mathcal{P}(A)$ and ∂A, denotes the interior of A, closure of A, power set and the boundary of A respectively. The following notions shall be used in addition:

$S(x, \xi) = \{y \in X : \rho(x, y) < \xi\}$ is the ξ-neighbourhood of point x, $S[x, \xi] = \{y \in X : \rho(x, y) \leq \xi\}$ is its closed ξ-neighbourhood, $S(M, \xi) = \{y \in X : \rho(y, M) < \xi\}$ is the ξ-neighbourhood of set M, $S[M, \xi] = \{y \in X : \rho(y, M) \leq \xi\}$ is its closed ξ-neighbourhood and $H(x, \xi) = \{y \in X : \rho(x, y) = \xi\}$ is the sphere-surface with radius ξ centred at x. If we refer to some neighbourhood of the origin then the first parameter is left.

2.1. Stability and attractivity of ODE systems

The mathematical notions related to the region of stability with respect to a given stationary point of an ODE are given below [2].

Definition 2.1. A set $M \subset X$ is called *invariant (positively invariant)* whenever $\phi(t, x) \in M$ for all $t \in \mathbb{R}$ ($t \in \mathbb{R}^+$) and $\phi(0, x) = x$, i.e. the set M is positively invariant if for every initial condition in M the trajectory $\phi(t, x)$ is contained in M.

Definition 2.2. For any given $x \in X$ the set $\Lambda^+(x)$ is called *positive or omega limit set for x* and $J^+(x)$ is called the *first positive prolongational limit set of x* where $\Lambda^+(x) = \{y \in X \mid \exists\{t_n\} \subset \mathbb{R} : t_n \to \infty \wedge x(t_n) \to y\}$ and $J^+(x) = \{y \in X \mid \exists\{x_n\} \subset X, \exists\{t_n\} \subset \mathbb{R}^+ : x_n \to x, t_n \to \infty \wedge x_n t_n \to y\}$ accordingly.

Definition 2.3. With a given compact set $\emptyset \neq M \subset X$ we associate the sets $A_\omega(M) = \{x \in X \mid \Lambda^+(x) \cap M \neq \emptyset\}$, $A(M) = \{x \in X \mid \Lambda^+(x) \neq \emptyset \wedge \Lambda^+(x) \subset M\}$ and $A_u(M) = \{x \in X \mid J^+(x) \neq \emptyset \wedge J^+(x) \subset M\}$ which are called the *domain of weak attraction, attraction and uniform attraction of M*, respectively.

A given set M is said to be a *local attractor* if $A(M)$ is some neighbourhood of M (i.e. for all point in M there is an open neighbourhood which is subset of $A(M)$).

A given set M is said to be (i) *stable* if every neighbourhood U of M contains a positively invariant neighbourhood $V(U)$ of M and (ii) *asymptotically stable* if it is stable and is an attractor. The union of $V(U)$'s for all neighbourhoods U is called the *domain of stability* of M.

The following theorem is classical (see, e.g. [14]).

Theorem 2.4. (LaSalle's principle to establish asymptotic stability) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $V(x) > 0$ if $x \neq 0$ and $\dot{V}(x) \leq 0$ on $\Omega_l = \{x \in \mathbb{R}^n : V(x) \leq l\}$ for some $l > 0$. Let $R = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$. If R contains no whole trajectory other than $x(t) = 0$ then the origin is asymptotically stable. The derivative should be considered in Dini-sense, if necessary (see [28]).

Characterization of stability by means of continuous functions is not in general possible, however strong theorems can be stated on asymptotic stability. Let the origin of system (1) be asymptotically stable equilibrium point. Satisfying this condition ensures (among other properties) that the DOA is subset of the domain of stability of the origin [9].

2.2. Maximal Lyapunov functions

It is well known that even if a Lyapunov function exists to an autonomous ODE, then it is not unique. A maximal Lyapunov function is a special Lyapunov function on A which indicates the DOA for a given locally asymptotically stable equilibrium point.

Definition 2.5. A function $V_M : \mathbb{R}^n \to \mathbb{R}_0^+$ is called *maximal Lyapunov function* for the system (1) if

   (i) $V_M(0) = 0$, $V_M(x) > 0$, $x \in A\backslash\{0\}$

   (ii) $V_M(x) < \infty$ if and only if $x \in A$,

   (iii) $\dot{V}_M$ is negative definite over A and

   (iv) $V_M(x) \to \infty$ as $x \to \partial A$ and/or $|x| \to \infty$,

with A being the domain of attraction of the origin for system (1).

Note that if the origin is an asymptotically stable equilibrium point its DOA cannot be closed unless it is the empty set or the whole space [2].

Theorem 2.6. If M is a weak attractor, attractor or uniform attractor then the corresponding domain of weak attraction, attraction or uniform attraction N is open (an open neighbourhood of M).

P r o o f. According to the definitions N is neighbourhood of M. So ∂N is invariant and disjoint from M and is indeed closed. Note that because $\forall x \in \partial N \colon \Lambda^+(x) \subset \partial N$ it is seen that $\forall x \in N \colon \Lambda^+(x) \cap M \neq \emptyset$. Since $\partial N \cap M = \emptyset$ we conclude that $N \cap \partial N = \emptyset$. Thus N is open. $\qquad\square$

The following theorem forms the basis of our further development.

Theorem 2.7. Suppose we can find a set $B \subseteq \mathbb{R}^n$ containing the origin in its interior and a continuous function $V : B \to \mathbb{R}^+$ and a positive definite function $\psi : \mathbb{R}^n \to \mathbb{R}^+$ such that

   (i) $V(0) = 0$ and $V(x) > 0$ for all $x \in B\backslash\{0\}$,

   (ii) the function $\dot{V}(x_0) = \lim_{t \to 0^+} \frac{V(x(t,x_0)) - V(x_0)}{t}$ is well defined at all $x \in B$ and satisfies the relation $\dot{V}(x) = -\psi(x), \forall x \in B$ and

   (iii) $V(x) \to \infty$ as $x \to \partial B$ and/or $|x| \to \infty$.

Then $B = A$.

P r o o f . See [25]. □

Theorems A.1 and A.2 in the Appendix show that the conditions on V imposed in theorem 2.7 are reasonable.

Suppose V is a continuous function on some ball $S(\delta)$ for some $\delta > 0$ such that $V(0) = 0$ and $\dot{V}$ is negative definite. Then one could prove that V is positive definite. This fact shows that if we can find a function V and a positive definite function ψ such that $V(0) = 0$ and $\partial V(x)' f(x) = -\psi(x)$ then V is guaranteed to be positive definite.

2.3. Hybrid systems

There are two possible fundamentally different ways of describing a hybrid system that possesses both continuous and discrete variables. The first one utilizes the concepts of theory of discrete event systems [18, 23] and embeds the continuous elements into a discrete system. In this article the other approach is followed which sometimes is called *switched systems approach* when some non-smooth variables are embedded into the continuous part [23] so that the dynamics by piecewise functions can be described which are continuous, however, not differentiable on the border between different dynamics. There also exists an extended approach called *systems with impulse effect* [16, 17] where the possibility is added to the states to jump when certain conditions are satisfied, often when some boundaries are crossed by the trajectory. These boundaries are generally subsets of the state-space, often being described by zeros of several functions.

Let function f in the autonomous system (1) be piecewise-defined over finite number of different domains X_i with no point belonging to more than one dynamics, more precisely $f(x) = f_i(x), x \in X_i, i \in \bar{n} = \{1, 2, \ldots, n\}$ and $\cup_{i \in \bar{n}} X_i = X$ and X_is are pairwise disjoint. The border of the dynamics $\{i_1, i_2, \ldots, i_m\} = I \in \mathcal{P}(\bar{n})$ is given by $\mathcal{D}_I = \cap_{i \in I} \overline{X_i}$. The border of all dynamics is then given by $\mathcal{D} = \cup_{I \in \mathcal{P}(\bar{n})} \mathcal{D}_I$. Furthermore let the DOA for the subsystem $\dot{x} = f_i(x)$ be $A_i (= A_i(\{0\})) \subseteq X_i$. Note that A_i is not restricted onto X_i (on which the given sub-dynamics is active).

2.3.1. DOA of hybrid systems

There are some not too rigorous conditions that the function f of a hybrid system model should meet in order to enable a relatively easy DOA analysis. In short it should be smooth enough to have only one trajectory going through any point in X. The detailed list of conditions can be found in Subsection 3.2.1.

It is easy to see that if for all $I \in \mathcal{P}(\bar{n})$ and for all $\{i_1, i_2\} \subseteq I : A_{i_1} \cap \mathcal{D}_I = A_{i_2} \cap \mathcal{D}_I$ then $A = \cup_{i \in \bar{n}} A_i$. Otherwise neither the union $\cup_{i \in \bar{n}} A_i$ nor the intersection $\cap_{i \in \bar{n}} A_i$ is necessarily invariant for each f_i, thus an appropriate subset of $\cup_{i \in \bar{n}} A_i$ has to be chosen as estimation of A. Precise description of the mathematical background of choosing this subset is detailed later in Section 3.3.

2.3.2. Stability analysis of hybrid systems

Several approaches have already been introduced in the literature for the stability analysis of hybrid systems. For a certain class of hybrid systems consisting of non-linear subsystems a linear matrix inequality (LMI) method has been proposed [20] which extends the classical Lyapunov theory for these hybrid systems. One of its assumptions is the same condition we have in this paper, namely to have the same equilibrium point of each subsystem, however, it allows the Lyapunov function to be discontinuous.

The emerging need of controlling hybrid systems in the industry yields results such as in [29] where set of sufficient conditions has been formulated for the stability of a switching control scheme by imposing a hierarchy among the controllers which is then in a suitable form for controller design.

Branicky introduced multiple Lyapunov functions in [3] which are used to analyse Lyapunov stability and then extended Bendixson's theorem for Lipschitz continuous vector fields, allowing limit cycle analysis of some class of 'continuous switched' systems.

For switched systems with impulse effects it is often not easy to construct multiple Lyapunov functions. In the paper [16] concepts of minimum holding time and redundancy have been used to establish sufficient conditions for stability, asymptotic stability and exponential stability in Lyapunov sense. For hybrid systems, condition of exponential stability can be formulated as LMIs by using piecewise quadratic forms of the Lyapunov function candidates, too [21].

3. NUMERICAL APPROXIMATION OF MAXIMAL LYAPUNOV FUNCTIONS

In this section an iterative method proposed by Vanelli and Vidyasagar is shown to estimate V with a sequence of V_ns. Results in this section are based upon the paper [25].

3.1. The Vanelli–Vidyasagar algorithm

Based on the properties of the maximal Lyapunov functions in Section 2.2, we seek for a function V_M and a positive definite function ψ satisfying $V_M(0) = 0$ and

$$\dot{V}_M(x) = -\psi(x) \tag{2}$$

over some neighbourhood of the origin such that the set ∂A is given by the relation $V_M(x) \to \infty$. If we could write V_M as $V_M(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials in x then ∂A would be given by $D(x) = 0$. So let us write $V_M(x)$ as

$$V_M(x) = \frac{\sum_{i=2}^{\infty} R_i(x)}{1 + \sum_{i=1}^{\infty} Q_i(x)} \tag{3}$$

where R_i and Q_i are homogeneous functions of degree i. The estimation V_n of V_M is

$$V_n = \frac{\sum_{i=2}^{n} R_i}{1 + \sum_{i=1}^{n-2} Q_i}. \tag{4}$$

In order to construct V_n the expansion $f(x) = \sum_{i=1}^{\infty} F_i(x)$ is used where functions $F_i, i \geq 1$ are again homogeneous functions of degree i. For $i = 1$ we have $F_1(x) = \Phi x, \Phi \in \mathbb{R}^{n \times n}$, where Φ is the Jacobian matrix of f at $x = 0$. For the sake of brevity, let $F_i(x) = 0, \ i \leq 0$. Having $\psi(x) = x'\mathfrak{Q}x, \ \mathfrak{Q} > 0$ we obtain $\dot{V}_M(x) = \nabla V_M(x)' f(x) = -\psi(x) = -x'\mathfrak{Q}x$ and thus finally we get the polynomial equality

$$\sum_{i=2}^{\infty} \sum_{k=1}^{\infty} \nabla R_i' F_k + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i \nabla R_j' F_k - \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i' R_j F_k$$

$$= -x'\mathfrak{Q}x \left(1 + 2\sum_{i=1}^{\infty} Q_i + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Q_i Q_j\right). \tag{5}$$

Equating the coefficients of the same degrees k of the two sides for $k = 2$ we get

$$\nabla R_2' F_1 = -x'\mathfrak{Q}x \tag{6}$$

and for $k \geq 3$ we have

$$\sum_{i=2}^{k} \nabla R_i' F_{k+1-i} + \sum_{i=1}^{k-2} \sum_{j=2}^{k-1} \left(Q_i \nabla R_j' - \nabla Q_i' R_j\right) F_{k+1-i-j}$$

$$= -x'\mathfrak{Q}x \left(2Q_{k-2} + \sum_{i=1}^{k-3} Q_i Q_{k-2-i}\right) \tag{7}$$

which can in short be expressed as and under-determined set of linear equations

$$A_n y = b_n \tag{8}$$

where A_n are matrices of appropriate dimension, and the vector y is composed of the coefficients of the homogeneous functions R_is and Q_is.

It can be shown that functions V_n are Lyapunov-functions, i.e. they are positive definite and $\dot{V}_n$ is negative definite about some neighbourhood of the origin. Proof of this non-obvious statement is detailed in [25] and [22].

The form of this result is similar to the one yielded by Zubov [30], but while there it was $\dim(A_n) = n \times n$, in our case it is $\dim(A_n) = m \times n, m < n$ which fact allows us to accelerate the convergence of V_n's to V_M by minimizing a certain expression $e_n(y)$ as it is detailed below.

Let us find homogeneous functions R_n and $Q_{n-2}, n \geq 3$, such that the coefficients of R_n and Q_n solve the constrained minimization problem

$$\begin{aligned} \min\ & e_n(y) \\ \text{s.t.}\ & A_n(y) = b_n \end{aligned} \tag{9}$$

where $e_n(y)$ is the squared 2-norm of the coefficients of degree greater than or equal to $n + 1$ in the expression of $\dot{V}_n$. Let

$$U = \left(1 + \sum_{i=1}^{n-2} Q_i(x)\right)^2. \tag{10}$$

Then

$$\dot{V}_n = \nabla V_n' f = \frac{1}{U}\left(\left(1+\sum_{i=1}^{n-2}Q_i\right)\sum_{j=2}^{n}\nabla R_j' - \left(\sum_{i=1}^{n-2}\nabla Q_i'\right)\left(\sum_{j=2}^{n}R_j\right)\right)f$$

$$= \frac{1}{U}\left(\sum_{i=2}^{n}\nabla R_i + \sum_{i=1}^{n-2}\sum_{j=2}^{n}\left(Q_i\nabla R_j' - \nabla Q_i'R_j\right)\right)\sum_{k=1}^{\infty}F_k = \frac{1}{U}\left(\dot{V}_{n\,\mathrm{main}} + \dot{V}_{n\,\mathrm{resid}}\right) \quad (11)$$

where

$$\dot{V}_{n\,\mathrm{main}} = \nabla R_2' F_1$$

$$+ \sum_{k=3}^{n}\left(\nabla R_2' F_{k-1} + \sum_{j=3}^{k}\left(\nabla R_j' + \sum_{i=1}^{j-2}\left(Q_i\nabla R_{j-i}' - \nabla Q_i'R_{j-i}\right)\right)F_{k-j+1}\right) \quad (12)$$

and

$$\dot{V}_{n\,\mathrm{resid}}$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=n-i+1}^{n-1}\nabla R_{j+1} + \sum_{r=2}^{n}\sum_{q=(n+1)-(i-1)-r}^{n-2}\left(Q_q\nabla R_r' - \nabla Q_q'R_r\right)\right)F_i$$

$$- \nabla R_1'\sum_{i=n+1}^{\infty}F_i. \quad (13)$$

For simpler notation let Q_i, R_i and F_i be zeros whenever $i \leq 0$. Then we get

$$\dot{V}_n = \frac{1}{U}\left(-x'\mathfrak{Q}x - x'\mathfrak{Q}x\left(2\sum_{k=3}^{n}Q_{k-2} + \sum_{k=3}^{n}\sum_{i=1}^{k-3}Q_iQ_{k-2-i}\right)\right) + \frac{1}{U}\dot{V}_{n\,\mathrm{resid}}$$

$$= \frac{1}{U}\left(-x'\mathfrak{Q}x\left(1 + 2\sum_{k=3}^{n}Q_{k-2} + \sum_{k=3}^{n}\sum_{i=1}^{k-3}Q_iQ_{k-2-i}\right)\right) + \frac{1}{U}\dot{V}_{n\,\mathrm{resid}}. \quad (14)$$

Since the numerator of the first term is part of the expression U of degree up to and including $n-2$, $\dot{V}_n$ can be written in the form of

$$\dot{V}_n = -x'\mathfrak{Q}x + \frac{U_g}{U} \quad (15)$$

where U_g is the sum of terms whose degree is greater than or equal to $n-1$. This, in addition, shows that $\dot{V}_n$ is negative definite about some neighbourhood of the origin.

According to the theorem of LaSalle [15] about invariant sets, one can choose the largest positive value $C^\star$ such that the sublevel set

$$A_{\mathrm{est}} = \{x : V_n(x) < C^\star\} \quad (16)$$

is contained in the region given by

$$\Omega = \left\{ x : \dot{V}_n(x) \leq 0 \right\}. \tag{17}$$

As soon as the desired accuracy (computed as $e_n(y^\star) = 0$ for the minimizer $y^\star$) has been reached (or the error starts growing) the iteration can be stopped. Relation $A_{\mathrm{est}} \subseteq A$ holds true in each step which means the more step one executes the more precise estimation of A can be computed. If $e_n(y^\star) = 0$ for some $y^\star$ and some n then the iteration can be stopped and due to the relation $\dot{V}_n = -x'\mathfrak{Q}x$ the set A is given by $D(x) = 0$, i. e.

$$A = \left\{ x : \sum_{i=1}^{n-2} Q_i(x) > -1 \right\}. \tag{18}$$

The choice of the quadratic form $-\phi(x) = -x'\mathfrak{Q}x$ may seem to be conservative for the time derivative of V_n. However, this form can guarantee that the coefficients of the homogeneous functions R_i and Q_i in (5) can be obtained as the solution of a set of linear equations (8). Moreover, $\dot{V}_n$ can be written as [25]

$$\dot{V}_n = -x'\mathfrak{Q}x + \frac{\{\text{terms of degree} \geq n+1\}}{\left(1 + \sum_{i=1}^{n-2} Q_i\right)^2} \tag{19}$$

and this ensures that $\dot{V}_n$ is negative definite over a neighbourhood of the origin. It is also proved in [25] that V_n is positive definite in some neighbourhood of the origin. These facts imply that V_n can be used as a Lyapunov function in a neighbourhood of the investigated equilibrium point for $n \geq 2$.

The concept of maximal Lyapunov function, basic ideas and some referred proofs are taken from [25]. However the authors of this article think that equation (63) on page 75 in [25] suffers from several typographical errors which may cause difficulties in applications of the algorithm. In this article and in [22] a similar reasoning in a rewritten form is followed which may be easier to understand and implement for application purposes.

Based on the above algorithm, a *Mathematica*-package has been implemented to find a sequence of Lyapunov functions V_n that can be used to estimate the set A. The given algorithm is able to estimate unbounded DOAs, too. Usually, the first few number of iterations show whether the domain is bounded or not.

3.2. Applying the algorithm to hybrid systems

3.2.1. Conditions for f

To find the common DOA $A(= A(\{0\}))$ using the Vanelli–Vidyasagar algorithm, the following criteria should be fulfilled for f:

  (i) f is continuous,

  (ii) $f_i(0) = 0, \forall i \in \bar{n}$,

(iii) $0 \in \mathcal{I}\left(\cap_{i \in \bar{n}} A_i\right),$

(iv) f should be Lipschitz continuous, which due to Lagrange-theorem, holds if f_i is Lipschitz-continuous on X_i for each $i \in \bar{n}$, and

(v) in order to directly use the implemented algorithm in *Mathematica*, f should be differentiable on some neighbourhood about the origin. Note that on set A_i the function V_i is maximal Lyapunov function for system f_i.

It is visible from conditions (i) – (v) that if f is continuously differentiable about the origin or the origin lies within $\mathcal{I} \cap_{i \in \bar{n}} X_i$ then the following algorithm (and the corresponding implemented *Mathematica*-package) can be used on the hybrid system determined by f.

3.2.2. Steps of the algorithm for hybrid systems

If the conditions detailed in Subsection 3.2.1 are satisfied, then the algorithm described in Section 3.1 can be adapted appropriately to find the DOA. Note that the first steps coincide with the ones described in [25].

Step 1. Since the origin is asymptotically stable, it follows that $\dot{x} = F_1(x) = \Phi x$ and the Lyapunov-equation $\Phi'P + PA = -\mathfrak{Q}$ can be solved such that $\mathfrak{Q} > 0$. That means for $n = 2$ it is $V_2(x) = R_2 = x'Px$.

Step 2. For $n \geq 3$ let us find the form of the minimization problem (8) based upon V_{n-1} and (7).

Step 3. Define $e_n(y)$ as in Equations (9) – (15) and solve the minimization problem (9). Denote the solution by $y^\star$.

Step 4. From the solution $y^\star$ we get the coefficients of R_n and Q_{n-2}. If $e_n(y^\star)$ is smaller than a threshold (which depends on the accuracy we want to accomplish) then go to Step 5. Otherwise increase n and repeat from Step 2.

Step 5. If $e_n(y^\star) < \epsilon$ then go to Step 6, otherwise go to Step 7. Here ϵ is a small number, since in numerical calculations numbers must not be compared directly to zero.

Step 6. The DOA is given by equation (18).

Step 7. Estimation of the DOA is given by equation (16). Finding of the appropriate $C^\star$ can be done manually or algorithmically. Differences between the two approaches are investigated among the examples. Basis of the algorithmic search is detailed in the following section.

3.3. Finding the widest possible sublevel set

The following theorem enables us to give a computationally feasible algorithm of finding a good approximation of the widest possible sublevel set for the DOA.

Theorem 3.1. Let us consider the system $\dot{x} = f_i(x)$ with its equilibrium point in $x = 0$. Let $\epsilon_i > 0$ such that the ball $S[\epsilon_i]$ is a compact subset of $\mathcal{I}(A_i)$ and define $m_i(\epsilon) = \min\{V_i(x) : x \in H(\epsilon)\}$ being the minimal value of the Lyapunov function

on the surface of the ball. So we have $0 < m_i(\epsilon_i)$. If we choose α_i such that $0 < \alpha_i < m(\epsilon_i)$, then $P_{\alpha_i} = K_{\alpha_i} \cap S[\epsilon_i]$ is a compact positively invariant subset of A_i where $K_{\alpha_i} = \{x \in A_i : V_i(x) \leq \alpha_i\}$ is the α_i sublevel set of the Lyapunov function $V_i(x)$.

P r o o f. Choose $\{x_n\}$ to be a sequence in P_{α_i}. Since $\{x_n\} \subseteq S[\epsilon_i]$ which is compact, it follows that $\{x_n\}$ (taking subsequences if necessary) converges to some $x \in S[\epsilon_i]$. Because of this, and because $\forall n : V_i(x_n) \leq \alpha_i$ then $V_i(x) \leq \alpha_i$, thus $x \in K_{\alpha_i}$. Then compactness follows from $x \in (K_{\alpha_i} \cap S[\epsilon_i]) = P_{\alpha_i}$.

Positive invariance can be proven by contradiction. Suppose $x(0) \in P_{\alpha_i}$ and $x(\mathbb{R}^+) \nsubseteq S(\epsilon_i)$. Then there is some $t > 0$ such that $x(t) \in H(\epsilon_i)$ and $x([0,t]) \subset S[\epsilon_i] \subset A_i$. Then it follows that $V_i(x(t)) \geq m_i(\epsilon_i) > \alpha_i$ but for V_i being Lyapunov function for the system f_i, the inequality $V_i(x(t)) < V_i(x(0)) \leq \alpha_i$ holds which is a contradiction. Thus $x(\mathbb{R}^+) \subset S(\epsilon_i) \subset A_i$ so for each $t \geq 0$ $x(\mathbb{R}^+) \subset K_{\alpha_i}$ and then $x(\mathbb{R}^+) \subset K_{\alpha_i} \cap S[\epsilon_i] = P_{\alpha_i}$. $\qquad\square$

With the above theorem the way of finding the widest possible sublevel set is as follows. Let us select a hypersphere $H(r)$ about the origin which contains subset of the DOA A for the whole hybrid system. First we choose the maximal $r > 0$ such that the hypersphere of radius r lies inside $\mathcal{D}$ that is positively invariant. More precisely, we select the maximal $r > 0$ such that $(H(r) \cap \mathcal{D}) \subseteq (\cup_{i \in \bar{n}} A_i \cap \mathcal{D})$ holds true which is possible since X is locally compact. As a next step, we find the maximal value C_i of V_i on $A_i \cap X_i \cap S[r]$ (this C_i is the value of $C^\star$ in equation (16)) so we can construct the set $\cup_{i \in \bar{n}} \{x \in A_i \cap X_i : V_i(x) \leq C_i\}$ which is a subset of A.

Note again that the origin has to be an asymptotically stable equilibrium point of the system and the f_is have to be Lipschitz-continuous on their domain to be able to perform the above algorithm. For details, we refer to Subsection 3.2.1.

Finally it is important to note that the above method of finding the widest possible sublevel set may be conservative because it considers only sphere-environments, so that it finds a sphere which contains a subset of the DOA.

3.3.1. Manually approximated DOA

The proposed algorithm can be applied for systems of any dimension. However, due to the automatic calculation of $C^\star$ in equation (16), the DOA will reside inside an appropriately chosen sphere-environment of the origin, which can be considered to be a conservative approximation of the widest sublevel set.

If the algorithmic determination of level $C^\star$ is not required, then in case of two-dimensional systems much better estimates of the DOA can be found by choosing manually the value $C^\star$. In most cases, this heuristic approach results in a better estimate, although the method can be carried out only by trial-and-error.

In Figures 1(b) and 2(d) one can see the difference between the quality of estimation by manually tuning the value of $C^\star$ and its automatic determination (as described above). The manual tuning includes the search for higher values of level $C^\star$ obeying equations (16) and (17).
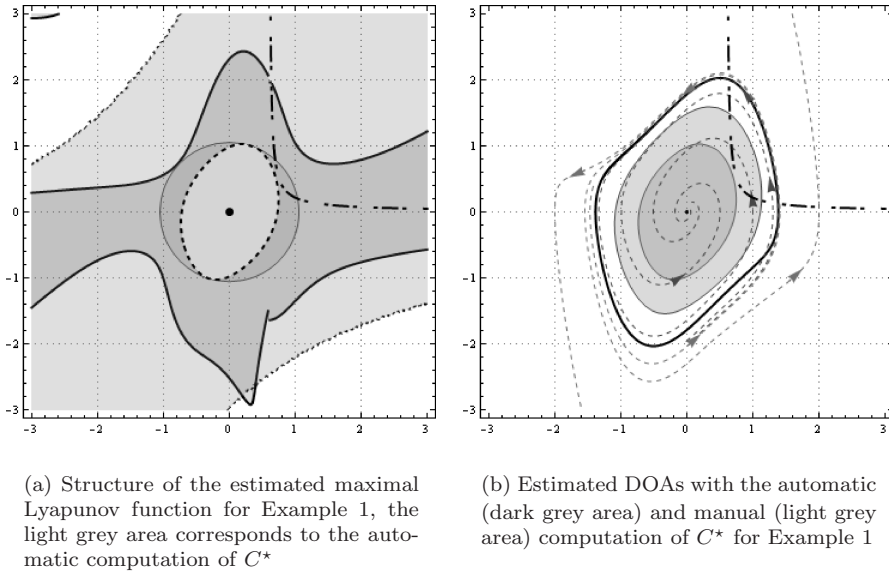
(a) Structure of the estimated maximal Lyapunov function for Example 1, the light grey area corresponds to the automatic computation of $C^\star$

(b) Estimated DOAs with the automatic (dark grey area) and manual (light grey area) computation of $C^\star$ for Example 1

Fig. 1. Estimated DOA of Example 1.

(c) Structure of the estimated maximal Lyapunov function for Example 2, the light grey area corresponds to the automatic computation of $C^\star$

(d) Estimated DOAs with the automatic (dark grey area) and manual (light grey area) computation of $C^\star$ for Example 2
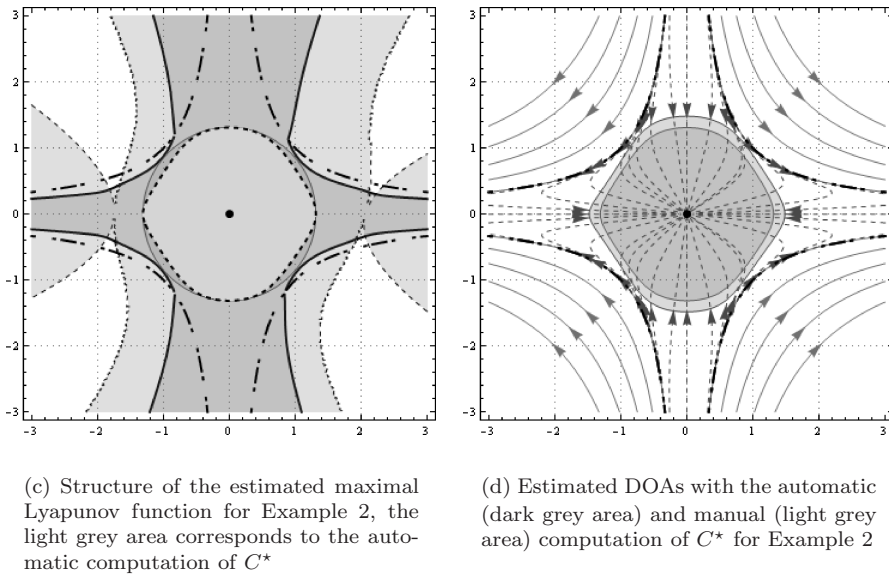
Fig. 2. Estimated DOA of Example 2.

4. EXAMPLES

In this section two examples are shown to demonstrate the operation of the extended Vanelli–Vidyasagar algorithm on hybrid systems. In order to be able to demonstrate the results graphically, both of our examples are dynamical systems defined on the real Euclidean plane described by the differential equations

$$
\begin{aligned}
\dot{x} &= g(x, y) \\
\dot{y} &= h(x, y)
\end{aligned}
$$

where $x, y \in \mathbb{R}$, $g, h \in \mathbb{R}^2 \to \mathbb{R}$.

4.1. Example 1: a hybrid Van der Pol model

In this section an example is shown where the DOA contains parts from both dynamics X_1 and X_2. This example is a hybrid version of the well known van der Pol model.

The van der Pol system [24] is a widely known example for non-linear closed loop system, which, for example, appears in the study of non-linear resonance circuits consisting of an inductor, a capacitor and a non-linear resistor. The given non-conservative non-linear system has a stable limit cycle. If the system shows hybrid behaviour (for example, certain elements in the circuit have different characteristics on different working domains) then a highly non-linear hybrid system is resulted.

Let us define the domains of different dynamics first:

$$
\begin{aligned}
X_1 &= \left\{ (x, y) \in \mathbb{R}^2 : x > 0.6 \wedge (0.36 - x^2)y > -0.14x \right\} \\
X_2 &= \mathbb{R}^2 \setminus X_1
\end{aligned}
$$

In order to construct a simple hybrid van der Pol model, let the functions g and h be

$$
\begin{aligned}
g(x, y) &= -1.2y \\
h(x, y) &= \begin{cases} h_1(x, y) & (x, y) \in X_1 \\ h_2(x, y) & (x, y) \in X_2 \end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(x, y) &= 2x - 0.65y + 1.1x^2 y \\
h_2(x, y) &= 1.86x - 1.01y + 2.1x^2 y
\end{aligned}
$$

Using the piecewise interpretation of hybrid systems, the functions f_is over their respective X_i's are:

$$
\begin{aligned}
f_1(x, y) &= \left(g\left(x, y\right), h_1\left(x, y\right) \right)' \\
f_2(x, y) &= \left(g\left(x, y\right), h_2\left(x, y\right) \right)'
\end{aligned}
$$

The equilibrium point, the origin, lies inside X_1. Since f is continuous on its whole domain and is differentiable about the origin, the algorithm can be applied and an estimated maximal Lyapunov function is obtained in 2 iterations.

The dashed-dotted line indicates the boundary between the two hybrid domains X_1 and X_2 in all figures.

The sublevel set corresponding to the estimated maximal Lyapunov function is shown in Figure 1(a). The area bounded by dotted line is the domain where the Lyapunov function is non-negative, while the thick line indicates the region where its derivative is negative. The value of sublevel set is chosen from the thin circle (it corresponds to $H(r)$ in Section 3.3). Choosing this value yields the estimated DOA shown by the thick dashed line.

In Figure 1(b) a few trajectories of the hybrid ODE can be seen demonstrating that the region we found (the dark grey area) is really a subset of the DOA. The manual tuning of the value of $C^\star$ leads us to a better estimate (the light grey area) which demonstrates the approximation capability of our algorithm more clearly. The two approaches can be compared to the real DOA which is denoted here by thick black line. The manually tuned approximated DOA is very close to it.

The estimated maximal Lyapunov function is found to be

$$(0.320906x^4 + x^3(0.350448y + 0.194919) + x^2(0.495843y^2 + 0.0874554y + 1.48863)$$
$$+ xy(-0.113556y^2 + 0.0496311y - 0.537634) + y^2(0.0385325y^2 + 0.0863618y + 0.814436))/$$
$$(-0.0156067x^2 + x(0.22978y + 0.130939) - 0.0775623(y - 4.33872)(y + 2.97158)).$$

Its time derivative is

$$((-0.0792071x^5 + x^4(0.0343035y - 0.0342947) + 0.146433x^3(y + 0.0500198)(y + 1.03264)$$
$$-0.0545911x^2(y-4.85856)(y-0.116165)(y+4.56961)+0.0353697x(y-1.68199)(y+3.42803)\cdot$$
$$\cdot(y^2-0.73423y+2.63625)-0.00597735(y-7.01123)y(y+4.88748)(y^2+1.19367y+7.95243))\cdot$$
$$\cdot\left(\begin{cases}1.1x^2y + 2x - 0.65y & x > 0.6 \wedge y(0.36 - 1.x^2) > -0.14x \\ 2.1x^2y + 1.86x - 1.01y & \text{otherwise}\end{cases}\right)$$
$$- 1.2y(-0.0100165x^5 + x^4(0.215744y + 0.123015) + x^3(0.0614912y^2 + 0.317465y + 1.33467)$$
$$+0.0306178x^2(y+0.563949)(y^2+4.39797y+45.1545)-0.0757147x(y-4.42848)(y+3.01079)\cdot$$
$$\cdot(y^2+0.172409y+2.94918)-0.0000463596y(y+6.43028)(y+873.11)(y^2+0.114472y+2.06561)))/$$
$$/((-0.0156067x^2 + x(0.22978y + 0.130939) - 0.0775623(y - 4.33872)(y + 2.97158))^2).$$

4.2. Example 2: stable system is embedded into an unstable one

Here an other simple hybrid system, where a stable subssystem is embedded into an unstable one, is discussed.

Let us define the domains of different dynamics first:

$$X_1 = \left\{(x, y) \in \mathbb{R}^2 : x^2 y^2 \geq 1\right\}$$
$$X_2 = \mathbb{R}^2 \setminus X_1$$

In order to construct the hybrid model, let the functions g and h be

$$g(x, y) \;=\; \begin{cases} g_1(x, y) & (x, y) \in X_1 \\ g_2(x, y) & (x, y) \in X_2 \end{cases}$$
$$h(x, y) \;=\; -y$$

where

$$g_1(x, y) = x$$
$$g_2(x, y) = 2x^3 y^2 - x$$

Using the piecewise interpretation of hybrid systems, the functions f_is over their respective X_i's are:

$$f_1(x, y) = (g_1(x, y), h(x, y))'$$
$$f_2(x, y) = (g_2(x, y), h(x, y))'$$

It can be shown that the origin is asymptotically stable for dynamics over X_2 (the area bounded by the dash-dotted lines), while over X_1 it is not, thus the conditions do not hold true as it is demanded in 3.2.1. In this case since X_2 itself is a positively invariant set thus the algorithm can still be executed yielding a maximal Lyapunov function in 4 iterations. Note also that the theoretical DOA coincides with domain X_2.

The dashed-dotted line indicates the boundary between the two hybrid domains X_1 and X_2 in all figures.

The sublevel set corresponding to the estimated maximal Lyapunov function is shown in Figure 2(c). The area bounded by dotted line is the domain where the Lyapunov function is non-negative, while the thick line indicates the region where its derivative is negative. The value of sublevel set is chosen from the thin circle (it corresponds to $H(r)$ in Section 3.3). Choosing this value yields the estimated DOA shown by the thick dashed line.

In Figure 2(d) a few trajectories of the hybrid ODE can be seen demonstrating that the region we found (the dark grey area) is really a subset of the DOA. Manually tuning the value of $C^\star$ leads us to a better estimate (the light grey area) which, again, demonstrates the strength of the algorithm. Here again, the manually tuned DOA is very close to the real DOA (thick dashed line here).

The estimated maximal Lyapunov function is found to be

$$(-0.0448653x^6 + x^5(0.0180341y + 0.0126314) + x^4(0.195264y^2 + 0.0158483y + 0.0235638)$$
$$+0.0360749x^3(y + 1.09455)(y^2 - 0.305092y + 1.26441) - 0.0492912x^2(y - 2.19048)(y + 1.69975)\cdot$$
$$\cdot(y^2 - 0.0870472y + 2.72444) + 0.0180408xy^2(y + 1.2174)(y^2 - 0.338931y + 2.2732) + 0.0439128y^2\cdot$$
$$(y^2 - 2.35279y + 3.40781)(y^2 + 2.64044y + 3.34121))/$$
$$/(-0.0897306x^4 + x^3(0.0360682y + 0.0252627) - 0.186408x^2(y - 0.594968)(y + 0.42493)$$
$$+ 0.0360816x(y + 1.2174)(y^2 - 0.338931y + 2.2732) + 0.0878257(y^2 - 2.35279y + 3.40781)\cdot$$
$$\cdot(y^2 + 2.64044y + 3.34121)).$$

The estimated Lyapunov function's time derivative is given by

$$
\begin{aligned}
&(x(0.00805158x^8 + x^7(-0.00647284y - 0.00453368) + 0.034754x^6(y - 0.533202)(y + 0.421965) \\
&-0.00789933x^5(y+1.19235)(y^2 - 0.635437y + 1.64973) - 0.102683x^4(y^2 - 1.69209y + 1.34265)\cdot \\
&\cdot(y^2 + 1.60552y + 1.24899) + 0.0289652x^3(y + 0.971945)(y^2 - 1.11574y + 1.96416\cdot \\
&\cdot(y^2 + 1.1252y + 1.08393) + 0.0856599x^2(y^2 - 2.33439y + 3.41084)(y^2 + 0.0772159y + 0.107608)\cdot \\
&+(y^2 + 2.63214y + 3.31506) + 0.00633779x(y + 1.2174)(y^2 - 2.35279y + 3.40781)(y^2 - 0.338931y \\
&2.2732)(y^2 + 2.64044y + 3.34121) + 0.00771335(y^2 - 2.35279y + 3.40781)(y^2 - 2.35279y + 3.40781)\cdot \\
&\cdot(y^2 + 2.64044y + 3.34121)(y^2 + 2.64044y + 3.34121))\left(
\begin{cases}
x & x^2y^2 \geq 1 \\
2x^3y^2 - x & \text{otherwise}
\end{cases}
\right) \\
&+ y^2(0.0517688x^8 - 0.00554989x^7(y + 2.21773) + x^6(-0.034754y^2 - 0.00669958 \\
&y - 0.023599) + 0.0319493x^5(y - 1.17169)(y^2 + 1.57296y + 1.36317) + 0.036961x^4(y - 1.77959) \\
&(y + 1.93546)(y^2 + 0.403155y + 3.88414) + 0.00711639x^3(y - 1.58435)(y + 0.621309)(y + 1.7218) \\
&(y^2 - 0.29928y + 4.96928) + 0.031441x^2(y - 0.657981)(y + 0.439392)(y^2 - 2.39252 \\
&y + 3.36771)(y^2 + 2.66083y + 3.4047) - 0.00633779x(y + 1.2174)(y^2 - 2.35279 \\
&y + 3.40781)(y^2 - 0.338931y + 2.2732)(y^2 + 2.64044y + 3.34121) - 0.00771335 \\
&(y^2 - 2.35279y + 3.40781)(y^2 - 2.35279y + 3.40781)(y^2 + 2.64044y + 3.34121) \\
&(y^2 + 2.64044y + 3.34121)))/ \\
&/((-0.0897306x^4 + x^3(0.0360682y + 0.0252627) - 0.186408x^2 \\
&(y - 0.594968)(y + 0.42493) + 0.0360816x(y + 1.2174)(y^2 - 0.338931y + 2.2732) + 0.0878257 \\
&(y^2 - 2.35279y + 3.40781)(y^2 + 2.64044y + 3.34121))^2).
\end{aligned}
$$

5. CONCLUSIONS AND FUTURE WORK

In this paper a method is proposed to systematically approximate the domain of attraction (DOA) of smooth hybrid non-linear systems. The proposed method is based on the algorithm of Vanelli and Vidyasagar [25], and it is capable to give a subset of the DOA for non-linear hybrid (switching) systems where the dynamics is continuous on the boundary of the different regimes of the state space.

The method first constructs the overall Lyapunov function from the individual Lyapunov functions of the sub-dynamics in a rational function form by approximating a maximal Lyapunov function V_i. All the necessary functions and algorithms have been implemented in a *Mathematica*-package. In addition, a potentially conservative but computationally feasible method is proposed to estimate the overall DOA from the individual sub-Lyapunov functions using a maximal fitting hypersphere. A manually tuned approximation method based on the manual tuning of the level value has also been proposed for two-dimensional systems.

The performance of the method was investigated by simulations using two two-dimensional example systems. The estimated DOA based on the maximal fitting hy-

persphere could be substantially improved by the proposed manual tuning method, and thus a good but still conservative estimate of the real DOA has been achieved.

Further work will be focused on using advanced optimization methods for the improvement of the automatic DOA computation part of the algorithm and defining less restrictive conditions for the system dynamics.

APPENDIX

Properties of maximal Lyapunov-functions

Proofs of the next two theorems can be found in [25].

Theorem A.1. Suppose f is continuously differentiable in some neighbourhood of the origin. Then there exists a continuous function $V : A \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which is continuous, monotonically increasing and $\gamma(0) = 0$ (γ belongs to class K due to Hahn in [11]) such that

(i) $V(0) = 0, V(x) > 0, \forall x \in A\backslash\{0\}$,

(ii) $\dot{V}(x) = -\gamma(|x|), \forall x \in A$,

(iii) $V(x) \to \infty$ as $x \to \partial A$.

Moreover, if f is Lipschitz-continuous on A then V can be selected to be continuously differentiable on A and

(iv) $V(x) \to \infty$ as $|x| \to \infty$.

Theorem A.2. Suppose f is Lipschitz-continuous on A. Then in order for an open set B containing the origin to be the DOA of system (1), it is necessary and sufficient that there exists a continuous function $V : B \to \mathbb{R}_0^+$ and a positive definite function ψ such that conditions of Theorem 2.7 hold true.

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