Yinggao Zhou; Min Wu

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ON EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF A KIND OF RAYLEIGH EQUATION WITH A DEVIATING ARGUMENT*

Yinggao Zhou, Min Wu, Changsha

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Abstract. The existence of positive periodic solutions for a kind of Rayleigh equation with a deviating argument

$$x''(t) + f(x'(t)) + g(t, x(t - \tau(t))) = p(t)$$

is studied. Using the coincidence degree theory, some sufficient conditions on the existence of positive periodic solutions are obtained.

Keywords: Rayleigh equations, positive periodic solution, a priori estimate

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1. Introduction

Consider the Rayleigh equation with a deviating argument

$$x''(t) + f(x'(t)) + g(t, x(t - \tau(t))) = p(t), \tag{1.1}$$

where $f, p, \tau \in C(\mathbb{R}, \mathbb{R})$ are T-periodic, $g \in C(\mathbb{R}^2, \mathbb{R})$ are T-periodic in its first argument, and $T > 0$.

In the past few years, the existence of positive periodic solutions for a kind of second order differential equations has received a lot of attention. For example, in [1], [3]–[5], [7]–[8] the differential equations

$$x''(t) + a(t)x(t) = f(t, x(t)),$$
$$x''(t) + f(t, x(t), x'(t)) = 0$$

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were studied. However, few results on the existence of positive periodic solutions for Rayleigh equations were found. Since the theory of existence of positive periodic solutions of the differential equation with retarded argument plays an important role in mathematical ecology and nonlinear elasticity, we discuss in this paper the existence of positive T-periodic solutions to Eq. (1.1). By using the coincidence degree theory and an improved a priori estimate, we obtain some sufficient conditions for the existence of positive T-periodic solution of Eq. (1.1).

For the sake of convenience, throughout this paper we will adopt the following notation. Let $X = \{x \in C^1(\mathbb{R},\mathbb{R})\colon x(t+T) = x(t)\}$ with the norm $\|x\|_1 = \max\limits_{t\in[0,T]}\{|x(t)|, |x'(t)|\}$ and $Z = \{x \in C(\mathbb{R},\mathbb{R})\colon x(t+T) = x(t)\}$ with the norm $\|x\|_0 = \max\limits_{t\in[0,T]} |x(t)|$. Then both $(X, \|\cdot\|_1)$ and $(Z, \|\cdot\|_0)$ are Banach spaces. Define respectively operators L and N as

$$L\colon X \cap C^2(\mathbb{R},\mathbb{R}) \to Z, \quad x(t) \mapsto x''(t), \tag{1.2}$$

and

$$N\colon X \to Z, \quad x(t) \mapsto -f(x'(t)) - g(t, x(t-\tau(t))) + p(t). \tag{1.3}$$

We know that $\operatorname{Ker} L = \mathbb{R}$, $\operatorname{Im} L = \bigl\{x \in Z\colon \int_0^T x(s)\,\mathrm{d}s = 0\bigr\}$. Define respectively projective operators P and Q as

$$P\colon X \to \operatorname{Ker} L, \quad x(t) \mapsto x(0),$$

and

$$Q\colon Z \to Z/\operatorname{Im} L, \quad x(t) \mapsto \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s.$$

Then we have $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$.

2. Preliminary results

For a positive number D, set

$$\Omega = \{x \in X\colon 0 < x(t) < D,\ |x'(t)| < D\}.$$

The following two lemmas which will be used in proofs of our main results are extracted from [2].

Lemma 2.1. *L is a Fredholm operator with null index.*

Lemma 2.2. *N is L-compact on $\bar{\Omega}$.*

In view of (1.2) and (1.3), the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1),$$

is equivalent to the equation

$$x''(t) + \lambda f(x'(t)) + \lambda g(t, x(t-\tau(t))) = \lambda p(t), \quad \lambda \in (0,1). \tag{2.1}$$

For convenience of the reader, we introduce the Continuation Theorem [2] as follows.

Lemma 2.3. *Let X and Z be two Banach spaces. Suppose that $L\colon \operatorname{dom}(L) \subset X \to Z$ is a Fredholm operator with index null and $N\colon X \to Z$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that the following conditions are satisfied:*
(1) *$Lx \neq \lambda Nx$ for $x \in \partial\Omega \cap \operatorname{dom}(L)$, $\lambda \in (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$;*
(3) *$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*
Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega}$.

Lemma 2.4. *Let $x(t)$ be a continuous differentiable T-periodic function ($T > 0$). Then for any $t_* \in (-\infty, \infty)$*

$$\max_{t\in[t_*,t_*+T]} |x(t)| \leqslant |x(t_*)| + \frac{1}{2}\int_0^T |x'(s)|\,\mathrm{d}s.$$

Proof. Choose $t^* \in [t_*, t_*+T]$ such that $|x(t^*)| = \max\limits_{t\in[t_*,t_*+T]} |x(t)|$. Then

$$|x(t^*)| = \left| x(t_*) + \int_{t_*}^{t^*} x'(s)\,\mathrm{d}s \right| \leqslant |x(t_*)| + \int_{t_*}^{t^*} |x'(s)|\,\mathrm{d}s$$

and

$$|x(t^*)| = |x(t^*-T)| = \left| x(t_*) - \int_{t^*-T}^{t_*} x'(s)\,\mathrm{d}s \right| \leqslant |x(t_*)| + \int_{t^*-T}^{t_*} |x'(s)|\,\mathrm{d}s.$$

Combining the above two inequalities, we have

$$|x(t^*)| \leqslant |x(t_*)| + \frac{1}{2}\int_{t^*-T}^{t^*} |x'(s)|\,\mathrm{d}s = |x(t_*)| + \frac{1}{2}\int_0^T |x'(s)|\,\mathrm{d}s.$$

The proof is complete. $\square$

Lemma 2.5 (Wirtinger inequality [6]). *Let $x(t)$ be a twice continuous differentiable T-periodic function. Then*

$$\int_0^T |x'(s)|^2 \, \mathrm{d}s \leqslant \Big(\frac{T}{2\pi}\Big)^2 \int_0^T |x''(s)|^2 \, \mathrm{d}s.$$

3. Main results

Theorem 3.1. *Assume that there exist constants $d > 0$, $r \geqslant 0$ and $K \geqslant 0$, not both zero, such that*

(H1) $-rx - K \leqslant g(t,x) - p(t) < 0$ *for* $t \in \mathbb{R}$, $x > d$;

(H2) $f(0) = 0$, $g(t,0) - p(t) > 0$ *for* $t \in \mathbb{R}$.

If $rT^2 < 4\pi$, then Eq. (1.1) *has at least one positive T-periodic solution.*

Proof. Let $x = x(t)$ be any positive T-periodic solution of Eq. (2.1). Then there exists $\xi \in [0,T]$ such that

$$x(\xi) = \max_{t\in[0,T]} x(t).$$

It follows that

$$x'(\xi) = 0, \quad x''(\xi) \leqslant 0. \tag{3.1}$$

By (2.1), (3.1), and (H2) we have

$$g(\xi, x(\xi - \tau(\xi))) - p(\xi) \geqslant 0.$$

Note that $x(t)$ is a positive T-periodic solution of Eq. (2.1). It follows from (H1) that

$$0 \leqslant x(\xi - \tau(\xi)) \leqslant d.$$

Let $\xi - \tau(\xi) = mT + t^*$, where $t^* \in [0,T]$ and m is an integer. Then

$$0 \leqslant x(t^*) \leqslant d. \tag{3.2}$$

By Lemma 2.4, we have

$$\|x\|_0 \leqslant d + \frac{1}{2}\int_0^T |x'(s)| \, \mathrm{d}s \leqslant d + \frac{T}{2}\|x'\|_0.$$

Let $E_1 = \{t\colon\ t \in [0,T],\ 0 \leqslant x(t-\tau(t)) \leqslant d\}$, $E_2 = \{t\colon\ t \in [0,T], x(t-\tau(t)) > d\}$. Applying the Schwarz inequality and Lemma 2.5, we have

$$\begin{aligned}
&\int_0^T |x''(s)|^2\,\mathrm{d}s \\
&\qquad = \lambda\Big[-\int_0^T f(x'(s))x''(s)\,\mathrm{d}s - \int_0^T [g(s,x(s-\tau(s))) - p(s)]x''(s)\,\mathrm{d}s\Big] \\
&\qquad \leqslant \int_0^T |g(s,x(s-\tau(s))) - p(s)||x''(s)|\,\mathrm{d}s \\
&\qquad = \Big(\int_{E_1} + \int_{E_2}\Big)|g(s,x(s-\tau(s))) - p(s)||x''(s)|\,\mathrm{d}s \\
&\qquad \leqslant g_d \int_{E_1} |x''(s)|\,\mathrm{d}s + (r\|x\|_0 + K)\int_{E_2} |x''(s)|\,\mathrm{d}s \\
&\qquad \leqslant (r\|x\|_0 + g_d + K)\int_0^T |x''(s)|\,\mathrm{d}s \\
&\qquad \leqslant \Big(\frac{r}{2}\int_0^T |x'(s)|\,\mathrm{d}s + rd + g_d + K\Big)\int_0^T |x''(s)|\,\mathrm{d}s \\
&\qquad \leqslant \sqrt{T}\Bigg[\frac{r\sqrt{T}}{2}\sqrt{\int_0^T |x'(s)|^2\,\mathrm{d}s} + rd + g_d + K\Bigg]\sqrt{\int_0^T |x''(s)|^2\,\mathrm{d}s} \\
&\qquad \leqslant \sqrt{T}\Bigg[\frac{rT\sqrt{T}}{4\pi}\sqrt{\int_0^T |x''(s)|^2\,\mathrm{d}s} + rd + g_d + K\Bigg]\sqrt{\int_0^T |x''(s)|^2\,\mathrm{d}s} \\
&\qquad = \frac{rT^2}{4\pi}\int_0^T |x''(s)|^2\,\mathrm{d}s + \sqrt{T}(rd + g_d + K)\sqrt{\int_0^T |x''(s)|^2\,\mathrm{d}s},
\end{aligned}$$

where $g_d = \max\limits_{0\leqslant t\leqslant T,\,0\leqslant x\leqslant d} |g(t,x) - p(t)|$. And so, we have

$$\int_0^T |x''(s)|^2\,\mathrm{d}s \leqslant T\Big(\frac{4\pi(rd + g_d + K)}{4\pi - rT^2}\Big)^2.$$

By Lemma 2.4, we get

$$\|x'\|_0 \leqslant \frac{1}{2}\int_0^T |x''(s)|\,\mathrm{d}s \leqslant \frac{\sqrt{T}}{2}\sqrt{\int_0^T |x''(s)|^2\,\mathrm{d}s} \leqslant \frac{2\pi T(rd + g_d + K)}{4\pi - rT^2} \triangleq M_1.$$

It follows that

$$\|x\|_0 \leqslant d + \frac{1}{2}M_1 T \triangleq M_0.$$

Let $M > \max\{M_0, M_1\}$ and

$$\Omega_1 = \{x \in X\colon\ 0 < x(t) < M,\ |x'(t)| < M\}.$$

By Lemma 2.1 and Lemma 2.2, we know that L is a Fredholm operator with index null and N is L-compact on $\bar{\Omega}_1$. Due to the bound of the positive periodic solution given above, we know that $Lx \neq \lambda Nx$ for any $x \in \partial\Omega_1 \cap \operatorname{dom} L$ and $\lambda \in (0,1)$. Since for any $x \in \partial\Omega_1 \cap \operatorname{Ker} L$, either $x = M$ $(> d)$ or $x = 0$, in view of (H1) and (H2) we have

$$\begin{aligned} QNx &= \frac{1}{T}\int_0^T [-f(0) - g(s,x) + p(s)]\,\mathrm{d}s \\ &= -\frac{1}{T}\int_0^T [g(s,x) - p(s)]\,\mathrm{d}s \neq 0. \end{aligned}$$

Define the transformation H as

$$H(x,\mu) = (1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(s,x) - p(s)]\,\mathrm{d}s, \quad \mu \in [0,1].$$

Then

$$H(0,\mu) = -\mu \cdot \frac{1}{T}\int_0^T [g(s,0) - p(s)]\,\mathrm{d}s < 0, \quad \mu \in [0,1]$$

and

$$H(M,\mu) = (1-\mu)M - \mu \cdot \frac{1}{T}\int_0^T [g(s,M) - p(s)]\,\mathrm{d}s > 0, \quad \mu \in [0,1].$$

Hence, for any $x \in \partial\Omega_1 \cap \operatorname{Ker} L$ and $\mu \in [0,1]$ we have $H(x,\mu) \neq 0$. Using the homotopic invariance theorem, we obtain

$$\begin{aligned} \deg\{QNx, \Omega_1 \cap \operatorname{Ker} L, 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g(s,x) - p(s)]\,\mathrm{d}s, \Omega_1 \cap \operatorname{Ker} L, 0\right\} \\ &= \deg\{x, \Omega_1 \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

By Lemma 2.3 there exists one positive T-periodic solution of (1.1). This completes the proof. □

Similarly, we can get the next theorem.

Theorem 3.2. *Assume that there exist constants $d > 0$, $r \geqslant 0$ and $K \geqslant 0$, not both zero, such that*

(H3) $0 < g(t,x) - p(t) \leqslant rx + K$ *for* $t \in \mathbb{R}$, $x > d$;

(H4) $f(0) = 0$, $g(t,0) - p(t) < 0$ *for* $t \in \mathbb{R}$.

If $rT^2 < 4\pi$, then Eq. (1.1) *has at least one positive T-periodic solution.*

According to the above proofs, we can get the following Corollaries 3.1 and 3.2.

Corollary 3.1. *Assume that* (H2) *holds. If there exist positive constants d and K such that*

(C1) $-K \leqslant g(t,x) - p(t) < 0$ *for* $t \in \mathbb{R}$, $x > d$,

then Eq. (1.1) *has at least one positive T-periodic solution.*

Corollary 3.2. *Assume that* (H4) *holds. If there exist positive constants d and K such that*

(C2) $0 < g(t,x) - p(t) \leqslant K$ *for* $t \in \mathbb{R}$, $x > d$,

then Eq. (1.1) *has at least one positive T-periodic solution.*

Example 3.1. Let $g(t,x) = x^{1/3} + 2$ for $t \in \mathbb{R}$, $x \leqslant 0$, and $g(t,x) = -\mathrm{e}^{1-x}$ for $t \in \mathbb{R}$, $x > 0$. Then the Rayleigh equation

$$x''(t) + x'^3(t) + g(t, x(t - \sin t)) = \sin^2 t \tag{3.3}$$

has at least one positive 2π-periodic solution.

Proof. From (3.3), we have $f(x) = x^3$ and $f(0) = 0$, $\tau(t) = \sin t$ and $p(t) = \sin^2 t$ are 2π-periodic, $0 > g(t,x) - p(t) \geqslant -2$ for $t \in \mathbb{R}$, $x > 1$, and $g(t,0) - p(t) > 1$ for $t \in \mathbb{R}$. Therefore, all the conditions needed in Corollary 3.1 are satisfied. From Corollary 3.1, Eq. (3.3) has at least one positive 2π-periodic solution. □

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Author's address: *Y. Zhou*, School of Mathematical Science and Computing Technology, Central South University, Changsha, Hunan 410083, P. R. China, e-mail: `ygzhou@mail.csu.edu.cn`, and School of Information Science and Engineering, Central South University, Changsha, Hunan 410083, P. R. China; *M. Wu*, School of Information Science and Enginering, Central Souh University, Changsha, Hunan 410083, P. R. China.