Czechoslovak Mathematical Journal

Ivan Chajda; Jan Kühr

Join-semilattices whose sections are residuated po-monoids

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1107–1127

Persistent URL: http://dml.cz/dmlcz/140444

Czechoslovak Mathematical Journal, 58 (133) (2008), 1107–1127

JOIN-SEMILATTICES WHOSE SECTIONS ARE RESIDUATED PO-MONOIDS

Ivan Chajda and Jan Kühr, Olomouc

(Received January 8, 2007)

Abstract. We generalize the concept of an integral residuated lattice to join-semilattices with an upper bound where every principal order-filter (section) is a residuated semilattice; such a structure is called a *sectionally residuated semilattice.* Natural examples come from propositional logic. For instance, implication algebras (also known as Tarski algebras), which are the algebraic models of the implication fragment of the classical logic, are sectionally residuated semilattices such that every section is even a Boolean algebra. A similar situation rises in case of the Łukasiewicz multiple-valued logic where sections are bounded commutative BCK-algebras, hence MV-algebras. Likewise, every integral residuated (semi)lattice is sectionally residuated in a natural way. We show that sectionally residuated semilattices can be axiomatized as algebras $(A, r, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 3, 2, 2, 0\rangle$ where $(A, \rightarrow, \rightsquigarrow, 1)$ is a $\{\rightarrow, \rightsquigarrow, 1\}$-subreduct of an integral residuated lattice. We prove that every sectionally residuated *lattice* can be isomorphically embedded into a residuated lattice in which the ternary operation r is given by $r(x, y, z) = (x \cdot y) \vee z$. Finally, we describe mutual connections between involutive sectionally residuated semilattices and certain biresiduation algebras.

Keywords: residuated lattice, residuated semilattice, biresiduation algebra, pseudo-MV-algebra, sectionally residuated semilattice, sectionally residuated lattice

MSC 2010: 06D35, 06F05, 06F35

1. Introduction

A *residuated partially ordered monoid* is a structure $\mathbf{A} = (A, \leqslant, \cdot, \rightarrow, \rightsquigarrow, 1)$ such that $(A, \cdot, 1)$ is a monoid, $(A, \leqslant)$ is a poset and

$$x \cdot y \leqslant z \quad \text{iff} \quad x \leqslant y \rightarrow z \quad \text{iff} \quad y \leqslant x \rightsquigarrow z \tag{1.1}$$

This work was supported by the Czech Government via the project MSM6198959214.

for all $x, y, z \in A$. We will assume in addition that $(A, \leqslant)$ is a join-semilattice or a lattice, and that the multiplicative identity 1 is its greatest element. In this case $\mathbf{A}$ is called an *integral residuated semilattice* or *lattice*, respectively.

In the last years residuated structures have become a subject of an intensive research in the logical context as well as in their own right. The algebraic model of a propositional logic usually is (or can equivalently be regarded as) a bounded residuated lattice, while the models of the implication fragment are join-semilattices with an upper bound such that every *section* (principal order-filter) is a bounded residuated lattice, but the whole semilattice is not residuated. This is the case at least for the classical propositional logic as well as for the Łukasiewicz many valued logic (and its non-commutative extension).

Indeed, the algebraic counterpart of implication in the classical calculus is given by implication algebras (also called Tarski algebras), which are join-semilattices where each section is a Boolean algebra. The algebras for the implication fragment of the Łukasiewicz logic are ŁBCK-algebras, i.e., commutative BCK-algebras satisfying prelinearity, which form join-semilattices whose sections are MV-algebras. Actually, semilattices with the property that every section is an MV-algebra lead to commutative BCK-algebras (weak implication algebras [5]) that need not be embedable into an MV-algebra.

Also every section of any residuated (semi)lattice is a residuated (semi)lattice. Thus there exist natural examples of *sectionally residuated* structures that generalize known integral residuated lattices.

In Section 2 we recall some relevant facts about residuated lattices and their $\{\rightarrow, \rightsquigarrow, 1\}$-subreducts that are called biresiduation algebras or pseudo-BCK-algebras, and about pseudo-MV-algebras, which are a non-commutative generalization of MV-algebras. In Section 3 we introduce the notion of a sectionally residuated semilattice and prove that sectionally residuated semilattices can alternatively be regarded as algebras $(A, r, \rightarrow, \rightsquigarrow, 1)$ of type $\langle 3, 2, 2, 0\rangle$ satisfying certain identities, such that the reduct $(A, \rightarrow, \rightsquigarrow, 1)$ is a biresiduation algebra. Section 4 is devoted to sectionally residuated lattices; we show that they can be embedded into bounded residuated lattices expanded with the ternary operation r which is defined by $r(x, y, z) = (x \cdot y) \vee z$. Finally, in Section 5 we are concerned with sectionally residuated semilattices the sections of which are involutive residuated lattices.

2. Preliminaries

A *residuated lattice* is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, \rightsquigarrow, 1)$ such that $(A, \vee, \wedge)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and the condition (1.1) is satisfied, i.e., for all $x, y, z \in L$ we have

$$x \cdot y \leqslant z \quad \text{iff} \quad x \leqslant y \to z \quad \text{iff} \quad y \leqslant x \rightsquigarrow z,$$

where $\leqslant$ is the partial order induced by the lattice operations. More generally, if $(A, \vee)$ is a join-semilattice, $(A, \cdot, 1)$ is a monoid and (1.1) is satisfied, then $\mathbf{A} = (A, \vee, \cdot, \to, \rightsquigarrow, 1)$ is called a *residuated (join-)semilattice*.

If, moreover, the monoid identity 1 is the greatest element of $(A, \leqslant)$, then $\mathbf{A}$ is said to be an *integral* residuated lattice or semilattice, respectively. Since we restrict ourselves exclusively to integral residuated lattices and semilattices, we will omit the adjective 'integral' unless we want to emphasize that 1 is a greatest element.

The concept of a *bounded* (integral) residuated lattice and semilattice is obtained by adding the least element 0 of $(A, \leqslant)$, provided it exists, to the similarity type as a new nullary operation.

Commutative residuated lattices were first studied by Ward and Dilworth [15] as a generalization of residuation in lattices of ideals of commutative rings with an identity element. For background on residuated lattices we refer to the survey paper [10] that contains an overview of recent results.

In the following lemma we collect basic properties of residuated semilattices:

Lemma 2.1. *In any residuated semilattice:*

(1) $x \leqslant y$ *implies* $x \cdot z \leqslant y \cdot z$ *and* $z \cdot x \leqslant z \cdot y$;

(2) *if* $\bigvee_{i \in I} x_i$ *exists then* $\left(\bigvee_{i \in I} x_i\right) \cdot y = \bigvee_{i \in I} x_i \cdot y$ *and* $y \cdot \left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} y \cdot x_i$;

(3) $x \to x = x \rightsquigarrow x = 1$, $x \to 1 = x \rightsquigarrow 1 = 1$ *and* $1 \to x = 1 \rightsquigarrow x = x$;

(4) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$, *the same for* $\rightsquigarrow$;

(5) $x \leqslant y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$;

(6) $(x \cdot y) \to z = x \to (y \to z)$, $(x \cdot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;

(7) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;

(8) $x \leqslant y \to z$ *iff* $y \leqslant x \rightsquigarrow z$;

(9) $y \leqslant x \to y$, $y \leqslant x \rightsquigarrow y$;

(10) $x \leqslant (x \to y) \rightsquigarrow y$, $x \leqslant (x \rightsquigarrow y) \to y$;

(11) $x \to y \leqslant (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leqslant (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;

(12) *if* $\bigvee_{i \in I} x_i$ *exists then so does* $\bigwedge_{i \in I} (x_i \to y)$ *and* $\left(\bigvee_{i \in I} x_i\right) \to y = \bigwedge_{i \in I} (x_i \to y)$, *and the same for* $\rightsquigarrow$; *in particular,* $(x \vee y) \to y = x \to y$ *and* $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$.

The residuation equivalences (1.1) can be captured by a few simple identities and hence residuated semilattices form a variety (another axiomatization of residuated semilattices can be found in [2]):

Lemma 2.2. *An algebra* $\mathbf{A} = (A, \vee, \cdot, \to, \rightsquigarrow, 1)$ *of type* $\langle 2, 2, 2, 2, 0\rangle$ *is a residuated semilattice if and only if* $(A, \vee)$ *is a join-semilattice with* 1 *as the greatest element,* $(A, \cdot, 1)$ *is a monoid, and* $\mathbf{A}$ *satisfies the identities*

$$(x \cdot y) \to z = x \to (y \to z), \quad (x \cdot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z), \tag{2.1}$$

$$((x \to y) \cdot x) \vee y = (x \cdot (x \rightsquigarrow y)) \vee y = y, \tag{2.2}$$

$$x \to (x \vee y) = x \rightsquigarrow (x \vee y) = 1. \tag{2.3}$$

Proof. It is easily seen that (2.1), (2.2) and (2.3) hold in any residuated semilattice. Conversely, let $\mathbf{A}$ be an algebra that fulfils (2.1), (2.2) and (2.3). Note that by (2.2) and (2.3) we obtain $x \leqslant y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$. Now, if $x \cdot y \leqslant z$ then $x \to (y \to z) = (x \cdot y) \to z = 1$, so that $x \leqslant y \to z$, and conversely, $x \leqslant y \to z$ entails $(x \cdot y) \to z = x \to (y \to z) = 1$, thus $x \cdot y \leqslant z$. Similarly, we can show $x \cdot y \leqslant z$ iff $y \leqslant x \rightsquigarrow z$. □

A *biresiduation algebra* [3] is an algebra $\mathbf{A} = (A, \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 0\rangle$ which is a $\{\to, \rightsquigarrow, 1\}$-subreduct (i.e., a subalgebra of the $\{\to, \rightsquigarrow, 1\}$-reduct) of an integral residuated lattice. Biresiduation algebras form a quasi-variety axiomatized by the following identities and quasi-identity:

$$(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1, \tag{2.4}$$

$$(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1, \tag{2.5}$$

$$1 \to x = x, \tag{2.6}$$

$$1 \rightsquigarrow x = x, \tag{2.7}$$

$$x \to 1 = 1, \tag{2.8}$$

$$x \to y = 1 \ \&\ y \to x = 1 \ \Rightarrow\ x = y. \tag{2.9}$$

We should note that if $\to$ and $\rightsquigarrow$ coincide, i.e., $x \to y = x \rightsquigarrow y$ for all $x, y \in A$, then $(A, \to, 1)$ becomes a BCK-algebra. As known, BCK-algebras do not form a variety, and hence neither do biresiduation algebras.

For any biresiduation algebra $\mathbf{A} = (A, \to, \rightsquigarrow, 1)$, the relation $\leqslant$ defined by $x \leqslant y$ iff $x \to y = 1$ (or equivalently, $x \rightsquigarrow y = 1$) is a partial order on A with 1 as the greatest element. The poset $(A, \leqslant)$ in general has no particular properties since any poset with a greatest element 1 can be made into a BCK-algebra by putting $x \to y := 1$ if $x \leqslant y$, and $x \to y := y$ otherwise.

Georgescu and Iogulescu [9] introduced a non-commutative extension of BCK-algebras under the name *pseudo-BCK-algebras*, which are essentially the same as van Alten's biresiduation algebras. As a matter of fact, it was proved by the second author [11] that pseudo-BCK-algebras are exactly the $\{\to, \rightsquigarrow, 1\}$-subreducts of integral residuated lattices.

Even strongly, if $(A, \leqslant)$ is a join-semilattice with the associated join operation $\vee$, then the algebra $\mathbf{A} = (A, \vee, \to, \rightsquigarrow, 1)$ is a $\{\vee, \to, \rightsquigarrow, 1\}$-subreduct of a residuated lattice (this was proved in [3], and independently in [12]). These *biresiduation semilattices* form a variety which can be axiomatized by (2.4)–(2.8) and

$$x \to (x \vee y) = 1, \tag{2.10}$$

$$x \vee ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow y. \tag{2.11}$$

As a consequence one readily sees that, in addition to (2.4)–(2.9), biresiduation algebras satisfy (3)–(5) and (7)–(12) of Lemma 2.1.

BCK-algebras satisfying the identity $(x \to y) \to y = (y \to x) \to x$ are traditionally referred to as *commutative BCK-algebras*. An appropriate generalization is the class of biresiduation algebras satisfying the identities

$$\begin{aligned} (x \to y) \rightsquigarrow y &= (y \to x) \rightsquigarrow x, \\ (x \rightsquigarrow y) \to y &= (y \rightsquigarrow x) \to x. \end{aligned} \tag{2.12}$$

Such algebras are called *commutative pseudo-BCK-algebras* in [12]. It can be easily shown that if $\mathbf{A} = (A, \to, \rightsquigarrow, 1)$ fulfil (2.12), then $(A, \leqslant)$ is a join-semilattice in which

$$x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y, \tag{2.13}$$

however, a biresiduation algebra which is a join-semilattice with respect to $\leqslant$ need not fulfil (2.12).

A *pseudo-MV-algebra* $\mathbf{A} = (A, \oplus, {}^-, {}^\sim, 0, 1)$ is a monoid $(A, \oplus, 0)$ endowed with a constant 1 and two unary operations ${}^-$ and ${}^\sim$, such that $\mathbf{A}$ satisfies the identities

$$\begin{aligned} &x \oplus 1 = 1 = 1 \oplus x, \\ &1^- = 0 = 1^\sim, \\ &(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-, \\ &x \oplus (y \cdot x^\sim) = y \oplus (x \cdot y^\sim) = (y^- \cdot x) \oplus y = (x^- \cdot y) \oplus x, \\ &(x^- \oplus y) \cdot x = y \cdot (x \oplus y^\sim), \\ &(x^-)^\sim = x, \end{aligned}$$

where the additional binary operation $\cdot$ is defined by $x \cdot y := (x^- \oplus y^-)^\sim$.

Pseudo-MV-algebras were introduced by Georgescu and Iorgulescu in [8], and independently by Rachůnek in [14] to be a non-commutative generalization of MV-algebras, the algebraic counterpart of the Łukasiewicz many-valued propositional calculus. In fact, MV-algebras agree with commutative pseudo-MV-algebras since the negations $^-$ and $^\sim$ coincide provided the addition $\oplus$ is commutative. For background on MV-algebras we refer to [6].

It is important to point out that pseudo-MV-algebras are termwise equivalent to bounded residuated lattices satisfying (2.13) [7], [10], and to bounded biresiduation algebras satisfying (2.12) [9]:

(a) Given a pseudo-MV-algebra $(A, \oplus, ^-, ^\sim, 0, 1)$, we define

$$\begin{aligned}
&x \cdot y := (x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-,\\
&x \to y := x^- \oplus y,\\
&x \rightsquigarrow y := y \oplus x^\sim,\\
&x \vee y := (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y,\\
&x \wedge y := (x \to y) \cdot x = x \cdot (x \rightsquigarrow y) = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-.
\end{aligned}$$

Then $(A, \vee, \wedge, \cdot, \to, \rightsquigarrow, 0, 1)$ is a bounded residuated lattice that obeys (2.13), and so the reduct $(A, \to, \rightsquigarrow, 0, 1)$ is a bounded biresiduation algebra satisfying (2.12). The lattice $(A, \vee, \wedge)$ is distributive.

(b) Let $(A, \to, \rightsquigarrow, 0, 1)$ be a bounded biresiduation algebra satisfying the identities (2.12). If we put

$$\begin{aligned}
x^- &:= x \to 0,\\
x^\sim &:= x \rightsquigarrow 0,\\
x \oplus y &:= x^\sim \to y = y^- \rightsquigarrow x,
\end{aligned}$$

then $(A, \oplus, ^-, ^\sim, 0, 1)$ is a pseudo-MV-algebra. Furthermore, this also means that whenever $(A, \vee, \wedge, \cdot, \to, \rightsquigarrow, 0, 1)$ is a bounded residuated lattice satisfying (2.13) and the operations $\oplus$, $^-$, $^\sim$ are defined as above, then $(A, \oplus, ^-, ^\sim, 0, 1)$ again is a pseudo-MV-algebra. Note that $x \oplus y = (x^- \cdot y^-)^\sim = (x^\sim \cdot y^\sim)^-$.

Another equivalent counterpart of pseudo-MV-algebras are Ceterchi's *pseudo-Wajsberg algebras* [4], which employ $\to$, $\rightsquigarrow$, $^-$, $^\sim$ and 1 as fundamental operations.

3. Sectionally residuated semilattices

We start with several natural examples. Throughout, by a *section* in a poset $(P, \leqslant)$ with a greatest element 1 we shall mean any principal order-filter $[a, 1] = \{x \in P \colon a \leqslant x\}$, $a \in P$.

Example 3.1. An *implication algebra* [1] is an algebra $(A, \to)$ with a single binary operation $\to$ satisfying the identities

$$(x \to y) \to x = x, \tag{3.1}$$

$$(x \to y) \to y = (y \to x) \to x, \tag{3.2}$$

$$x \to (y \to z) = y \to (x \to z). \tag{3.3}$$

Implication algebras (also known as *Tarski algebras*) are the algebraic models of the implication fragment of the classical propositional calculus. That is, if $(B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then $(B, \to)$ is an implication algebra with $x \to y := x' \vee y$, and every implication algebra $(A, \to)$ embeds into $(B, \to)$ for some Boolean algebra.

Every implication algebra is a join-semilattice with a greatest element 1 where the supremum $x \vee y$ of $x, y \in A$ is given by $x \vee y = (x \to y) \to y$. Moreover, for all $a \in A$, the section $[a, 1]$ forms a Boolean lattice in which $x^a := x \to a$ is the complement of $x \in [a, 1]$, and $x \wedge_a y := (x^a \vee y^a)^a = (x \to (y \to a)) \to a$ is the infimum of $x, y \in [a, 1]$.

Thus every section $[a, 1]$ in an implication algebra is a commutative residuated lattice where the multiplication $\cdot_a$ agrees with the meet $\wedge_a$.

Example 3.2. Recall that a BCK-algebra $(A, \to, 1)$ is commutative if it satisfies the identity (3.2); commutative BCK-algebras form an equational class axiomatized e.g. by the identities (3.2), (3.3), $x \to x = 1$ and $x \to 1 = 1$. The models of the implication in the infinite-valued logic of Łukasiewicz are commutative BCK-algebras satisfying in addition the identity

$$(x \to y) \to (y \to x) = y \to x. \tag{3.4}$$

(These algebras are sometimes called *Łukasiewicz BCK-algebras*, in short *ŁBCK-algebras.*) Indeed, for any MV-algebra $(M, \oplus, \neg, 0)$, the algebra $(M, \to, 1)$—where $x \to y := \neg x \oplus y$ and $1 := \neg 0$—is a commutative BCK-algebra satisfying (3.4), and every commutative BCK-algebra that fulfils (3.4) can be embedded into $(M, \to, 1)$ for a suitable MV-algebra $(M, \oplus, \neg, 0)$.

Regardless of the identity (3.4), all commutative BCK-algebras enjoy the property of being a sectionally residuated semilattice. Indeed, given any commutative BCK-algebra $(A, \to, 1)$, $x \vee y = (x \to y) \to y$ is the join of $x, y \in A$, and for an arbitrary $a \in A$, the section $[a, 1]$ is the carrier of an MV-algebra $([a, 1], \oplus_a, \neg_a, a)$, where $x \oplus_a y := (x \to a) \to y$ and $\neg_a x := x \to a$. The multiplication $\cdot_a$ on $[a, 1]$ is given by $x \cdot_a y := (x \to (y \to a)) \to a$.

We will show later in Theorem 5.3 that a similar situation rises in case of pseudo-MV-algebras that are algebras for the non-commutative version of the Łukasiewicz sentential logic, i.e., we will have join-semilattices whose sections are pseudo-MV-algebras.

Example 3.3. Let $(S, \vee, \cdot, \rightarrow, \rightsquigarrow, 1)$ be a residuated semilattice, and $a \in S$. Define

$$x \cdot_a y := (x \cdot y) \vee a,$$

for $x, y \in [a, 1]$. Then $([a, 1], \vee, \cdot_a, \rightarrow, \rightsquigarrow, 1)$ is a residuated semilattice. Analogously, if we are given a residuated lattice $(L, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, 1)$ and an arbitrary $a \in L$, then the structure $([a, 1], \vee, \wedge, \cdot_a, \rightarrow, \rightsquigarrow, 1)$ is again a residuated lattice.

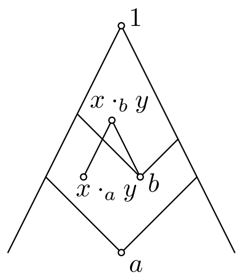

Figure 3.1. The compatibility condition (C)

In any case, we have a join-semilattice whose sections are residuated lattices (Boolean algebras or MV-algebras) or, more generally, residuated semilattices. These observations provide a motivation of the following extension of residuated (semi)lattices:

Definition 3.4. A *sectionally residuated semilattice* is a system

$$\mathbf{S} = (S, \vee, (\cdot_a, \rightarrow_a, \rightsquigarrow_a)_{a \in S}, 1)$$

such that

(i) $(S, \vee)$ is a join-semilattice with 1 at the top,
(ii) for every $a \in S$, $([a, 1], \vee, \cdot_a, \rightarrow_a, \rightsquigarrow_a, 1)$ is a residuated semilattice,
(iii) the following *compatibility condition* holds for all $a, b \in S$:

(C) If $a \leqslant b$ then $x \cdot_b y = (x \cdot_a y) \vee b$ for all $x, y \in [b, 1]$.

It is not hard to verify that the compatibility condition (C), which can be visualized by Fig. 3.1, is satisfied in Examples 3.1 and 3.2. Obviously, every residuated semilattice is sectionally residuated (Example 3.3), but the converse fails to be true, i.e., a sectionally residuated semilattice need not admit a residuated structure, as shown in the following simple examples:

Example 3.5. Let $S = \{a, b, 1\}$ and consider the join-semilattice $(S, \vee)$, where $a < 1$, $b < 1$ and $a \parallel b$. It is clear that $\{a, 1\}$ and $\{b, 1\}$ are carriers of two isomorphic commutative residuated lattices (Boolean algebras), and hence S is the carrier of a sectionally residuated semilattice. Suppose that there are binary operations $\cdot$, $\to$ and $\rightsquigarrow$ on S such that $(S, \vee, \cdot, \to, \rightsquigarrow, 1)$ is a residuated semilattice. In view of Lemma 2.1 (1), $a, b < 1$ would imply $a \cdot b \leqslant a, b$, which is impossible since the elements a, b have no common lower bound.

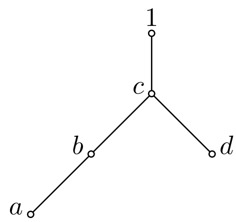

Figure 3.2

Example 3.6. Let $(S, \vee)$ be the join-semilattice from Fig. 3.2. The section $[a, 1] = \{a, b, c, 1\}$ is a residuated (semi)lattice with the operations $\cdot_a$, $\to_a$ and $\rightsquigarrow_a$ given by

$\cdot_a$	a	b	c	1
a	a	a	a	a
b	a	a	a	b
c	a	b	c	c
1	a	b	c	1

$\to_a$	a	b	c	1
a	1	1	1	1
b	b	1	1	1
c	b	b	1	1
1	a	b	c	1

$\rightsquigarrow_a$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	a	b	1	1
1	a	b	c	1

and $[d, 1] = \{d, c, 1\}$ forms a commutative residuated (semi)lattice with the operations $\cdot_d$ and $\to_d = \rightsquigarrow_d$ defined by

$\cdot_d$	d	c	1
d	d	d	d
c	d	d	c
1	d	c	1

$\to_d$	d	c	1
d	1	1	1
c	c	1	1
1	d	c	1

The section $[b, 1] = \{b, c, 1\}$ is equipped with the multiplication $\cdot_b$ inherited from $[a, 1]$ by the rule $x \cdot_b y := (x \cdot_a y) \vee b$. Therefore $(S, \vee, (\cdot_p, \to_p, \rightsquigarrow_p)_{p \in S}, 1)$ is a (non-commutative) sectionally residuated semilattice which is not residuated again because of the absence of a least element.

Lemma 3.7. *The compatibility condition* (C) *is equivalent to either of the following conditions:*

(C1) *If $a \leqslant b$ then $x \to_a y = x \to_b y$ for all $x, y \in [b, 1]$.*

(C2) *If $a \leqslant b$ then $x \rightsquigarrow_a y = x \rightsquigarrow_b y$ for all $x, y \in [b, 1]$.*

Proof. (C) $\Rightarrow$ (C1). Let $a \leqslant b$ and $x, y \in [b, 1]$. From $b \cdot_a x \leqslant b \leqslant y$ it follows $b \leqslant x \to_a y$, so $x \to_a y \in [b, 1]$. Now, from $(x \to_a y) \cdot_b x = ((x \to_a y) \cdot_a x) \vee b \leqslant y \vee b = y$ we obtain $x \to_a y \leqslant x \to_b y$. On the other hand, $(x \to_b y) \cdot_a x \leqslant ((x \to_b y) \cdot_a x) \vee b = (x \to_b y) \cdot_b x \leqslant y$ yields $x \to_b y \leqslant x \to_a y$. Altogether, $x \to_a y = x \to_b y$.

(C1) $\Rightarrow$ (C). Again, let $a \leqslant b$ and $x, y, z \in [b, 1]$. If $x \cdot_b y \leqslant z$ then $x \leqslant y \to_b z = y \to_a z$, whence $x \cdot_a y \leqslant z$ and so $(x \cdot_a y) \vee b \leqslant z \vee b = z$. Conversely, from $x \cdot_a y \leqslant (x \cdot_a y) \vee b \leqslant z$ it follows $x \leqslant y \to_a z = y \to_b z$ whence $x \cdot_b y \leqslant z$. Thus for any $z \in [b, 1]$, $x \cdot_b y \leqslant z$ iff $(x \cdot_a y) \vee b \leqslant z$, which settles $x \cdot_b y = (x \cdot_a y) \vee b$.

We can analogously show that (C) is equivalent to (C2). $\square$

It is more convenient to have some total operations on S instead of plenty of partial operations $(\cdot_a, \to_a, \rightsquigarrow_a)_{a \in S}$. The difficulty concerning the number of the partial operations $(\to_a, \rightsquigarrow_a)_{a \in S}$ on S can be overcome in the following way:

Given a sectionally residuated semilattice $\mathbf{S} = (S, \vee, (\cdot_a, \to_a, \rightsquigarrow_a)_{a \in S}, 1)$, we define two new total binary operations $\to$ and $\rightsquigarrow$ on S via

$$x \to y := (x \vee y) \to_y y \quad \text{and} \quad x \rightsquigarrow y := (x \vee y) \rightsquigarrow_y y. \tag{3.5}$$

We are going to show that, for each $a \in S$, the operations $\to_a$ and $\rightsquigarrow_a$ on the section $[a, 1]$ are the restrictions to $[a, 1]$ of $\to$ and $\rightsquigarrow$, respectively. Hence any sectionally residuated semilattice $(S, \vee, (\cdot_a, \to_a, \rightsquigarrow_a)_{a \in S}, 1)$ can be equivalently defined as a structure $(S, \vee, (\cdot_a)_{a \in S}, \to, \rightsquigarrow, 1)$ such that $(S, \vee)$ is a join-semilattice with an upper bound 1, and for every $a \in S$, $([a, 1], \vee, \cdot_a, \to, \rightsquigarrow, 1)$ is a residuated semilattice. This is a consequence of the following

Lemma 3.8. *Let $\mathbf{S} = (S, \vee, (\cdot_a, \to_a, \rightsquigarrow_a)_{a \in S}, 1)$ be a sectionally residuated semilattice and let $\to$ and $\rightsquigarrow$ be defined by* (3.5). *Then*

(i) $(x \vee y) \to y = x \to y$ *and* $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$ *for all $x, y \in S$,*

(ii) *for any $a \in S$, if $x, y \in [a, 1]$ then $x \to y = x \to_a y$ and $x \rightsquigarrow y = x \rightsquigarrow_a y$.*

Proof. We have $(x \vee y) \to y = (x \vee y \vee y) \to_y y = (x \vee y) \to_y y = x \to y$. Using the condition (C1) for $a \leqslant y \leqslant x \vee y$, we obtain $x \to y = (x \vee y) \to_y y = (x \vee y) \to_a y = x \to_a y$. The argument for $\rightsquigarrow$ is parallel. $\square$

Now we summarize basic properties of the operations $\to$ and $\rightsquigarrow$ defined by (3.5):

Lemma 3.9. *Let* $\mathbf{S} = (S, \vee, (\cdot_a)_{a\in S}, \rightarrow, \rightsquigarrow, 1)$ *be a sectionally residuated semilattice. The following hold:*

(1) $x \rightarrow x = 1$, $x \rightarrow 1 = 1$ *and* $1 \rightarrow x = x$, *and the same for* $\rightsquigarrow$;

(2) $x \leqslant y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$;

(3) $x \leqslant (x \rightarrow y) \rightsquigarrow y$, $x \leqslant (x \rightsquigarrow y) \rightarrow y$;

(4) *if* $x \leqslant y$ *then* $y \rightarrow z \leqslant x \rightarrow z$ *and* $z \rightarrow x \leqslant z \rightarrow y$, *and the same for* $\rightsquigarrow$;

(5) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$;

(6) $x \leqslant y \rightarrow z$ *iff* $y \leqslant x \rightsquigarrow z$;

(7) $x \rightarrow y \leqslant (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leqslant (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;

(8) *if* $\bigvee_{i\in I} x_i$ *exists then so does* $\bigwedge_{i\in I}(x_i \rightarrow y)$ *and* $\left(\bigvee_{i\in I} x_i\right) \rightarrow y = \bigwedge_{i\in I}(x_i \rightarrow y)$; *the same holds for* $\rightsquigarrow$.

Proof. (1) By (3.5) we have $x \rightarrow x = (x \vee x) \rightarrow_x x = x \rightarrow_x x = 1$, $x \rightarrow 1 = (x \vee 1) \rightarrow_1 1 = 1$ and $1 \rightarrow x = (1 \vee x) \rightarrow_x x = x$.

(2) If $x \leqslant y$ then $x \rightarrow y = (x \vee y) \rightarrow y = y \rightarrow y = 1$. Conversely, if $x \rightarrow y = 1$ then $(x \vee y) \rightarrow_y y = 1$ which entails $x \vee y \leqslant y$, so $x \leqslant y$.

(3) In the section $[y, 1]$ we have $x \vee y \leqslant ((x \vee y) \rightarrow_y y) \rightsquigarrow_y y = (x \rightarrow y) \rightsquigarrow y$ by Lemma 3.8.

(4) From $x \leqslant y$ it follows $x \vee z \leqslant y \vee z$ whence $x \rightarrow z = (x \vee z) \rightarrow_z z \geqslant (y \vee z) \rightarrow_z z = y \rightarrow z$ proving the first part of (4). Furthermore, $x \leqslant y$ implies

$$\begin{aligned} z \rightarrow x &= (z \vee x) \rightarrow_x x \leqslant (z \vee x) \rightarrow_x y = ((z \vee x) \rightarrow_x y) \wedge (y \rightarrow_x y) \\ &= (z \vee x \vee y) \rightarrow_x y = (z \vee y) \rightarrow_x y = (z \vee y) \rightarrow_y y = z \rightarrow y. \end{aligned}$$

(5) Using Lemma 3.8 we get

$$\begin{aligned} x \rightarrow (y \rightsquigarrow z) &= (x \vee ((y \vee z) \rightsquigarrow z)) \rightarrow ((y \vee z) \rightsquigarrow z) \\ &= ((x \vee z) \vee ((y \vee z) \rightsquigarrow z)) \rightarrow ((y \vee z) \rightsquigarrow z) \\ &= (x \vee z) \rightarrow ((y \vee z) \rightsquigarrow z) \\ &= (x \vee z) \rightarrow_z ((y \vee z) \rightsquigarrow_z z) \\ &= (y \vee z) \rightsquigarrow_z ((x \vee z) \rightarrow_z z) \\ &= (y \vee z) \rightsquigarrow ((x \vee z) \rightarrow z) \\ &= y \rightsquigarrow (x \rightarrow z). \end{aligned}$$

(6) This follows immediately from (2) and (5).

(7) By (3) we have $y \leqslant (y \rightarrow z) \rightsquigarrow z$, whence it follows $x \rightarrow y \leqslant x \rightarrow ((y \rightarrow z) \rightsquigarrow z) = (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$ by (4) and (5).

(8) Put $x = \bigvee_{i\in I} x_i$. By (4) we have $x \to y \leqslant x_i \to y$ for all $i \in I$. But if z is another common lower bound of all $x_i \to y$'s then by (6), $x_i \leqslant z \rightsquigarrow y$ for every $i \in I$, so $x \leqslant z \rightsquigarrow y$, which is equivalent to $z \leqslant x \to y$. Thus $x \to y = \bigwedge_{i\in I} (x_i \to y)$. □

Corollary 3.10. *Let* $\mathbf{S} = (S, \vee, (\cdot_a)_{a\in S}, \to, \rightsquigarrow, 1)$ *be a sectionally residuated semilattice. Then* $(S, \to, \rightsquigarrow, 1)$ *is a biresiduation algebra and* $(S, \vee, \to, \rightsquigarrow, 1)$ *is a biresiduation semilattice, i.e., it is a* $\{\vee, \to, \rightsquigarrow, 1\}$*-subreduct of an integral residuated lattice.*

P r o o f. By Lemma 3.9 (1), (3) and (7) $(S, \vee, \to, \rightsquigarrow, 1)$ satisfies (2.4)–(2.8), (2.10) and (2.11). □

One way of interpreting this result is that the operations $\to$, $\rightsquigarrow$ in sectionally residuated semilattices have essentially the same properties as the residua in residuated lattices. The converse of Corollary 3.10 does not hold (even when the semilattice is a lattice); the following is an example of a biresiduation semilattice that is not the reduct of any sectionally residuated semilattice:

Example 3.11. Let us come back to the sectionally residuated semilattice from Example 3.6. The operations $\to$ and $\rightsquigarrow$ defined by (3.5) are given by the tables

$\to$	a	b	c	d	1
a	1	1	1	c	1
b	b	1	1	c	1
c	b	b	1	c	1
d	b	b	1	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	c	1
b	c	1	1	c	1
c	a	b	1	c	1
d	a	b	1	1	1
1	a	b	c	d	1

If we add a new bottom element 0 (see Fig. 3.3) and define $x \to 0 = x \rightsquigarrow 0 := 0$ and $0 \to x = 0 \rightsquigarrow x := 1$ for all $x \in S$, then $(S \cup \{0\}, \to, \rightsquigarrow, 0, 1)$ is a bounded biresiduation algebra (lattice) which is not (sectionally) residuated. Indeed, if it were (sectionally) residuated then necessarily $a \cdot_0 d = 0$, which would yield $a \leqslant d \to 0 = 0$ by (1.1), a contradiction.

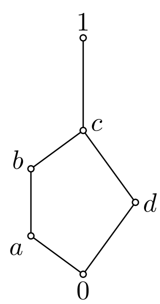

Figure 3.3

Let $\mathbf{S} = (S, \vee, (\cdot_a)_{a \in S}, \to, \rightsquigarrow, 1)$ be a sectionally residuated semilattice. Similarly to the case of partial operations $(\to_a, \rightsquigarrow_a)_{a \in S}$, which can be replaced by the operations $\to, \rightsquigarrow$ defined by (3.5), we would like to capture the family $(\cdot_a)_{a \in S}$ by a single total operation on S. Unfortunately, Examples 3.5 and 3.6 show that this is not manageable by means of a single *binary* operation. Instead, we define a ternary operation $r\colon S^3 \to S$ via

$$r(x, y, z) := (x \vee z) \cdot_z (y \vee z). \tag{3.6}$$

It is worth noticing that in any residuated semilattice (regarded as a sectionally residuated semilattice as in 3.3) we have

$$r(x, y, z) = (x \cdot y) \vee z. \tag{3.7}$$

Indeed, $r(x, y, z) = ((x \vee z) \cdot (y \vee z)) \vee z = (x \cdot y) \vee (x \cdot z) \vee (z \cdot y) \vee (z \cdot z) \vee z = (x \cdot y) \vee z$.

It turns out that sectionally residuated semilattices can be axiomatized using the ternary operation r:

Theorem 3.12.

(i) *Let* $\mathbf{S} = (S, \vee, (\cdot_a)_{a \in S}, \to, \rightsquigarrow, 1)$ *be a sectionally residuated semilattice. Then the algebra* $\mathscr{A}(\mathbf{S}) = (S, \vee, r, \to, \rightsquigarrow, 1)$*—where* r *is the operation defined by* (3.6)*—satisfies the following identities:*

$$\begin{aligned} & x \to (x \vee y) = 1, \\ \text{(R1)} \qquad & x \rightsquigarrow (x \vee y) = 1, \\ \text{(R2)} \qquad & x \vee y = r(1, x, y) = r(x, 1, y), \\ \text{(R3)} \qquad & r(r(x, y, w), z, w) = r(x, r(y, z, w), w), \\ & r(x, y, w) \to (z \vee w) = x \to (y \to (z \vee w)), \\ \text{(R4)} \qquad & r(x, y, w) \rightsquigarrow (z \vee w) = y \rightsquigarrow (x \rightsquigarrow (z \vee w)), \\ & r(x \to (y \vee z), x, z) \leqslant y \vee z, \\ \text{(R5)} \qquad & r(x, x \rightsquigarrow (y \vee z), z) \leqslant y \vee z, \\ \text{(R6)} \qquad & r(x, y, z \vee w) = r(x \vee w, y \vee w, z) \vee w. \end{aligned}$$

(ii) *Let* $\mathbf{A} = (A, \vee, r, \to, \rightsquigarrow, 1)$ *be an algebra of type* $\langle 2, 3, 2, 2, 0 \rangle$ *such that* $(A, \vee)$ *is a join-semilattice with a greatest element* 1 *and* $\mathbf{A}$ *satisfies* (R1)–(R6). *If we define*

$$x \cdot_a y := r(x, y, a)$$

for any $a \in A$ *and* $x, y \in [a, 1]$, *then* $\mathscr{S}(\mathbf{A}) = (A, \vee, (\cdot_a)_{a \in A}, \to, \rightsquigarrow, 1)$ *is a sectionally residuated semilattice.*

(iii) *The correspondence is one-to-one, i.e., if* $\mathbf{S}$ *is a sectionally residuated semilattice then* $\mathscr{S}(\mathscr{A}(\mathbf{S}))=\mathbf{S}$*, and if* $\mathbf{A}=(A,\vee,r,\rightarrow,\rightsquigarrow,1)$ *is an algebra satisfying* (R1)–(R6) *then* $\mathscr{A}(\mathscr{S}(\mathbf{A}))=\mathbf{A}$.

Proof. (i) (R1) This is an immediate consequence of Lemma 3.9 (2).

(R2) We have $r(1,x,y)=(1\vee y)\cdot_y(x\vee y)=x\vee y$ and $r(x,1,y)=(x\vee y)\cdot_y(1\vee y)=x\vee y$.

(R3) We calculate

$$\begin{aligned}r(r(x,y,w),z,w)&=(((x\vee w)\cdot_w(y\vee w))\vee w)\cdot_w(z\vee w)\\&=((x\vee w)\cdot_w(y\vee w))\cdot_w(z\vee w)\\&=(x\vee w)\cdot_w((y\vee w)\cdot_w(z\vee w))\\&=r(x,r(y,z,w),w).\end{aligned}$$

(R4) By Lemma 3.9 (8) we have

$$\begin{aligned}&(x\vee w)\rightarrow((y\vee w)\rightarrow(z\vee w))\\&\quad=(x\vee w)\rightarrow((y\rightarrow(z\vee w))\wedge(w\rightarrow(z\vee w)))\\&\quad=(x\vee w)\rightarrow(y\rightarrow(z\vee w))\\&\quad=(x\rightarrow(y\rightarrow(z\vee w)))\wedge(w\rightarrow(y\rightarrow(z\vee w)))\\&\quad=x\rightarrow(y\rightarrow(z\vee w)\end{aligned}$$

since $w\leqslant z\vee w\leqslant y\rightarrow(z\vee w)$ yields $w\rightarrow(y\rightarrow(z\vee w))=1$. Therefore

$$\begin{aligned}r(x,y,w)\rightarrow(z\vee w)&=((x\vee w)\cdot_w(y\vee w))\rightarrow_w(z\vee w)\\&=(x\vee w)\rightarrow_w((y\vee w)\rightarrow_w(z\vee w))\\&=(x\vee w)\rightarrow((y\vee w)\rightarrow(z\vee w))\\&=x\rightarrow(y\rightarrow(z\vee w)).\end{aligned}$$

(R5) We have $r(x\rightarrow(y\vee z),x,z)=((x\rightarrow(y\vee z))\vee z)\cdot_z(x\vee z)=(x\rightarrow(y\vee z))\cdot_z(x\vee z)=((x\vee z)\rightarrow_z(y\vee z))\cdot_z(x\vee z)\leqslant y\vee z$. Analogously, $r(x,x\rightsquigarrow(y\vee z),z)\leqslant y\vee z$.

(R6) By the compatibility condition (C) applied to $z\leqslant z\vee w\leqslant x\vee z\vee w, y\vee z\vee w$ we obtain $r(x,y,z\vee w)=(x\vee z\vee w)\cdot_{z\vee w}(y\vee z\vee w)=((x\vee z\vee w)\cdot_z(y\vee z\vee w))\vee z\vee w=r(x\vee w,y\vee w,z)\vee w$.

(ii) Let $a\in A$. In order to show that $([a,1],\vee,\cdot_a,\rightarrow,\rightsquigarrow,1)$ is a residuated semilattice we have to verify the identities (2.1)–(2.3) of Lemma 2.2. Let $x,y,z\in[a,1]$. By (R3), the operation $\cdot_a$ is associative since $(x\cdot_a y)\cdot_a z=r(r(x,y,a),z,a)=r(x,r(y,z,a),a)=x\cdot_a(y\cdot_a z)$. By (R2) $x\cdot_a 1=r(x,1,a)=x\vee a=x$ and

$1 \cdot_a x = x$ for each $x \in [a,1]$. Thus $([a,1], \cdot_a, 1)$ is a monoid. Further, by (R4), $(x \cdot_a y) \to z = r(x,y,a) \to (z \vee a) = x \to (y \to (z \vee a)) = x \to (y \to z)$ and similarly $(x \cdot_a y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$. Finally, by (R5) we have $(x \to y) \cdot_a x = r(x \to y, x, a) = r(x \to (y \vee a), x, a) \leqslant y \vee a = y$; similarly $x \cdot_a (x \rightsquigarrow y) \leqslant y$. This shows that the section $[a,1]$ is a residuated semilattice.

The compatibility condition is captured by the identity (R6). Indeed, if $a \leqslant b \leqslant x, y$ then $x \cdot_b y = r(x,y,b) = r(x,y,a \vee b) = r(x \vee b, y \vee b, a) \vee b = r(x,y,a) \vee b = (x \cdot_a y) \vee b$.

(iii) Let $\mathbf{S} = (S, \vee, (\cdot_a)_{a \in S}, \to, \rightsquigarrow, 1)$ is a sectionally residuated semilattice and let $\mathscr{S}(\mathscr{A}(\mathbf{S})) = (S, \vee, (\cdot'_a)_{a \in S}, \to, \rightsquigarrow, 1)$. For any $a \in S$ and $x, y \in [a,1]$ we have $x \cdot'_a y = r(x,y,a) = (x \vee a) \cdot_a (y \vee a) = x \cdot_a y$. Thus $\mathscr{S}(\mathscr{A}(\mathbf{S})) = \mathbf{S}$.

Let now $\mathbf{A} = (A, \vee, r, \to, \rightsquigarrow, 1)$ be an algebra fulfilling the equations (R1)–(R6). Let $\mathscr{A}(\mathscr{S}(\mathbf{A})) = (A, \vee, r', \to, \rightsquigarrow, 1)$. Then using (R6) twice with $w = z$, we obtain $r'(x,y,z) = (x \vee z) \cdot_z (y \vee z) = r(x \vee z, y \vee z, z) = r(x \vee z, y \vee z, z) \vee z = r(x,y,z)$. So $\mathscr{A}(\mathscr{S}(\mathbf{A})) = \mathbf{A}$. $\square$

Consequently, sectionally residuated semilattices are in fact algebras $(A, \vee, r, \to, \rightsquigarrow, 1)$ of type $\langle 2,3,2,2,0 \rangle$ and as such algebras form a variety. However, $\vee$ is a term operation in r and 1, explicitly, we have $x \vee y = r(1,x,y) = r(x,1,y)$, and so sectionally residuated semilattices can alternatively be treated as algebras $(A, r, \to, \rightsquigarrow, 1)$ of type $\langle 3,2,2,0 \rangle$.

Let us recall several universal algebraic notions. As usual, $\mathrm{Con}(\mathbf{A})$ stands for the set of all congruences of an algebra $\mathbf{A} = (A, \tau)$, and for any $\theta \in \mathrm{Con}(\mathbf{A})$ and $a \in A$, $[a]_\theta = \{x \in A \colon (x,a) \in \theta\}$.

A variety $\mathscr{V}$ with a nullary fundamental operation 1 is said to be

(a) *weakly regular* if every congruence θ on any algebra $\mathbf{A}$ in $\mathscr{V}$ is determined by its kernel $[1]_\theta$, and *regular* if θ is determined by any single class $[a]_\theta$;

(b) *distributive at* 1 if $[1]_{(\theta \vee \varphi) \cap \psi} = [1]_{(\theta \cap \psi) \vee (\varphi \cap \psi)}$ for all $\theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$ and $A \in \mathscr{V}$, and *distributive* if for every $\mathbf{A} \in \mathscr{V}$, the congruence lattice $(\mathrm{Con}(\mathbf{A}), \subseteq)$ is distributive;

(c) *permutable at* 1 if $[1]_{\theta \circ \varphi} = [1]_{\varphi \circ \theta}$, and *permutable* if $\theta \circ \varphi = \varphi \circ \theta$ for all $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ and for each $\mathbf{A} \in \mathscr{V}$;

(d) *arithmetical at* 1 if it is both distributive and permutable at 1, and *arithmetical* if $\mathscr{V}$ is both distributive and permutable.

Theorem 3.13. *The variety of all sectionally residuated semilattices is weakly regular, arithmetical at* 1, *and hence distributive.*

P r o o f. *Weak regularity:* It is well known that a variety $\mathscr{V}$ is weakly regular if and only if there exist binary terms $t_1, \ldots, t_n$ for some $n \in \mathbb{N}$ such that

$t_1(x,y) = \ldots = t_n(x,y) = 1$ is equivalent to $x = y$. For the variety $\mathscr{S}$ of sectionally residuated semilattices, let $t_1(x,y) = x \to y$ and $t_2(x,y) = y \to x$. Clearly, $t_1(x,y) = t_2(x,y) = 1$ iff $x = y$.

Arithmeticity at 1*:* This property is captured by a simple Maltsev type condition—$\mathscr{V}$ is arithmetical at 1 if and only if there exists a binary term t with $t(x,x) = t(1,x) = 1$ and $t(x,1) = x$. In the case of $\mathscr{S}$, the term $t(x,y) = y \to x$ satisfies $t(x,x) = 1$, $t(1,x) = 1$ and $t(x,1) = x$.

Distributivity: Since $\mathscr{S}$ is weakly regular and distributive at 1, it follows at once that $\mathscr{S}$ is distributive. □

4. Sectionally residuated lattices

Definition 4.1. A structure $\mathbf{L} = (L, \vee, \wedge, (\cdot_a)_{a\in L}, \to, \rightsquigarrow, 1)$ is called a *sectionally residuated lattice* if $(L, \vee, \wedge)$ is a lattice with a greatest element 1 and $(L, \vee, (\cdot_a)_{a\in L}, \to, \rightsquigarrow, 1)$ is a sectionally residuated semilattice.

In the light of Theorem 3.12 likewise sectionally residuated lattices form a variety since they can be equivalently regarded as algebras $(A, \vee, \wedge, r, \to, \rightsquigarrow, 1)$ of type $\langle 2,2,3,2,2,0\rangle$.

Theorem 4.2. *The variety of all sectionally residuated lattices is weakly regular and arithmetical.*

P r o o f. *Weak regularity:* Again, we take the terms $t_1(x,y) = x \to y$ and $t_2(x,y) = y \to x$.

Arithmeticity: It is well-known that a variety $\mathscr{V}$ is arithmetical if and only if it has a Pixley term t satisfying $t(x,y,y) = t(x,y,x) = t(y,y,x) = x$. For the variety of all sectionally residuated lattice we can use the same Pixley term as for residuated lattices, namely, $t(x,y,z) = ((x \to y) \to z) \wedge ((z \to y) \to x) \wedge (x \vee z)$. It can be easily seen that $t(x,y,y) = t(x,y,x) = t(y,y,x) = x$. □

Example 4.3. Let $L = \{a_i \colon i \in \mathbb{N}_0\}$ be a countable chain with $1 = a_0 > a_1 > a_2 > \ldots$, i.e., $a_i \leqslant a_j$ iff $i \geqslant j$. For every $a_k \in L$, we can define a multiplication $\cdot_{a_k}$ on $[a_k, 1]$ as follows: $a_i \cdot_{a_k} a_j := a_k$ if $a_k \leqslant a_i, a_j < 1$, and $1 \cdot_{a_k} a_i = a_i \cdot_{a_k} 1 := a_i$ for $a_k \leqslant a_i$. The section $[a_k, 1]$ is then the carrier of a commutative residuated lattice in which

$$a_i \to a_j = \begin{cases} 1 & \text{for } a_i \leqslant a_j, \\ a_j & \text{for } a_i = 1, \\ a_1 & \text{otherwise.} \end{cases}$$

The compatibility condition is clearly satisfied, hence $\mathbf{L} = (L, \vee, \wedge, (\cdot_{a_k})_{k\in\mathbb{N}_0}, \rightarrow, 1)$ is a commutative sectionally residuated lattice which is not residuated. Indeed, given any $a_i < a_j < a_1$, the product $a_i \cdot a_j$ in L has to be the least element $a_l \in L$ such that $a_i \leqslant a_j \rightarrow a_l$. But such an element does not exist since we have $a_j \rightarrow a_l \in \{1, a_1\}$ and $a_i \leqslant a_j \rightarrow a_l$ holds for all a_l.

Nevertheless, $\mathbf{L}$ embeds into a residuated lattice—it suffices to add a new least element a_∞ and define $a_i \cdot a_j := a_\infty$ for all $a_i, a_j < 1$, and $1 \cdot a_i = a_i \cdot 1 := a_i$ for every a_i.

In the case of sectionally residuated lattices, it is always possible to derive the operations $(\cdot_a)_{a\in L}$ or the ternary operation r, respectively, from a certain residuated lattice in the following manner:

Theorem 4.4. *Let $\mathbf{L} = (L, \vee, \wedge, r, \rightarrow, \rightsquigarrow, 1)$ be a sectionally residuated lattice. Then there exists a bounded residuated lattice $\mathbf{M} = (M, \vee, \wedge, \cdot, \rightarrow, \rightsquigarrow, 0, 1)$ such that $(L, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a subalgebra of the reduct $(M, \vee, \wedge, \rightarrow, \rightsquigarrow, 1)$ and for every $x, y, z \in L$,*

$$r(x, y, z) = (x \cdot y) \vee z.$$

In other words, $(L, \vee, \wedge, r, \rightarrow, \rightsquigarrow, 1)$ is a subalgebra of $(M, \vee, \wedge, r, \rightarrow, \rightsquigarrow, 1)$.

Proof. It is easy to see that $(a] \cap (b] = (a \wedge b] \neq \varnothing$ for all $a, b \in L$, so the set $\{(a]\colon a \in L\}$ has the finite intersection property and hence there exists an ultrafilter $\mathcal{U}$ in the Boolean algebra $\mathbf{Exp}(L)$ of all subsets of L such that $\{(a]\colon a \in L\} \subseteq \mathcal{U}$.

Let $\mathbf{L}[a]$ denote the bounded residuated lattice $([a, 1], \vee, \wedge_a, \cdot_a, \rightarrow, \rightsquigarrow, a, 1)$, for every $a \in L$. Let

$$\mathbf{M} = \prod_{a\in L} \mathbf{L}[a]/\mathcal{U}$$

be the ultraproduct of $\{\mathbf{L}[a]\colon a \in L\}$ over $\mathcal{U}$. Of course, $\mathbf{M}$ is a bounded residuated lattice. Recall that the ultraproduct $\mathbf{M}$ is the quotient algebra $\prod\limits_{a\in L} \mathbf{L}[a]/\theta_\mathcal{U}$, where $\theta_\mathcal{U}$ is a congruence on the direct product $\prod\limits_{a\in L} \mathbf{L}[a]$ given by $(\alpha, \beta) \in \theta_\mathcal{U}$ iff $\{a \in L\colon \alpha(a) = \beta(a)\} \in \mathcal{U}$; the elements of $\mathbf{M}$ are denoted $\alpha/\mathcal{U}$ or, more detailed, $(\alpha(a)\colon a \in L)/\mathcal{U}$.

Now, define a mapping $f\colon L \longrightarrow M$ by

$$f(x) = (x \vee a\colon a \in L)/\mathcal{U}.$$

This mapping has the following properties:

(1) f is injective,

(2) f preserves the operations $\vee$, $\wedge$, $\rightarrow$ and $\rightsquigarrow$,

(3) for every $p \in L$ and $x, y \in [p, 1]$,

$$f(x \cdot_p y) = (f(x) \cdot f(y)) \vee f(p).$$

For (1), note that for any $x, y \in L$, $f(x) = f(y)$ iff $\{a \in L\colon\ x \vee a = y \vee a\} \in \mathcal{U}$. Let $x \neq y$. It is clear that whenever $a \leqslant x \wedge y$ then $x \vee a \neq y \vee a$ and hence $(x \wedge y] \subseteq \{a \in L\colon\ x \vee a \neq y \vee a\}$. Since $(x \wedge y] \in \mathcal{U}$, it follows that $\{a \in L\colon\ x \vee a \neq y \vee a\} \in \mathcal{U}$. But $\{a \in L\colon\ x \vee a \neq y \vee a\}$ is the complement of $\{a \in L\colon\ x \vee a = y \vee a\}$ in the Boolean algebra $\mathbf{Exp}(L)$ and $\mathcal{U}$ is an ultrafilter in $\mathbf{Exp}(L)$, and consequently, $\{a \in L\colon\ x \vee a = y \vee a\} \notin \mathcal{U}$, showing that $f(x) \neq f(y)$.

The parts of (2) are similar to one another, as a sample we show that f preserves $\to$. We have $f(x) \to f(y) = ((x \vee a) \to (y \vee a)\colon\ a \in L)/\mathcal{U}$ and $f(x \to y) = ((x \to y) \vee a\colon\ a \in L)/\mathcal{U}$. If $a \leqslant y$ then $(x \vee a) \to (y \vee a) = (x \vee a) \to y = (x \to y) \wedge (a \to y) = (x \to y) \wedge 1 = x \to y$ and likewise $(x \to y) \vee a = x \to y$ since $x \to y \geqslant y \geqslant a$. Thus $(y] \subseteq \{a \in L\colon\ (x \vee a) \to (y \vee a) = (x \to y) \vee a\}$ which yields $((x \vee a) \to (y \vee a)\colon\ a \in L)/\mathcal{U} = ((x \to y) \vee a\colon\ a \in L)/\mathcal{U}$, so that $f(x) \to f(y) = f(x \to y)$.

It remains to verify (3). We have $f(x \cdot_p y) = ((x \cdot_p y) \vee a\colon\ a \in L)/\mathcal{U}$ and $(f(x) \cdot f(y)) \vee f(p) = ((x \vee a\colon\ a \in L)/\mathcal{U} \cdot (y \vee a\colon\ a \in L)/\mathcal{U}) \vee (p \vee a\colon\ a \in L)/\mathcal{U} = (((x \vee a) \cdot_a (y \vee a)) \vee p\colon\ a \in L)/\mathcal{U}$. Thus we wish to show that $\{a \in L\colon\ (x \cdot_p y) \vee a = ((x \vee a) \cdot_a (y \vee a)) \vee p\} \in \mathcal{U}$.

If $a \leqslant p$ then $(x \cdot_p y) \vee a = x \cdot_p y$ and $((x \vee a) \cdot_a (y \vee a)) \vee p = (x \cdot_a y) \vee p = x \cdot_p y$ due to the compatibility condition (C). So $(p] \subseteq \{a \in L\colon\ (x \cdot_p y) \vee a = ((x \vee a) \cdot_a (y \vee a)) \vee p\}$ yielding $f(x \cdot_p y) = (f(x) \cdot f(y)) \vee f(p)$.

Therefore, for any $x, y, z \in L$, $f(r(x, y, z)) = f((x \vee z) \cdot_z (y \vee z)) = (f(x \vee z) \cdot f(y \vee z)) \vee f(z) = ((f(x) \vee f(z)) \cdot (f(y) \vee f(z))) \vee f(z) = (f(x) \cdot f(y)) \vee f(z) = r(f(x), f(y), f(z))$. Now, when identifying $\mathbf{L}$ with the corresponding subalgebra of $\mathbf{M}$, the statement follows directly from (1)–(3). $\square$

5. Involutive sectionally residuated semilattices

We turn our attention to a narrower class of sectionally residuated semilattices in which the operations r and $(\cdot_a)_{a \in S}$ can be expressed by means of $\to$, $\rightsquigarrow$ as polynomial functions.

Definition 5.1. A sectionally residuated semilattice $\mathbf{S} = (S, \vee, r, \to, \rightsquigarrow, 1)$ is said to be *involutive* if every residuated semilattice $([a, 1], \vee, \cdot_a, \to, \rightsquigarrow, a, 1)$ obeys the *law of double negation*

$$x = (x \to a) \rightsquigarrow a = (x \rightsquigarrow a) \to a$$

for all $x \in [a, 1]$.

For instance, the sectionally residuated semilattice from Example 3.6 is not involutive since in the section $[a,1]$ we have $(b \to a) \rightsquigarrow a = c \neq b$.

The term 'law of double negation' reflects the fact that on each section $[a,1]$ we have two negations defined by $x^{-_a} := x \to a$ and $x^{\sim_a} := x \to a$. Thus $\mathbf{S}$ is involutive if

$$x = x^{-_a \sim_a} = x^{\sim_a -_a}$$

for all $a \in S$ and $x \in [a,1]$. It is also obvious that $\mathbf{S}$ is involutive if and only if it satisfies the identities

$$x \vee y = ((x \vee y) \to y) \rightsquigarrow y = ((x \vee y) \rightsquigarrow y) \to y, \tag{5.1}$$

which are nothing else than (2.13) rewritten using $(x \vee y) \to y = x \to y$ and $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$. Furthermore, it can be easily seen that for each $a \in S$, $([a,1], \leqslant)$ is a lattice as it follows from (8) of Lemma 3.9 that

$$x \wedge_a y := (x^{-_a} \vee y^{-_a})^{\sim_a} = (x^{\sim_a} \vee y^{\sim_a})^{-_a}$$

is the infimum of $\{x,y\} \subseteq [a,1]$. Hence $([a,1], \vee, \wedge_a, \cdot_a, \to, \rightsquigarrow, 1)$ is a residuated lattice; since it fulfils (2.13), it is distributive by [7], [10]. Consequently, we may say that $(S, \leqslant)$ is a *distributive nearlattice*, i.e., a join-semilattice, where all principal filters are distributive lattices.

Theorem 5.2. *Let* $\mathbf{S} = (S, \vee, r, \to, \rightsquigarrow, 1)$ *be an involutive sectionally residuated semilattice. Then* $(S, \to, \rightsquigarrow, 1)$ *is a biresiduation algebra satisfying* (2.12).

Proof. As we have pointed out $\mathbf{S}$ satisfies the equations $x \vee y = (x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y$ which imply (2.12), thus the biresiduation algebra $(S, \to, \rightsquigarrow, 1)$ fulfils (2.12). $\square$

A natural question arises whether also every biresiduation algebra which satisfies (2.12) can be converted into a sectionally residuated semilattice. In view of the equivalence between bounded biresiduation algebras satisfying the equations (2.12), pseudo-MV-algebras and bounded residuated lattices satisfying (2.13) we have briefly discussed in Section 2, one can expect the positive answer. Roughly speaking, every section of a biresiduation algebra satisfying (2.12) is a pseudo-MV-algebra and consequently a residuated lattice satisfiyng (2.13) [9], thus the initial biresiduation algebra is a sectionally residuated semilattice. For the reader's convenience we give more detailed proof below:

Theorem 5.3. *Let* $(A, \to, \rightsquigarrow, 1)$ *be a biresiduation algebra that satisfies* (2.12). *Then for every* $a \in A$, *upon defining*

$$x \cdot_a y := (x \to (y \to a)) \rightsquigarrow a = (y \rightsquigarrow (x \rightsquigarrow a)) \to a,$$

$([a, 1], \vee, \wedge_a, \cdot_a, \to, \rightsquigarrow, a, 1)$ *is a bounded residuated semilattice satisfying the law of double negation, hence* $(A, \vee, r, \to, \rightsquigarrow, 1)$ *is an involutive sectionally residuated semilattice, where*

$$r(x, y, z) = (x \to (y \to z)) \rightsquigarrow z = (y \rightsquigarrow (x \rightsquigarrow z)) \to z.$$

Proof. For every $a \in A$, $([a, 1], \to, \rightsquigarrow, a, 1)$ obviously is a bounded biresiduation algebra satisfying (2.12). Hence $([a, 1], \oplus_a, {}^{-_a}, {}^{\sim_a}, a, 1)$ is a pseudo-MV-algebra where $x^{-_a} := x \to a$, $x^{\sim_a} := x \rightsquigarrow a$ and $x \oplus_a y := x^{\sim_a} \to y = y^{-_a} \rightsquigarrow x$. It follows that $([a, 1], \vee, \wedge_a, \cdot_a, \to, \rightsquigarrow, a, 1)$—where $x \cdot_a y := (x^{-_a} \oplus_a y^{-_a})^{\sim_a}$—is a bounded residuated lattice; it is easily seen that $x \cdot_a y = (x^{-_a \sim_a} \to y^{-_a})^{\sim_a} = (x \to y^{-_a})^{\sim_a} = (x \to (y \to a)) \rightsquigarrow a$ and similarly $x \cdot_a y = (y \rightsquigarrow (x \rightsquigarrow a)) \to a$. Finally, we have

$$r(x, y, z) = (x \vee z) \cdot_z (y \vee z) = ((x \vee z) \to ((y \vee z) \to z)) \rightsquigarrow z = (x \to (y \to z)) \rightsquigarrow z$$

since $(y \vee z) \to z = y \to z$ and $(x \vee z) \to (y \to z) = (x \to (y \to z)) \wedge (z \to (y \to z)) = (x \to (y \to z)) \wedge 1 = x \to (y \to z)$. Analogously $r(x, y, z) = (y \rightsquigarrow (x \rightsquigarrow z)) \to z$. $\square$

Corollary 5.4. *Involutive sectionally residuated semilattices and biresiduation algebras satisfying the identities* (2.12) *are term equivalent.*

Proof. The only point is the passage from involutive sectionally residuated semilattices to biresiduation algebras and back to involutive sectionally residuated semilattices.

Let $(S, \vee, r, \to, \rightsquigarrow, 1)$ be an involutive sectionally residuated semilattice. Then $(S, \to, \rightsquigarrow, 1)$ is a biresiduation algebra satisfying (2.12) and $(S, \vee, r', \to, \rightsquigarrow, 1)$ is an involutive sectionally residuated semilattice in which $r'(x, y, z) = (x \to (y \to z)) \rightsquigarrow z$. But by (2.13) and (R4) of Theorem 3.12 we have $r(x, y, z) = r(x, y, z) \vee z = (r(x, y, z) \to z) \rightsquigarrow z = (x \to (y \to z)) \rightsquigarrow z$, so $r' = r$ as desired. $\square$

Remark. If a bounded residuated lattice is an involutive sectionally residuated lattice then it is a pseudo-MV-algebra, but we stress that this does *not* mean that a bounded residuated lattice satisfying the law of double negation is a pseudo-MV-algebra—there are many different classes of bounded biresiduation algebras satisfying the law of double negation.

Remark. Biresiduation algebras satisfying (2.12) are *not* the models of implication in the non-commutative version of the Łukasiewicz propositional logic [13], because a biresiduation algebra is embedable into a pseudo-MV-algebra if and only if it fulfils (2.12) together with the identities

$$(x \to y) \rightsquigarrow (y \to x) = y \to x,$$
$$(x \rightsquigarrow y) \to (y \rightsquigarrow x) = y \rightsquigarrow x.$$

References

[1] *J. C. Abbott*: Semi-boolean algebra. Matem. Vestnik *4* (1967), 177–198.

[2] *C. J. van Alten*: Representable biresiduated lattices. J. Algebra *247* (2002), 672–691.

[3] *C. J. van Alten*: On varieties of biresiduation algebras. Stud. Log. *83* (2006), 425–445.

[4] *R. Ceterchi*: Pseudo-Wajsberg algebras. Mult.-Valued Log. *6* (2001), 67–88.

[5] *I. Chajda, R. Halaš and J. Kühr*: Implication in MV-algebras. Algebra Univers. *52* (2004), 377–382.

[6] *R. L. O. Cignoli, I. M. L. D'Ottaviano and D. Mundici*: Algebraic Foundations of Many-Valued Reasoning. Kluwer Acad. Publ., Dordrecht, 2000.

[7] *N. Galatos and C. Tsinakis*: Generalized MV-algebras. J. Algebra *283* (2005), 254–291.

[8] *G. Georgescu and A. Iorgulescu*: Pseudo-MV algebras. Mult.-Valued Log. *6* (2001), 95–135.

[9] *G. Georgescu and A. Iorgulescu*: Pseudo-BCK algebras: An extension of BCK algebras. Proc. of DMTCS'01: Combinatorics, Computability and Logic, London, 2001, pp. 97–114.

[10] *P. Jipsen and C. Tsinakis*: A survey of residuated lattices. Ordered Algebraic Structures (J. Martinez, ed.), Kluwer Acad. Publ., Dordrecht, 2002, pp. 19–56.

[11] *J. Kühr*: Pseudo BCK-algebras and residuated lattices. Contr. Gen. Algebra *16* (2005), 139–144.

[12] *J. Kühr*: Commutative pseudo BCK-algebras. To appear in Southeast Asian Bull. Math.

[13] *I. Leuştean*: Non-commutative Łukasiewicz propositional logic. Arch. Math. Log. *45* (2006), 191–213.

[14] *J. Rachůnek*: A non-commutative generalization of MV-algebras. Czech. Math. J. *52* (2002), 255–273.

[15] *M. Ward and R. P. Dilworth*: Residuated lattices. Trans. Am. Math. Soc. *45* (1939), 335–354.

Authors' addresses: I. Chajda, Department of Algebra and Geometry, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `chajda@inf.upol.cz`; J. Kühr, Department of Algebra and Geometry, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `kuhr@inf.upol.cz`.