Czechoslovak Mathematical Journal

Valentin Skvortsov; Francesco Tulone

Kurzweil-Henstock type integral on zero-dimensional group and some of its applications

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1167–1183

Persistent URL: http://dml.cz/dmlcz/140448

Czechoslovak Mathematical Journal, 58 (133) (2008), 1167–1183

KURZWEIL-HENSTOCK TYPE INTEGRAL ON ZERO-DIMENSIONAL GROUP AND SOME OF ITS APPLICATIONS

VALENTIN SKVORTSOV, Moscow, and FRANCESCO TULONE, Palermo

(Received March 24, 2007)

Abstract. A Kurzweil-Henstock type integral on a zero-dimensional abelian group is used to recover by generalized Fourier formulas the coefficients of the series with respect to the characters of such groups, in the compact case, and to obtain an inversion formula for multiplicative integral transforms, in the locally compact case.

Keywords: Kurzweil-Henstock integral, derivation basis, locally compact zero-dimensional abelian group, characters of a group, multiplicative integral transform, inversion formula.

MSC 2010: 26A39, 42C10, 43A25

1. INTRODUCTION

In this paper we introduce a Kurzweil-Henstock type integral on compact subsets of a locally compact zero-dimensional abelian group and use this integral to recover by generalized Fourier formulas the coefficients of the series with respect to the characters of such groups, in the compact case, and to obtain an inversion formula for multiplicative integral transforms, in the locally compact case.

The present results are generalizations of our previous ones obtained in [6] and [7]. In comparison with those papers a Kurzweil-Henstock type integral is defined here directly on the group instead of using a mapping of this group on the real line and defining an integral on intervals. That mapping was connected with the introduction of a certain ordering in this group.

An advantage of the present new approach is that it permits to obtain some more general results on the coefficient problem and on the inversion formula which do not

Supported by RFFI-08-01-00669.

depend on a particular numeration (the so called Vilenkin-Palley numeration) or, respectively, on ordering of characters, as was the case in the previous papers.

In Section 2 we present some known facts from the theory of Kurzweil-Henstock integral with respect to a general basis and prove some sufficient conditions for integrability of a function in the sense of this integral. In Section 3 we define a differential basis on the zero-dimensional group and consider some properties of Kurzweil-Henstock type integral with respect to this basis. Section 4 is devoted to the problem of recovering the coefficients of the series with respect to the characters of a compact zero-dimensional abelian group. In Section 5 we consider a generalization of this problem to the case of a locally compact zero-dimensional abelian group obtaining an inversion formula for integral transforms with kernel expressed in terms of characters of this group.

2. Preliminaries

We start with the general definition of a derivation basis (see [5] and [8]). A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ such that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. In this paper we suppose that $\mu(I) > 0$ for each $I \in \mathcal{I}$ and all the pairs (I, x) constituting each $\beta \in \mathcal{B}$ are such that $x \in I$, although it is not the case in the general theory (see [4], [5]). For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta \colon\ I \subset E\} \quad \text{and} \quad \beta[E] := \{(I, x) \in \beta \colon\ x \in E\}.$$

We suppose that the basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$. We assume also that the basis $\mathcal{B}$ has a *local character* by which we mean that for any family of basis sets $\{\beta_\tau\}$, $\beta_\tau \in \mathcal{B}$ and for any pairwise disjoint sets E_τ there exists $\beta \in \mathcal{B}$ such that $\beta\Big[\bigcup_\tau E_\tau\Big] \subset \bigcup_\tau \beta_\tau[E_\tau]$.

Assuming that X is a topological space we shall suppose that the measure μ is *regular from above* by which we mean that for any measurable set E we have $\mu(E) = \inf \mu(G)$, where infimum is taken over all open sets G such that $E \subset G$. We shall also suppose that $\mathcal{B}$ is a *Vitali basis* by which we mean that for any x and for any neighborhood $U(x)$ of x there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

A β-*partition* is a finite collection π of elements of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' nonoverlapping, i.e., $\mu(I' \cap I'') = 0$. Let $L \in \mathcal{I}$. If $\pi \subset \beta(L)$ then π is called a β-partition *in* L, if $\bigcup_{(I,x)\in\pi} I = L$ then π is called a β-partition *of* L.

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: (i) for each finite collection $I_0, I_1, \dots, I_n$ of $\mathcal{B}$-intervals with $I_1, \dots, I_n \subset I_0$ the difference $I_0 \setminus \bigcup_{i=1}^{n} I_i$ can be expressed as a finite union of pairwise nonoverlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I.

Definition 2.1 (see [5])**.** Let $\mathcal{B}$ be a basis having the partitioning property and $L \in \mathcal{I}$. A complex-valued function f on L is said to be *Kurzweil-Henstock integrable with respect to the basis* $\mathcal{B}$ (or $H_{\mathcal{B}}$-*integrable*) on L, with $H_{\mathcal{B}}$-*integral* A, if for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

It is clear that a complex-valued function is $H_{\mathcal{B}}$-integrable if and only if both its real and imaginary parts are $H_{\mathcal{B}}$-integrable.

It is easy to check, with $\mathcal{B}$ being a Vitali basis, that a function which is equal to zero almost everywhere on $L \in \mathcal{I}$, is $H_{\mathcal{B}}$-integrable on L with value zero. This justifies the following extension of Definition 2.1 to the case of functions defined only almost everywhere on L.

Definition 2.2. A complex valued function f defined almost everywhere on $L \in \mathcal{I}$ is said to be $H_{\mathcal{B}}$-*integrable* on L, with integral value A, if the function

$$f_1(g) := \begin{cases} f(g), & \text{where } f \text{ is defined,} \\ 0, & \text{otherwise} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L to A in the sense of Definition 2.1.

We note that if f is $H_{\mathcal{B}}$-integrable on L then it is $H_{\mathcal{B}}$-integrable also on any $\mathcal{B}$-interval $J \subset L$. It can be easily proved that the $\mathcal{B}$-interval function $F\colon J \mapsto (H_{\mathcal{B}}) \int_J f$ is additive and we call it the *indefinite* $H_{\mathcal{B}}$-*integral* of f.

An essential part of the theory of the Kurzweil-Henstock integral is based on the following proposition known as Henstock lemma (see a version of it in [5, Theorem 1.6.1]).

Lemma 2.1. *If a function f is $H_{\mathcal{B}}$-integrable on L, with F being its indefinite $H_{\mathcal{B}}$-integral, then for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that for any β-partition π in L we have*

$$\sum_{(I,x)\in\pi} |f(x)\mu(I) - F(I)| < \varepsilon.$$

Let F be an additive set function on $\mathcal{I}$ and E an arbitrary subset of X. For a fixed $\beta \in \mathcal{B}$, we set

$$\operatorname{Var}(E, F, \beta) := \sup_{\pi\subset\beta[E]} \sum |F(I)|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} \operatorname{Var}(E, F, \beta).$$

The extended real-valued set function $V_F(\cdot)$ is called the *variational measure* generated by F, with respect to the basis $\mathcal{B}$. Following the proof given in [9] for the interval bases in $\mathbb{R}$ it is possible to show that $V_F(\cdot)$ is an outer measure and a metric outer measure in the case of a metric space X (in the last case the definition of Vitali basis should be used).

Given a set function $F\colon \mathcal{I} \to \mathbb{R}$ we define the *upper* and the *lower* $\mathcal{B}$*-derivative* at a point x, with respect to the basis $\mathcal{B}$ and measure μ, as

$$(2.1)\qquad \overline{D}_{\mathcal{B}}F(x) := \inf_{\beta\in\mathcal{B}} \sup_{(I,x)\in\beta} \frac{F(I)}{\mu(I)} \quad\text{and}\quad \underline{D}_{\mathcal{B}}F(x) := \sup_{\beta\in\mathcal{B}} \inf_{(I,x)\in\beta} \frac{F(I)}{\mu(I)},$$

respectively. As we have assumed that $\mathcal{B}$ ignores no point, it is always true that $\overline{D}_{\mathcal{B}}F(x) \geqslant \underline{D}_{\mathcal{B}}F(x)$. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$ we call this common value the $\mathcal{B}$-*derivative* $D_{\mathcal{B}}F(x)$. For a complex-valued set function $F = \operatorname{Re} F + i \operatorname{Im} F$ we define the $\mathcal{B}$-derivative at a point x as $D_{\mathcal{B}}F(x) = D_{\mathcal{B}} \operatorname{Re} F(x) + D_{\mathcal{B}} \operatorname{Im} F(x)$.

We say that a set function F, real- or complex-valued, is $\mathcal{B}$-*continuous* at a point x with respect to the basis $\mathcal{B}$, if $V_F(\{x\}) = 0$.

It is easy to check that for any $H_{\mathcal{B}}$-integrable function the indefinite $H_{\mathcal{B}}$-integral is $\mathcal{B}$-continuous at each point.

We shall need the following (see [5, Proposition 1.6.4])

Proposition 2.1. *Let an additive complex-valued function F defined on $\mathcal{I}$ be $\mathcal{B}$-differentiable on $L \in \mathcal{I}$ outside a set $E \subset L$ such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

We get the next theorem as a corollary of the above proposition.

Theorem 2.1. *Let an additive function $F\colon \mathcal{I} \to \mathbb{R}$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$ outside of a set E with $\mu(E) = 0$ and let $-\infty < \underline{D}_{\mathcal{B}}F(x) < \overline{D}_{\mathcal{B}}F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}$-continuous. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

Proof. To apply Proposition 2.1 we need to prove only that $V_F(E) = 0$.

We note first that $\mathcal{B}$-continuity of F at each point of M and the fact that $V_F(\cdot)$ is an outer measure imply $V_F(M) = 0$.

Now let

$$H := E \setminus M = \bigcup_{m \in \mathbb{N}} H_m$$

where

$$H_m := \{\xi \in E \setminus M\colon -m < \underline{D}_{\mathcal{B}}F(\xi) < \overline{D}_{\mathcal{B}}F(\xi) < +m\}.$$

As $\mu(H_m) = 0$ and the measure μ is assumed to be regular from above, there exists for any $\varepsilon > 0$ an open set $G_m \supset H_m$ such that $\mu(G_m) < \varepsilon/m$. Then, the definition of the upper and the lower $\mathcal{B}$-derivatives and the definition of the Vitali basis imply that for any $x \in H_m$ there exists β_x such that for any pair $(I, x) \in \beta_x$ we have

$$I \subset G_m, \quad \text{and} \quad |F(I)| \leqslant m\mu(I). \tag{2.2}$$

Now choose β such that $\beta[H_m] \subset \bigcup_{x \in H_m} \beta_x[\{x\}]$ and take any β-partition $\pi \subset \beta[H_m]$. Using (2.2) we compute

$$\sum_{(I,x)\in\pi} |F(I)| \leqslant m \sum_{(I,x)\in\pi} \mu(I) \leqslant m\mu(G_m) \leqslant m \cdot \frac{\varepsilon}{m} = \varepsilon.$$

Since ε is arbitrary we get $V_F(H_m) = 0$. Then, once again using the property of the outer measure, we obtain

$$V_F(E) \leqslant V_F(M) + \sum_m V_F(H_m) = 0.$$

□

3. Derivation basis in zero dimensional group and the corresponding Kurzweil-Henstock integral

Let G be a zero-dimensional locally compact abelian group G which satisfies the second countability axiom. We suppose also that the group G is periodic. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$\ldots \supset G_{-n} \supset \ldots \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots \tag{3.1}$$

with $G = \bigcup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=-\infty}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is periodic, the factor group G_n/G_{n+1} is finite for each n and this implies that G_n (and so also all its cosets) is compact. Note that the factor group G_n/G_0 is also finite for any $n < 0$ and so the factor group G/G_0 is countable. We denote by K_n any coset of the subgroup G_n and by $K_n(g)$ the coset of the subgroup G_n which contains the element g, i.e.,

$$K_n(g) = g + G_n. \tag{3.2}$$

For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

Now for each coset K_n of G_n we choose and fix for the rest of the paper an element g_{K_n}. Then for each $n \in \mathbb{Z}$ we can represent any element $g \in G$ in the form

$$g = g_{K_n} + \{g\}_n \tag{3.3}$$

where $\{g\}_n \in G_n$. We agree to put $g_{G_n} = 0$, so that $g = \{g\}_n$ if $g \in G_n$.

Let Γ denote the dual group of G, i.e., the group of characters of the group G. It is known (see [1]) that under the assumption imposed on G the group Γ is also a periodic locally compact zero-dimensional abelian group (with respect to the pointwise multiplication of characters) and we can represent it as a sum of increasing sequence of subgroups

$$\ldots \supset \Gamma_{-n} \supset \ldots \supset \Gamma_{-2} \supset \Gamma_{-1} \supset \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_n \supset \ldots \tag{3.4}$$

introducing a topology in Γ. Then $\Gamma = \bigcup_{i=-\infty}^{+\infty} \Gamma_i$ and $\bigcap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ where $(g, \gamma^{(0)}) = 1$ for all $g \in G$ (here and below (g, γ) denotes the value of a character γ at a point g). For each $n \in \mathbb{Z}$ the group Γ_{-n} is the annulator of G_n, i.e.,

$$\Gamma_{-n} = G_n^{\perp} := \{\gamma \in \Gamma\colon\ (g, \gamma) = 1 \ \text{ for all } \ g \in G_n\}.$$

Lemma 3.1. *If $\gamma \in \Gamma_{-n}$ then γ is constant on each coset K_n of G_n.*

Proof. The representation (3.3), the properties of a character and the definition of the annulator imply

$$(g,\gamma) = (g_{K_n},\gamma)(\{g\}_n,\gamma) = (g_{K_n},\gamma).$$

So with a fixed element g_{K_n}, the value (g,γ) is constant for all $g \in K_n$. □

The factor groups $\Gamma_{-n-1}/\Gamma_{-n} = G^{\perp}_{n+1}/G^{\perp}_{n}$ and G_n/G_{n+1} are isomorphic (see [1]) and so they are of finite order for each $n \in \mathbb{Z}$. This implies that the group Γ_{-n}/Γ_0 is also finite for any $n > 0$ and Γ/Γ_0 is countable.

Now, as we have done above for the group G, for each coset J of Γ_0 we choose and fix an element γ_J. Then we can represent any element $\gamma \in \Gamma$ in the form

$$\gamma = \gamma_J \cdot \{\gamma\} \tag{3.5}$$

where $\{\gamma\} \in \Gamma_0$. We agree to put $\gamma_{\Gamma_0} = \gamma^{(0)}$, so that $\gamma = \{\gamma\}$ if $\gamma \in \Gamma_0$.

We denote by μ_G and μ_Γ the Haar measures on the groups G and Γ, respectively, and normalize them so that $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$. We can make these measures complete by including all subsets of the sets of measure zero into the respective class $\mathcal{M}$ of measurable sets.

The functions γ have also the following property.

Lemma 3.2. *If $\gamma \in \Gamma \setminus \Gamma_{-n}$ then $\int_{K_n} (g,\gamma)\,\mathrm{d}\mu_G = 0$ for each coset K_n.*

Proof. The above integral is understood in the sense of the Lebesgue integral with respect to the measure μ_G. As γ does not belong to the annulator of the subgroup G_n there exists an element $g_0 \in G_n$ such that $(g_0,\gamma) \neq 1$. If $g \in K_n$ then $g + g_0 \in K_n$. Due to the invariance with respect to translation of the Haar measure μ_G and also of the integral we have the equality

$$\int_{K_n} (g,\gamma)\,\mathrm{d}\mu_G = \int_{K_n} (g+g_0,\gamma)\,\mathrm{d}\mu_G = (g_0,\gamma)\int_{K_n} (g,\gamma)\,\mathrm{d}\mu_G$$

which implies

$$\int_{K_n} (g,\gamma)\,\mathrm{d}\mu_G = 0.$$

□

It follows from this lemma that if γ_1 and γ_2 are not equal identically on K_n, then they are orthogonal on K_n, i.e.,

$$\int_{K_n} (g, \gamma_1\overline{\gamma_2})\,\mathrm{d}\mu_G = 0.$$

Now we define a derivation basis $\mathcal{B}_G$ on the measure space $(G, \mathcal{M}, \mu_G)$. Take any function $\nu\colon\ G \to \mathbb{Z}$ and define a basis set by

$$\beta_\nu = \{(I, g)\colon\ g \in G, I = K_n(g), n \geqslant \nu(g)\}.$$

So our basis $\mathcal{B}_G$ in G is the family $\{\beta_\nu\}_\nu$ where ν runs over the set of all integer-valued functions on G. This basis has all the properties described in Section 2 for the general derivation basis, in particular it is a Vitali basis. Note that in our case the set $\mathcal{I}_G$ of all $\mathcal{B}_G$-intervals is composed by all the cosets $K_n,\ n \in \mathbb{Z}$. The partitioning property of $\mathcal{B}_G$ follows easily from compactness of any $\mathcal{B}_G$-interval and from the fact that any two $\mathcal{B}_G$-intervals K' and K'' are either disjoint or one of them is contained in the other.

Definition 2.1 of the $H_\mathcal{B}$-integral can be rewritten in our particular case in the following form:

Definition 3.1. Let $L \in \mathcal{I}_G$. A complex-valued function f on L is said to be *Kurzweil-Henstock integrable with respect to basis* $\mathcal{B}_G$ (or H_G*-integrable*) on L, with H_G*-integral* A, if for every $\varepsilon > 0$ there exists a function $\nu\colon\ L \mapsto \mathbb{Z}$ such that for any β_ν-partition π of L we have

$$\Bigl|\sum_{(I,g)\in\pi} f(g)\mu_G(I) - A\Bigr| < \varepsilon.$$

We denote the integral value A by $(H_G)\int_L f$.

Remark 3.1. We note that all the above definitions depend on the structure of the sequence of subgroups (3.1). So if we consider for the group Γ the definitions of the $\mathcal{B}_\Gamma$-basis and the H_Γ-integral, then we should use the sequence (3.4) in our construction.

Remark 3.2. It is easy to check that the H_G-integral is invariant under translation given by some element $g \in G$.

The *upper* and the *lower* $\mathcal{B}_G$*-derivative* of a set function $F\colon\ \mathcal{I}_G \to \mathbb{R}$ at a point g can be rewritten, in the case of the basis $\mathcal{B}_G$ and measure μ_G, as

$$(3.6)\qquad \overline{D}_G F(g) := \limsup_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}, \quad \underline{D}_G F(g) := \liminf_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}.$$

The $\mathcal{B}_G$-*derivative* at g is

$$D_G F(g) := \lim_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{3.7}$$

Note that in the case of our basis $\mathcal{B}_G$, given a point g, any β_ν-partition contains only one pair (I, g) with this point g. Because of this we can reformulate the definition of $\mathcal{B}$-continuity in a simpler way, saying that a set function F is $\mathcal{B}_G$-*continuous* at a point g with respect to the basis $\mathcal{B}_G$ if $\lim\limits_{n\to\infty} F(K_n(g)) = 0$.

As in the general case considered in Section 2, the indefinite H_G-integral on $L \in \mathcal{I}_G$ is an additive $\mathcal{B}_G$-continuous function on the set of all $\mathcal{B}_G$-subintervals of L.

The property of differentiation of the indefinite $H_{\mathcal{B}}$-integral almost everywhere is not true in the general case (even for some basis in the plane, see for example [3]), but in our case of the H_G-integral it can be proved if we use the following version of a covering lemma.

Lemma 3.3. *Let E be an arbitrary subset of $L \in \mathcal{I}_G$ and for each $g \in E$ let there exist an increasing sequence of natural numbers $\{n_i\} = \{n_i(g)\}$. Put $\alpha := \{(K_{n_i(g)}(g), g)\colon g \in E,\ K_{n_i(g)}(g) \subset L\}$. Then there exists a countable family of pairs $\{(I_j, g_j)\}_{j=1}^{\infty}$ such that $(I_j, g_j) \in \alpha$ for each $j = 1, 2, \ldots$, with I_j being pairwise disjoint and $E \subset \bigcup\limits_{j=1}^{\infty} I_j$.*

P r o o f. Let $L = K_{n_0}$ for some n_0. If for some $g \in E$ we get $(K_{n_0}, g) \in \alpha$, we are done. If not, we shall proceed by induction having in mind that for each $n \geqslant n_0$ the family of all cosets $K_n \subset L$ is finite and that any two K_s and K_r are either disjoint or one of them is contained in the other. In the n-th step, $n > n_0$, we consider all $K_n \subset L$ which are not covered by any I_j chosen in the previous steps and for which $(K_n, g) \in \alpha$ with some $g \in E$. We include into our sequence of pairs all such K_n combined with one of the admissible g. In this way each point $g \in E$ will be covered in a certain step because otherwise we should get a contradiction with the fact that $(K_{n_i(g)}(g), g) \in \alpha$ for each $n_i(g)$ sufficiently great and this $K_{n_i(g)}(g)$ should have been chosen in the n_i-th step if it was not already covered earlier. □

Theorem 3.1. *If a function f is H_G-integrable on $L \in \mathcal{I}_G$ then the indefinite H_G-integral $F(K) = (H_G)\int_K f$ as an additive function on the set of all $\mathcal{B}_G$-subintervals of L, is $\mathcal{B}_G$-differentiable almost everywhere on L and*

$$D_G F(g) = f(g) \quad \text{a.e. on } L. \tag{3.8}$$

P r o o f. Having got the previous lemma we can prove this theorem by standard argument, similar to the one used for the usual Kurzweil-Henstock integral on the interval (see [4, Theorem 8.2]).

Let E be the set where F is not differentiable or (3.8) does not hold. Then $E = \bigcup\limits_{m=1}^{\infty} E_m$ where E_m stands for the set of all $g \in E$ for each of which there exists a sequence $\{n_i(g)\}$ of natural numbers such that

$$\big|f(g)\mu_G(K_{n_i(g)}(g)) - F(K_{n_i(g)}(g))\big| > \frac{1}{m}\mu_G(K_{n_i(g)}(g)). \tag{3.9}$$

So it is enough to show that $\mu_G(E_m) = 0$ for each m. Having fixed m, we apply Lemma 2.1 to our basis $\mathcal{B}_G$ to find for every $\varepsilon > 0$ a function $\nu\colon L \to \mathbb{Z}$ such that for any β_ν-partition π in L we have

$$\sum_{(I,g)\in\pi} |f(g)\mu_G(I) - F(I)| < \frac{\varepsilon}{2m}. \tag{3.10}$$

Without loss of generality we can assume that $n_i(g) > \nu(g)$ for each $g \in E_m$ and each natural i. Now we apply Lemma 3.3 to the set E_m and to the above sequence $\{n_i(g)\}$. We obtain a sequence of pairs $\{(I_j, g_j)\}_{j=1}^{\infty}$ such that, according to (3.9), for each $j = 1, 2, \ldots$

$$|f(g_j)\mu_G(I_j) - F(I_j)| > \frac{1}{m}\mu_G(I_j), \tag{3.11}$$

$(I_j, g_j) \in \beta_\nu[E_m]$, sets I_j are pairwise disjoint and $E_m \subset \bigcup\limits_{j=1}^{\infty} I_j$. Having chosen k such that $\sum\limits_{j=1}^{k} \mu_G(I_j) > \mu_G(E_m)/2$ we obtain a β_ν-partition $\{(I_j, g_j)\}_{j=1}^{k}$ for which the inequality (3.10) holds. Using (3.11) we finally get an estimate

$$\frac{1}{2m}\mu_G(E_m) < \frac{1}{m}\sum_{j=1}^{k}\mu_G(I_j) < \sum_{j=1}^{k}|f(g_j)\mu_G(I_j) - F(I_j)| < \frac{\varepsilon}{2m}$$

which implies that $\mu_G(E_m) < \varepsilon$. As ε is arbitrary this completes the proof. □

4. Application to the series with respect to the characters

We consider here the case when the group G is compact and so the chain (3.1) is reduced to the one-sided sequence

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots . \tag{4.1}$$

In this case the H_G-integral is defined on the whole group G. Moreover, the group Γ of characters of the group G is discrete now (see [1]) and can be represented as a sum of an increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_{-1} \subset \Gamma_{-2} \subset \ldots \subset \Gamma_{-n} \subset \ldots \tag{4.2}$$

where $\Gamma_0 = \{\gamma^{(0)}\}$ with $(g, \gamma^{(0)}) = 1$ for all $g \in G$.

So the characters γ constitute a countable orthogonal system on G with respect to the normalized measure μ_G and we can consider a series

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \tag{4.3}$$

with respect to this system. We define a convergence of this series at a point g as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma \in \Gamma_{-n}} a_\gamma (g, \gamma) \tag{4.4}$$

when n tends to infinity.

With the series (4.3) we associate a function F defined on each coset K_n by

$$F(K_n) := \int_{K_n} S_n(g)\, \mathrm{d}\mu_G. \tag{4.5}$$

It follows easily from Lemma 3.2 that F is an additive function on the family of all $\mathcal{B}_G$-intervals.

By Lemma 3.1 the sum S_n, defined by (4.4), is constant on each K_n. Then (4.5) implies

$$S_n(g) = \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{4.6}$$

Theorem 4.1. *The series* (4.3) *is the H_G-Fourier series of an H_G-integrable function f if and only if the function F associated with this series by expression* (4.5) *coincides on each $\mathcal{B}_G$-interval I with the indefinite integral* $(H_G)\int_I f$.

Proof. This can be easily proved by the argument used in [2, Theorem 2.8.1] for the Lebesgue integral and the Vilenkin-Price system. □

The following two lemmas are immediate consequences of the equality (4.6).

Lemma 4.1. *If the series* (4.3) *converges at a point $g \in G$ to a value $f(g)$ then the associated function F (see* (4.5)*) is $\mathcal{B}_G$-differentiable at g and $D_G F(g) = f(g)$. Moreover, if the series* (4.3) *satisfies at a point g the conditions*

$$-\infty < \liminf_{n\to\infty} \operatorname{Re} S_n(g) \leqslant \limsup_{n\to\infty} \operatorname{Re} S_n(g) < +\infty \tag{4.7}$$

$$-\infty < \liminf_{n\to\infty} \operatorname{Im} S_n(g) \leqslant \limsup_{n\to\infty} \operatorname{Im} S_n(g) < +\infty \tag{4.8}$$

then the associated function F satisfies the inequalities

$$-\infty < \underline{D}_G \operatorname{Re} F(g) \leqslant \overline{D}_G \operatorname{Re} F(g) < +\infty, \tag{4.9}$$

$$-\infty < \underline{D}_G \operatorname{Im} F(g) \leqslant \overline{D}_G \operatorname{Im} F(g) < +\infty. \tag{4.10}$$

Lemma 4.2. *If the partial sums* (4.4) *satisfy at a point g the condition*

$$S_n(g) = o\Big(\frac{1}{\mu_G(K_n(g))}\Big) \tag{4.11}$$

then the associated function F is $\mathcal{B}_G$-continuous at the point g.

The next lemma gives a sufficient condition for the assumption (4.11) of the previous lemma to hold.

Lemma 4.3. *Suppose that the coefficients $\{a_\gamma\}$ of a series* (4.3) *satisfy the condition*

$$\max_{\gamma\in\Gamma_{-(n+1)}\setminus\Gamma_{-n}} |a_\gamma| \to 0 \quad \textit{if } n \to \infty. \tag{4.12}$$

Then (4.11) *holds for partial sums $S_n(g)$ at each point $g \in G$.*

Proof. We start with denoting the order of the factor group G_n/G_{n+1} by p_n. Then the order of G_0/G_1 is p_0, the order of G_0/G_2 is $p_0 \cdot p_1$ and by induction the order of G_0/G_n, $n = 1, 2, \ldots$, is $m_n := p_0 \cdot p_1 \cdot \ldots \cdot p_{n-1}$, with $p_i \geqslant 2$ for all i (we agree that $m_0 := 1$). Due to the isomorphism between G_0/G_n and Γ_{-n}/Γ_0 the order of the subgroup Γ_{-n} is m_n.

Since $\mu_G(G_0) = 1$ and μ_G is translation invariant we have

$$\mu_G(G_n) = \mu_G(K_n) = \frac{1}{m_n} \tag{4.13}$$

for all cosets K_n, $n \geqslant 0$.

Fix a point $g \in G_0$ and let $\{K_n(g)\}$ be a sequence of cosets convergent to g. In view of (4.12) for any $\varepsilon > 0$ there exists k such that for any $j \geqslant k$ we have

$$\max_{\gamma \in \Gamma_{-(j+1)} \setminus \Gamma_{-j}} |a_\gamma| < \varepsilon. \tag{4.14}$$

Fixing this k we choose n such that $|S_k(g)| \cdot (m_n)^{-1} < \varepsilon$. Since $|(g, \gamma)| = 1$, we get for any $j \geqslant k$

$$S_{j+1}(g) - S_j(g) \leqslant \sum_{\gamma \in \Gamma_{-(j+1)} \setminus \Gamma_{-j}} |a_\gamma| < m_{j+1} \cdot \varepsilon.$$

Then for any $n > k$ we obtain

$$\begin{aligned}
|S_n(g)|\mu_G(K_n(g)) &\leqslant \frac{1}{m_n}\Big(|S_k(g)| + \sum_{j=k}^{n-1} |S_{j+1}(g) - S_j(g)|\Big) \\
&\leqslant \varepsilon + \varepsilon \sum_{j=k}^{n-1} \frac{m_{j+1}}{m_n} \\
&\leqslant \varepsilon + \varepsilon\Big(1 + \frac{1}{p_{n-1}} + \frac{1}{p_{n-2}p_{n-1}} + \ldots + \frac{1}{p_{k+1} \cdot \ldots \cdot p_{n-1}}\Big) \\
&\leqslant \varepsilon + \varepsilon\Big(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-k-1}}\Big) \leqslant 2\varepsilon.
\end{aligned}$$

This inequality implies (4.11). □

Theorem 4.2. *Suppose that the partial sums* (4.4) *of the series* (4.3) *converge almost everywhere on G to a function f and satisfy the conditions* (4.7) *and* (4.8) *everywhere on G except on a countable set M where* (4.11) *holds. Then f is H_G-integrable in the sense of Definition* 2.2 (*applied to the basis $\mathcal{B}_G$*) *and* (4.3) *is the H_G-Fourier series of f.*

P r o o f. Applying (4.6) we get that for any point g at which the series (4.3) converges to $f(g)$, the function F defined by (4.5) is $\mathcal{B}_G$-differentiable at g with $D_G F(g) = f(g)$.

According to Lemma 4.1, (4.7) and (4.8) imply inequalities (4.9) and (4.10) everywhere on G except on the set M where by Lemma 4.2 F, together with $\operatorname{Re} F$ and $\operatorname{Im} F$, is $\mathcal{B}_G$-continuous.

Therefore, by Theorem 2.1 (used for the basis $\mathcal{B}_G$), $\operatorname{Re} f$ and $\operatorname{Im} f$ are H_G-integrable and $\operatorname{Re} F$ and $\operatorname{Im} F$ are their H_G-integrals. Hence f is H_G-integrable, with F being its indefinite H_G-integral. Finally, using Theorem 4.1 we complete the proof. □

Remark 4.1. In view of Lemma 4.3 we can replace the condition (4.11) by the condition (4.12) in the assumption of the above theorem.

The following theorem is a particular case of Theorem 4.2.

Theorem 4.3. *Suppose that the partial sums* (4.4) *of the series* (4.3) *converge to a function f everywhere on G. Then f is H_G-integrable on G and the series* (4.3) *is the H_G-Fourier series of f.*

Let $f\colon G \to \mathbb{C}$ be H_G-integrable on G. Then the partial sums $S_n(f, g)$ of the H_G-Fourier series of f with respect to the system of characters can be represented, according to Theorem 4.1 and formula (4.6), as

$$S_n(f, g) = \frac{1}{\mu_G(K_n(g))}(H_G)\int_{K_n(g)} f.$$

This equality together with the differentiability property of the indefinite H_G-integral (see Theorem 3.1) yields

Theorem 4.4. *The partial sums $S_n(f, g)$ of the H_G-Fourier series of an H_G-integrable on G function f are convergent to f almost everywhere on G.*

5. Inversion formula for the transform in the locally compact case

To simplify our notation we shall put in this section $K = K_0$, $[g] := g_K$, $\{g\} := \{g\}_0$, so that the representation (3.3) with $n = 0$ for any element g of a coset K of G_0 can be rewritten in the form $g = [g] + \{g\}$ where $[g]$ is a fixed element of K and $\{g\} \in G_0$. Similarly we shall use sometimes the notation $[\gamma] := \gamma_J$ to underline duality, so the representation (3.5) for any element γ of a coset J of Γ_0 can be rewritten in the form $\gamma = [\gamma] \cdot \{\gamma\}$ where $[\gamma]$ is a fixed element of J and $\{\gamma\} \in \Gamma_0$.

Using this notation and the properties of a character γ we can write

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot (\{g\}, \{\gamma\}) \cdot ([g], \{\gamma\}). \tag{5.1}$$

Now we observe:

1) $\{g\} \in G_0$ and $\{\gamma\} \in \Gamma_0 = G_0^{\perp}$. So $(\{g\}, \{\gamma\}) = 1$ and we can eliminate $(\{g\}, \{\gamma\})$ from the representation (5.1) getting

$$(g, \gamma) = (\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}). \tag{5.2}$$

2) $[\gamma] \in \Gamma_{-m(\gamma)} = G_{m(\gamma)}^{\perp}$ where $m(\gamma) \geqslant 0$ and $[\gamma] \restriction_{G_0}$ is a character of the subgroup G_0.

3) $([g], [\gamma])$ is constant if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0.

4) Using the duality between G and Γ we can state that g represents a character of Γ and, similarly to the property 2), $[g] \restriction_{\Gamma_0}$ is a character of Γ_0. So $([g], \{\gamma\})$ is a value of this character at the point $\{\gamma\}$.

Therefore, according to (5.2), if g belongs to a fixed coset of G_0 and γ belongs to a fixed coset of Γ_0, we can represent (g, γ), up to a constant multiplier $([g], [\gamma])$, as a product of $(\{g\}, [\gamma])$ considered as the value of the character $[\gamma]$ at $\{g\}$, and $([g], \{\gamma\})$ considered as the value of the character $[g]$ at $\{\gamma\}$.

Now we obtain a generalization of Theorem 4.3 in the locally compact case.

Theorem 5.1. *If the limit*

$$\lim_{n\to\infty} (H_\Gamma) \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) \, d\mu_\Gamma$$

exists at each $g \in G$ and its value is $f(g)$, where $a(\gamma)$ is a locally H_Γ-integrable function, then f is H_G-integrable on G_{-n} for each n and

$$a(\gamma) = \lim_{n\to\infty} (H_G) \int_{G_{-n}} f(g)\overline{(g, \gamma)} \, d\mu_G \quad \text{a.e. on } \Gamma. \tag{5.3}$$

P r o o f. We follow the lines of the proof in [6, Theorem 9] using the present definition of the H_G- and H_Γ-integral and having in mind the convergence of a series as it is understood in Section 4 (see (4.4)). Suppose that $g \in K$ and J denotes any coset of Γ_0. Then by (5.2)

$$\begin{aligned} f(g) &= \lim_{n\to\infty} (H_\Gamma) \int_{\Gamma_{-n}} a(\gamma)(g, \gamma) \, d\mu_\Gamma \\ &= \lim_{n\to\infty} \sum_{J \subset \Gamma_{-n}} (H_\Gamma) \int_J a(\gamma)(\{g\}, [\gamma]) \cdot ([g], [\gamma]) \cdot ([g], \{\gamma\}) \, d\mu_\Gamma \\ &= \lim_{n\to\infty} \sum_{J \subset \Gamma_{-n}} (\{g\}, \gamma_J) \cdot (H_\Gamma) \int_J a(\gamma)([g], [\gamma]) \cdot ([g], \{\gamma\}) \, d\mu_\Gamma. \end{aligned}$$

So if $g \in K$, the function $f(g)$ is the sum of series, with respect to the system of characters γ_J, at the point $\{g\}$, with coefficients

$$b_J^{(K)} = (H_\Gamma) \int_J a(\gamma)([g],[\gamma])(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma,$$

and this series is convergent everywhere on K. Then by Theorem 4.3 the function $p(t) = f(g_K + t)$ with $t = \{g\} \in G_0$ is H_G-integrable on G_0 and the coefficients $b_J^{(K)}$ are the H_G-Fourier coefficients of $p(t)$, i.e.,

$$\begin{aligned} b_J^{(K)} &= (H_\Gamma) \int_J a(\gamma)([g],[\gamma])(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma \\ &= (H_G) \int_{G_0} p(t)\overline{(\{g\},\gamma_J)}\, \mathrm{d}\mu_G = (H_G) \int_K f(g)\overline{(\{g\},\gamma_J)}\, \mathrm{d}\mu_G \end{aligned} \tag{5.4}$$

(the last equality is justified by Remark 3.2). By observation 3), $([g],[\gamma])$ is constant when $g \in K$ and $\gamma \in J$. Hence (5.4) implies

$$(H_\Gamma) \int_J a(\gamma)(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma = (H_G) \int_K f(g)\overline{([g],[\gamma])(\{g\},\gamma_J)}\, \mathrm{d}\mu_G. \tag{5.5}$$

We notice now that for each fixed J the value

$$(H_\Gamma) \int_J a(\gamma)(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma$$

is the Fourier coefficient, with respect to the character $\overline{g_K}$, of the H_Γ-integrable function $a(\gamma) = a([\gamma]+\{\gamma\})$ considered as a function of $\{\gamma\} \in \Gamma_0$. Applying Theorem 4.4 to this H_Γ-Fourier series, we get

$$\lim_{n\to\infty} \sum_{K \subset G_{-n}} (H_\Gamma) \int_J a(\gamma)(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} = a(\gamma) \text{ a.e. on } J.$$

Now using (5.5) and (5.2) we compute

$$\begin{aligned} &\lim_{n\to\infty} \sum_{K \subset G_{-n}} (H_\Gamma) \int_J a(\gamma)(g_K, \{\gamma\})\, \mathrm{d}\mu_\Gamma \cdot \overline{(g_K, \{\gamma\})} \\ &\quad = \lim_{n\to\infty} \sum_{K \subset G_{-n}} (H_G) \int_K f(g)\overline{([g],[\gamma])(\{g\},\gamma_J)}\, \mathrm{d}\mu_G \cdot \overline{(g_K, \{\gamma\})} \\ &\quad = \lim_{n\to\infty} (H_G) \int_{G_{-n}} f(g)\overline{(\{g\},\gamma_J)\cdot(g_K,\{\gamma\})\cdot([g],[\gamma])}\, \mathrm{d}\mu_G \\ &\quad = \lim_{n\to\infty} (H_G) \int_{G_{-n}} f(g)\overline{(g,\gamma)}\, \mathrm{d}\mu_G = a(\gamma) \quad \text{a.e. on } J. \end{aligned}$$

The last equality is true for any J, so we get (5.3), completing the proof. □

References

[1] *G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli and A. I. Rubistein*: Multiplicative system of functions and harmonic analysis on zero-dimensional groups. Baku, 1981. (In Russian.)
[2] *B. Golubov, A. Efimov and V. Skvortsov*: Walsh Series and Transforms: Theory and Applications. Kluwer Academic Publishers, 1991.
[3] *M. de Guzman*: Differentiation of Integral in $\mathbb{R}^n$. Springer-Verlag, Berlin, 1975.
[4] *P. Y. Lee and R. Výborný*: The Integral: An Easy Approach after Kurzweil and Henstock. Austral. Math. Soc. Lectures Series 14, University Press, Cambridge (2000).
[5] *K. M. Ostaszewski*: Henstock integration in the plane. Memoirs of the AMS, Providence *63* (1986).
[6] *V. A. Skvortsov and F. Tulone*: Henstock type integral in harmonic analysis on zero-dimensional groups. J. Math. Anal. Appl. *322* (2006), 621–628.
[7] *V. A. Skvortsov and F. Tulone*: $\mathcal{P}$-adic Henstock integral in the theory of series with respect to characters of zero-dimensional groups. Vestnik Moskov. Gos. Univ. Ser. Mat. Mekh. *1* (2006), 25–29; , Engl. transl. Moscow Univ. Math. Bull. *61* (2006), 27–31.
[8] *B. S. Thomson*: Derivation bases on the real line. Real Anal. Exchange *8* (1982/83), 67–207 and 278–442.
[9] *B. S. Thomson*: Derivates of interval functions. Memoirs of the AMS, Providence *93* (1991).

Authors' addresses: Valentin Skvortsov, Department of Math., Moscow State University, Moscow 119 992, Russia, and Institute of Mathematics, Casimirus the Great University, pl. Weyssenhoffa 11, 85-079 Bydgoszcz, Poland, e-mail: vaskvor2000@yahoo.com; Francesco Tulone, Dipartimento di Matematica ed Applicazioni, Università degli Studi di Palermo, via Archirafi 34, 901 23 Palermo, Italy, e-mail: tulone@math.unipa.it.