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Existence, uniqueness and regularity of stationary solutions to inhomogeneous Navier-Stokes equations in $\mathbb{R}^n$

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EXISTENCE, UNIQUENESS AND REGULARITY OF STATIONARY SOLUTIONS TO INHOMOGENEOUS NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

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Abstract. For a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, we use the notion of very weak solutions to obtain a new and large uniqueness class for solutions of the inhomogeneous Navier-Stokes system $-\Delta u + u \cdot \nabla u + \nabla p = f$, $\operatorname{div} u = k$, $u_{|\partial\Omega} = g$ with $u \in L^q$, $q \geqslant n$, and very general data classes for f, k, g such that u may have no differentiability property. For smooth data we get a large class of unique and regular solutions extending well known classical solution classes, and generalizing regularity results. Moreover, our results are closely related to those of a series of papers by Frehse & Růžička, see e.g. Existence of regular solutions to the stationary Navier-Stokes equations, Math. Ann. 302 (1995), 669–717, where the existence of a weak solution which is locally regular is proved.

Keywords: stationary Stokes and Navier-Stokes system, very weak solutions, existence and uniqueness in higher dimensions, regularity classes in higher dimensions

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1. Introduction and main result

We consider the stationary Navier-Stokes system

$$-\Delta u + u \cdot \nabla u + \nabla p = f,\ \operatorname{div} u = k,\ u_{|\partial\Omega} = g \tag{1.1}$$

in a bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$, with boundary $\partial\Omega$ of class $C^{2,1}$ and with data $f = \operatorname{div} F$, k, g satisfying

$$\begin{gathered} F = (F_{i,j})_{i,j=1}^n \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-1/q,q}(\partial\Omega), \\ \int_\Omega k \, \mathrm{d}x = \int_{\partial\Omega} N \cdot g \, \mathrm{d}S \quad \text{where} \ \ n \leqslant q < \infty,\ q' < r \leqslant q,\ \frac{1}{n} + \frac{1}{q} \geqslant \frac{1}{r}. \end{gathered} \tag{1.2}$$

Here $N = N(x) = (N_1(x), \ldots, N_n(x))$ denotes the outer normal at $x = (x_1, \ldots, x_n) \in \partial\Omega$, the surface integral is well defined in the generalized sense

$$\int_{\partial\Omega} N \cdot g \, \mathrm{d}S = \langle g, N \rangle_{\partial\Omega} = \langle N \cdot g, 1 \rangle_{\partial\Omega}$$

of a boundary distribution, and $q' = q/(q-1)$.

The aim of this paper is to prove existence, uniqueness and regularity of solutions $u \in L^q(\Omega)$ to the system (1.1) for the general data class (1.2) with very low regularity. Note that u need *not be differentiable* excepting $\operatorname{div} u = k$; in particular u need not have a finite Dirichlet integral. Thus this solution class is different from the usual class of weak solutions which have more differentiability properties but no uniqueness in general. A scaling argument shows that the data class (1.2) is optimal for the solution class $L^q(\Omega)$. In particular, (1.2) extends the class introduced in [20] for $n = 3$ where $k \in L^q(\Omega)$, $q \geqslant r$, is supposed.

Our largest solution class is obtained for $q = n$ by $u \in L^n(\Omega)$. We cannot expect that there is any larger solution class $L^q(\Omega)$ with $1 < q < n$, keeping the regularity property. Note in this context that the condition $q = n$ corresponds to Serrin's regularity condition $2/\infty + n/q = 1$ in the nonstationary case.

Our first result, Theorem 1.3 below, shows the existence of a unique solution $u \in L^q(\Omega)$, $q \geqslant n$, with data (1.2) under the smallness condition

$$\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-1/q,q}(\partial\Omega)} \leqslant K \tag{1.3}$$

with some constant $K = K(\Omega, q, r) > 0$. The next result, Theorem 1.4, states the uniqueness of any solution $u \in L^q(\Omega)$ with data (1.2), if the smallness condition

$$\|u\|_{L^q(\Omega)} + \|k\|_{L^r(\Omega)} \leqslant K \tag{1.4}$$

is satisfied with some constant $K = K(\Omega, q, r) > 0$. Finally, Theorem 1.5 shows the regularity of such a solution $u \in L^q(\Omega)$, $q \geqslant n$, if the data (1.2) are correspondingly smooth.

These results extend classical results, see [19], essentially in two directions. First we obtain a new existence and uniqueness class $u \in L^q(\Omega)$ without any differentiability property. Further, since the norms in (1.3), (1.4) are weaker than those in the usual conditions, we obtain a new uniqueness class even for regular solutions. In particular, we extend in this way regularity results of Galdi [19], Ch. VIII, Gerhardt [21] and von Wahl [34], where finite Dirichlet integrals are supposed. Note that the objective of this paper is different from that in a series of papers by Frehse & Růžička [10]–[15]; those authors prove for large data f and $k = 0$, $g = 0$ the existence of at least one weak L^2-solution satisfying the maximum type estimate $\sup\limits_{\Omega_0} \frac{1}{2}|u|^2 + p \leqslant c(\Omega_0)$ for every subdomain $\Omega_0 \subset\subset \Omega$ and being a strong solution. For a result on local regularity of solutions with finite Dirichlet integral we refer to Frehse & Růžička [16].

The notion of very weak solutions, introduced in principle by Amann [2], [3] for the $3D$-nonstationary case with $k = 0$, and generalized in [9], [20] to $k \neq 0$, rests on the use of test functions in the space

$$C^2_{0,\sigma}(\overline{\Omega}) := \{v = (v_1, \ldots, v_n) \in C^2(\overline{\Omega}); \operatorname{div} v = 0, v_{|\partial\Omega} = 0\}. \tag{1.5}$$

When we apply a test function $w \in C^2_{0,\sigma}(\overline{\Omega})$ formally to (1.1) we obtain the following relation well defined for $u \in L^q$, $q \geqslant n$, and data as in (1.2):

$$\begin{aligned} -\langle u, \Delta w\rangle_\Omega + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} - \langle uu, \nabla w\rangle_\Omega - \langle ku, w\rangle_\Omega \\ = -\langle F, \nabla w\rangle_\Omega, \quad w \in C^2_{0,\sigma}(\overline{\Omega}). \end{aligned} \tag{1.6}$$

Here $\langle \cdot, \cdot\rangle_\Omega$ means the usual L^q-$L^{q'}$-pairing in Ω, $\langle g, N \cdot \nabla w\rangle_{\partial\Omega}$ denotes the value of the distribution $g = (g_1, \ldots, g_n) \in W^{-1/q,q}(\partial\Omega)$ at the normal derivative $N \cdot \nabla w_{|\partial\Omega}$, and $uu = (u_i u_j)^n_{i,j=1}$. Further we use the relation $u \cdot \nabla u = (u \cdot \nabla)u = \operatorname{div}(uu) - ku$, and the notation $f = \operatorname{div} F := \Bigl(\sum\limits_{i=1}^n D_i F_{ij}\Bigr)^n_{j=1}$, $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$.

To clarify the meaning of all terms in (1.6) let $\tau = \tau(x) = (\tau_1(x), \ldots, \tau_{n-1}(x))$ be a system of unit tangential vectors at $x \in \partial\Omega$ such that $(\tau(x), N(x)) = (\tau_1(x), \ldots, \tau_{n-1}(x), N(x))$ defines a Cartesian basis at x. Then $g(x)$ has the form

$$g(x) = \mathscr{L}_g(\tau(x)) + (N \cdot g)N(x)$$

where $\mathscr{L}_g(\tau(x)) \in \mathbb{R}^n$ means a suitable linear combination of $\tau_1(x), \ldots, \tau_{n-1}(x)$ contained in the tangential plane at x, and $N \cdot g = N_1 g_1 + \ldots + N_n g_n$ denotes the normal component of $g(x)$. An elementary calculation, using $\operatorname{div} w = 0$ and assuming without loss of generality that $(\tau(x), N(x))$ is the standard basis of $\mathbb{R}^n$, shows that $N \cdot \nabla w_{|\partial\Omega}$ is contained in the tangential plane. Thus we obtain that

$$\langle g, N \cdot \nabla w\rangle_{\partial\Omega} = \langle \mathscr{L}_g(\tau_1, \ldots, \tau_{n-1}), N \cdot \nabla w\rangle_{\partial\Omega};$$

hence (1.6) contains only the tangential components of g.

For the normal component $N \cdot g$ of g we have to require the additional (well defined) conditions

$$\operatorname{div} u = k \text{ in } \Omega, \quad N \cdot u = N \cdot g \text{ on } \partial\Omega. \tag{1.7}$$

Thus, if (1.6) is satisfied for some vector field $u \in L^q(\Omega)$, we say that

$$\mathscr{L}_{u_{|\partial\Omega}}(\tau_1, \dots, \tau_{n-1}) := \mathscr{L}_g(\tau_1, \dots, \tau_{n-1}) \in W^{-1/q,q}(\partial\Omega)$$

is the tangential trace of u at $\partial\Omega$ in the sense of boundary distributions. Since the trace $N \cdot u_{|\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ is well defined in the usual sense we get a precise meaning of the trace $u_{|\partial\Omega} = g$ in (1.1).

Definition 1.1. Let data f, k, g be given as in (1.2). Then a vector field $u \in L^q(\Omega)$ is called a *very weak solution of* (1.1) if and only if the relation (1.6) and the conditions (1.7) are satisfied.

For the linearized system

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = k, \quad u_{|\partial\Omega} = g \tag{1.8}$$

we may omit the condition $q' < r$ in (1.2), caused by the nonlinear term $u \cdot \nabla u$, and suppose that the data $f = \operatorname{div} F$, k, g satisfy

$$\begin{gathered} F \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in W^{-1/q,q}(\partial\Omega), \\ \int_\Omega k \,\mathrm{d}x = \int_{\partial\Omega} N \cdot g \,\mathrm{d}S \quad \text{with} \quad n \leqslant q < \infty, \quad 1 < r \leqslant q, \quad \frac{1}{n} + \frac{1}{q} \geqslant \frac{1}{r}. \end{gathered} \tag{1.9}$$

Definition 1.2. Let data f, k, g be given as in (1.9). Then a vector field $u \in L^q(\Omega)$ is called a *very weak solution of* (1.8) if and only if the relation

$$-\langle u, \Delta w\rangle_\Omega + \langle g, N \cdot \nabla w\rangle_{\partial\Omega} = -\langle F, \nabla w\rangle_\Omega \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}) \tag{1.10}$$

and the conditions $\operatorname{div} u = k$, $N \cdot u_{|\partial\Omega} = N \cdot g$ are satisfied.

Our main result reads as follows.

Theorem 1.3 (Existence for small data)**.** *Suppose the data* $f = \operatorname{div} F$*,* k*,* g *satisfy* (1.2)*. Then there exists a constant* $K = K(\Omega, q, r) > 0$ *such that in the case*

$$\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-1/q,q}(\partial\Omega)} \leqslant K \tag{1.11}$$

there is a unique very weak solution $u \in L^q(\Omega)$ *of* (1.1) *satisfying the estimate*

$$\|u\|_{L^q(\Omega)} \leqslant C(\|F\|_{L^r(\Omega)} + \|k\|_{L^r(\Omega)} + \|g\|_{W^{-1/q,q}(\partial\Omega)}) \tag{1.12}$$

with $C = C(\Omega, q, r) > 0$*. Moreover, there exists a pressure* $p \in W^{-1,q}(\Omega)$ *such that* $-\Delta u + u \cdot \nabla u + \nabla p = f$ *is satisfied in the sense of distributions.*

Our uniqueness and regularity results are described in the following two theorems.

Theorem 1.4 (Uniqueness of small solutions). *Suppose the data* $f = \operatorname{div} F, k, g$ *satisfy* (1.2), *and let* $u \in L^q(\Omega)$ *be a very weak solution of* (1.1). *Then there exists a constant* $K = K(\Omega, q, r) > 0$ *such that under the condition*

$$\|u\|_q + \|k\|_r \leqslant K \tag{1.13}$$

there is no other very weak solution $v \in L^q(\Omega)$ *of* (1.1) *for the same data* f, k, g.

Theorem 1.5 (Regularity for smooth data). *Let* $u \in L^q(\Omega)$ *be a very weak solution of the Navier-Stokes system* (1.1) *with data* $f = \operatorname{div} F$ *and* k, g *as in* (1.2).

(i) *Assume that the data* f, k, g *satisfy the additional conditions*

$$F \in L^q(\Omega), \quad k \in L^q(\Omega) \quad \text{and} \quad g \in W^{1-1/q,q}(\partial\Omega).$$

Then $u \in W^{1,q}(\Omega)$, *the equation* $-\Delta u + u \cdot \nabla u + \nabla p = f$ *holds in the sense of distributions with some pressure function* $p \in L^q(\Omega)$, *and* $u_{|\partial\Omega} = g$ *holds in the sense of the usual trace theorem.*

(ii) *Assume that the data* $f = \operatorname{div} F$, k, g *satisfy the additional conditions*

$$f \in L^q(\Omega), \quad k \in W^{1,q}(\Omega) \quad \text{and} \quad g \in W^{2-1/q,q}(\partial\Omega).$$

Then $u \in W^{2,q}(\Omega)$, *the equation* $-\Delta u + u \cdot \nabla u + \nabla p = f$ *holds strongly in* $L^q(\Omega)$ *with some pressure function* $p \in W^{1,q}(\Omega)$ *and* $u_{|\partial\Omega} = g$ *holds in the sense of traces.*

Remark 1.6. The regularity result in Theorem 1.5 (ii) can be slightly extended as follows: Assume $u \in L^q(\Omega)$ is a very weak solution of (1.1) with data $f = \operatorname{div} F$, k, g satisfying (1.2) and additionally

$$f \in L^s(\Omega), \; F \in L^q(\Omega), \; k \in W^{1,q}(\Omega) \; \text{ and } \; g \in W^{2-1/q,q}(\partial\Omega)$$

where $\frac{1}{2}n \leqslant s < \infty$. Then $u \in D(A_s) + W^{2,q}(\Omega)$, where $D(A_s)$ denotes the domain of the Stokes operator, see §2 below, the equation $-\Delta u + u \cdot \nabla u + \nabla p = f$ holds strongly in $L^{\tilde{q}}(\Omega)$, $\tilde{q} = \min(q, s)$, with some pressure function $p \in W^{1,\tilde{q}}(\Omega)$ and $u_{|\partial\Omega} = g$ holds in the sense of traces.

2. Preliminaries

Let $1 < q < \infty$ and $q' = q/(q-1)$. We need the usual function spaces $L^q(\Omega)$, $L^q(\partial\Omega)$, $W^{\alpha,q}(\Omega)$, $W_0^{\alpha,q}(\Omega)$, $W^{-\alpha,q}(\Omega) = (W_0^{\alpha,q'}(\Omega))'$, $W^{\alpha,q}(\partial\Omega)$, and $W^{-\alpha,q}(\partial\Omega) = (W^{\alpha,q'}(\partial\Omega))'$, $0 \leqslant \alpha \leqslant 2$. The corresponding pairings are denoted by $\langle\cdot,\cdot\rangle_\Omega$ or $\langle\cdot,\cdot\rangle_{\partial\Omega}$, resp., and the corresponding norms are denoted by $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$, $\|\cdot\|_{\pm\alpha;q,\Omega} = \|\cdot\|_{\pm\alpha;q}$, $\|\cdot\|_{q,\partial\Omega}$, and $\|\cdot\|_{\pm\alpha;q,\partial\Omega}$, respectively.

The spaces of smooth functions on Ω are denoted by $C^j(\Omega)$, $C_0^j(\Omega)$, $C^j(\overline{\Omega})$ for $j = 0, 1, 2, \ldots$ and $j = \infty$. We set

$$C_0^j(\overline{\Omega}) := \{v \in C^j(\overline{\Omega});\ v_{|\partial\Omega} = 0\},$$
$$C_{0,\sigma}^j(\overline{\Omega}) := \{v = (v_1, \ldots, v_n) \in C_0^j(\overline{\Omega});\ \operatorname{div} v = 0\},$$

and $C_{0,\sigma}^j(\Omega) := \{v \in C_0^j(\Omega);\ \operatorname{div} v = 0\}$. The space of distributions $C_0^\infty(\Omega)'$ is the dual space of $C_0^\infty(\Omega)$ in the usual topology, the duality pairing of which is again denoted by $\langle\cdot,\cdot\rangle_\Omega$. Correspondingly, we use the test function space $C^j(\partial\Omega)$, $j = 1, 2$, on the boundary $\partial\Omega$, and the corresponding distribution spaces $C^j(\partial\Omega)'$ with pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

Let $L_\sigma^q(\Omega)$ be the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_{L^q(\Omega)}$. The dual space $L_\sigma^q(\Omega)'$ of $L_\sigma^q(\Omega)$ is identified with $L_\sigma^{q'}(\Omega)$ by the pairing $\langle f, v\rangle_\Omega = \int_\Omega f \cdot v \, \mathrm{d}x$. By analogy, we identify $L^q(\partial\Omega)' = L^{q'}(\partial\Omega)$ with pairing $\langle f, v\rangle_{\partial\Omega} = \int_{\partial\Omega} f \cdot v \, \mathrm{d}S$.

We need some trace and extension properties for $\alpha = 1, 2$. The trace map $f \mapsto f_{|\partial\Omega}$ is a well defined bounded linear operator from $W^{\alpha,q}(\Omega)$ onto $W^{\alpha-1/q,q}(\partial\Omega)$. Conversely, there exists a bounded linear operator $E^1\colon W^{1-1/q,q}(\partial\Omega) \to W^{1,q}(\Omega)$ with $E^1(h)_{|\partial\Omega} = h$, and a bounded linear operator $E^2\colon W^{2-1/q,q}(\partial\Omega) \times W^{1-1/q,q}(\partial\Omega) \to W^{2,q}(\Omega)$ satisfying $E^2(h_1,h_2)_{|\partial\Omega} = h_1$, $N \cdot \nabla E^2(h_1,h_2)_{|\partial\Omega} = h_2$; see [28], Theorem 5.8, [33], 5.4.4.

Let $1 < r \leqslant q$, $1/n + 1/q \geqslant 1/r$, and let $f = (f_1, \ldots, f_n) \in L^q(\Omega)$, $\operatorname{div} f \in L^r(\Omega)$. Then, using E^1 with q replaced by q', from the embedding estimate

$$\|E^1(h)\|_{r',\Omega} \leqslant C(\|E^1(h)\|_{q',\Omega} + \|\nabla E^1(h)\|_{q',\Omega}), \quad C = C(\Omega, q, r) > 0,$$

and Green's identity $\langle \operatorname{div} f, E^1(h)\rangle_\Omega = \langle N \cdot f, h\rangle_{\partial\Omega} - \langle f,\ \nabla E^1(h)\rangle_\Omega$ for $h \in W^{1/q,q'}(\partial\Omega)$, we get $N \cdot f_{|\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ and the estimate

$$\|N \cdot f\|_{-\frac{1}{q};q,\partial\Omega} \leqslant C(\|f\|_{q,\Omega} + \|\operatorname{div} f\|_{r,\Omega}) \tag{2.1}$$

with $C = C(\Omega, q, r) > 0$.

Conversely, there is a linear operator $\widehat{E}\colon W^{-1/q,q}(\partial\Omega) \to L^q(\Omega)$ satisfying $\operatorname{div}\widehat{E}(h) \in L^r(\Omega)$, $N\cdot\widehat{E}(h)_{|\partial\Omega} = h$ and the estimate

$$\|\widehat{E}(h)\|_{q,\Omega} + \|\operatorname{div}\widehat{E}(h)\|_{r,\Omega} \leqslant C\|h\|_{-1/q;q,\partial\Omega}, \quad h \in W^{-1/q,q}(\partial\Omega), \tag{2.2}$$

with $C = C(\Omega, q, r) > 0$; see [29], Corollary 4.6, (4.10).

As an application we consider some gradient $\nabla H = (D_1 H, \dots, D_n H) \in L^q(\Omega)$ with $\Delta H \in L^r(\Omega)$, and apply (2.1) to ∇H and to the vector fields $f^{i,j} = (f_1^{i,j}, \dots, f_n^{i,j})$, $1 \leqslant i < j \leqslant n$, satisfying $f_i^{i,j} := D_j H$, $f_j^{i,j} := -D_i H$ but $f_k^{ij} = 0$ if $i \neq k \neq j$. Obviously $\operatorname{div} f^{i,j} = D_i D_j H - D_j D_i H = 0$ in the sense of distributions. Then $N\cdot\nabla H_{|\partial\Omega}$ and $N\cdot f^{i,j}_{|\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ are well defined by (2.1), and a calculation shows that each $D_k H$, $k = 1,\dots,n$, at $\partial\Omega$ is a linear combination of $N\cdot\nabla H_{|\partial\Omega}$ and $N\cdot f^{i,j}_{|\partial\Omega}$ with $1 \leqslant i < j \leqslant n$. Therefore we conclude from (2.1) that $\nabla H_{|\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ is well defined and satisfies the estimate

$$\|\nabla H\|_{-1/q;q,\partial\Omega} \leqslant C(\|\nabla H\|_{q,\Omega} + \|\Delta H\|_{r,\Omega}) \tag{2.3}$$

with $C = C(\Omega, q, r) > 0$.

Consider the data $f = \operatorname{div} F$, k, g as in (1.9), and the weak Neumann problem

$$\Delta H = k, \quad N\cdot\nabla H_{|\partial\Omega} = N\cdot g \tag{2.4}$$

where $\nabla H \in L^q(\Omega)$ is considered as a solution. Then we use $\widehat{E}(h)$ with $h = N\cdot g \in W^{-1/q,q}(\partial\Omega)$, and choose a solution $b(h) \in W_0^{1,r}(\Omega)$ of the equation $\operatorname{div} b(h) = \operatorname{div}\widehat{E}(h) - k \in L^r(\Omega)$. Since

$$\int_\Omega (\operatorname{div}\widehat{E}(h) - k)\,\mathrm{d}x = \int_{\partial\Omega} N\cdot g\,\mathrm{d}S - \int_\Omega k\,\mathrm{d}x = 0,$$

such a solution exists, see [18], Theorem III, 3.2, or [31], II, Lemma 2.1.1, and satisfies

$$\|b(h)\|_{q,\Omega} \leqslant C_1\|\nabla b(h)\|_{r,\Omega} \leqslant C_2(\|\operatorname{div}\widehat{E}(h)\|_{r,\Omega} + \|k\|_{r,\Omega}) \tag{2.5}$$

with $C_j = C_j(\Omega, q, r) > 0$, $j = 1, 2$. Writing (2.4) in the form

$$\Delta H = \operatorname{div}(\widehat{E}(h) - b(h)), \quad N\cdot(\nabla H - \widehat{E}(h) - b(h))_{|\partial\Omega} = 0, \tag{2.6}$$

we find, see [17], [29], a unique solution $\nabla H \in L^q(\Omega)$ satisfying

$$\|\nabla H\|_{q,\Omega} \leqslant C_1(\|\widehat{E}(h)\|_{q,\Omega} + \|b(h)\|_{q,\Omega}) \leqslant C_2(\|N\cdot g\|_{-1/q;q,\partial\Omega} + \|k\|_{r,\Omega}), \tag{2.7}$$

and therefore

$$(2.8)\qquad \|\nabla H\|_{-1/q;q,\partial\Omega} \leqslant C(\|N\cdot g\|_{-1/q;q,\partial\Omega} + \|k\|_{r,\Omega})$$

with $C = C(\Omega,q,r) > 0$, $C_j = C_j(\Omega,q,r) > 0$, $j = 1,2$.

For the proof of the identity (2.9) below we will approximate the data k, g in (2.4) by smooth functions k_j, g_j, $j \in \mathbb{N}$, such that

$$\lim_{j\to\infty} \|k-k_j\|_{r,\Omega} = 0,\ \lim_{j\to\infty} \|N\cdot(g-g_j)\|_{-1/q;q,\partial\Omega} = 0, \text{ and } \int_\Omega k_j\,\mathrm{d}x = \int_{\partial\Omega} N\cdot g_j\,\mathrm{d}S.$$

To prove their existence we use (2.6), $F = \widehat{E}(h) - b(h) \in L^r(\Omega)$, and construct by a standard mollification procedure smooth functions F_j, $j \in \mathbb{N}$, satisfying

$$\lim_{j\to\infty} \|F_j - F\|_{q,\Omega} = 0 \text{ and } \lim_{j\to\infty} \|\operatorname{div}(F_j - F)\|_{r,\Omega} = 0.$$

Setting $k_j = \operatorname{div} F_j$, $g_j = F_{j|\partial\Omega}$ and using (2.1) with f replaced by $F - F_j$ we get the desired properties. Let $\nabla H_j \in L^q(\Omega)$ be the corresponding smooth solutions of (2.4). Using (2.7), (2.8) with $\nabla H, g, k$ replaced by $\nabla H - \nabla H_j, g - g_j, k - k_j$ we see that $\lim_{j\to\infty} \|\nabla H - \nabla H_j\|_{q,\Omega} = 0$ and $\lim_{j\to\infty} \|\nabla H - \nabla H_j\|_{-1/q;q,\partial\Omega} = 0$. Then, using the Stokes operator $A_{q'}$ and its inverse $A_{q'}^{-1}$, see below, we get the important identity

$$\begin{aligned}(2.9)\qquad \langle \nabla H, \Delta A_{q'}^{-1} v\rangle_\Omega &= \lim_{j\to\infty} \langle \nabla H_j, \Delta A_{q'}^{-1} v\rangle_\Omega \\ &= \lim_{j\to\infty} (\langle \nabla H_j, N\cdot\nabla A_{q'}^{-1} v\rangle_{\partial\Omega} + \langle \nabla\Delta H_j, A_{q'}^{-1} v\rangle_\Omega) \\ &= \langle \nabla H, N\cdot\nabla A_{q'}^{-1} v\rangle_{\partial\Omega}\end{aligned}$$

for all $v \in L^{q'}_\sigma(\Omega)$ since $\operatorname{div} A_{q'}^{-1} v = 0$ and $A_{q'}^{-1} v_{|\partial\Omega} = 0$.

Let $f = (f_1,\dots,f_n) \in L^q(\Omega)$. Then as in (2.6) the weak Neumann problem

$$\Delta H = \operatorname{div} f, \quad N\cdot(\nabla H - f)_{|\partial\Omega} = 0$$

has a unique solution $\nabla H \in L^q(\Omega)$, see [17], [29], satisfying

$$(2.10)\qquad \|\nabla H\|_{q,\Omega} \leqslant C\|f\|_{q,\Omega}$$

with $C = C(\Omega,q) > 0$. Setting $P_q f := f - \nabla H$ we get the Helmholtz projection as a bounded linear operator from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$ satisfying $P_q^2 = P_q$ and $P_q' = P_{q'}$ where P_q' means the dual operator.

The Stokes operator A_q with domain $D(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$ defined by $A_q u := -P_q\Delta u$, $u \in D(A_q)$, is a densely defined closed operator satisfying $\langle A_q u, v\rangle_\Omega = \langle u, A_{q'} v\rangle_\Omega$ for $u \in D(A_q)$, $v \in D(A_{q'})$, and $A_q u = A_\gamma u$ for $1 < q, \gamma < \infty$, $u \in D(A_q) \cap D(A_\gamma)$. The fractional power A_q^β: $D(A_q^\beta) \to L^q_\sigma(\Omega)$, $0 \leqslant \beta \leqslant 1$, with $D(A_q) \subseteq D(A_q^\beta) \subseteq L^q_\sigma(\Omega)$, is well defined and bijective; its inverse $A_q^{-\beta} = (A_q^\beta)^{-1}$ is bounded from $L^q_\sigma(\Omega)$ onto $R(A_q^{-\beta}) = D(A_q^\beta)$. Moreover, it holds $(A_q^\beta)' = A_{q'}^\beta$. We note that the norms $\|u\|_{2;q,\Omega}$ and $\|A_q u\|_{q,\Omega}$ are equivalent for $u \in D(A_q)$, as well as the norms $\|u\|_{1;q,\Omega}$ and $\|A_q^{1/2} u\|_{q,\Omega}$ are equivalent for $u \in D(A_q^{1/2})$. Further the embedding estimate

$$\|u\|_{q,\Omega} \leqslant C\|A_\gamma^\beta u\|_{\gamma,\Omega}, \quad u \in D(A_\gamma^\beta),\ 1 < \gamma \leqslant q < \infty,\ 2\beta + \frac{n}{q} = \frac{n}{\gamma}, \tag{2.11}$$

holds with $C = C(\Omega, q, \gamma) > 0$. Using $A_q^{1/2}$ we define the Yosida operators $J_m = (I + m^{-1} A_q^{1/2})^{-1}$ for $m \in \mathbb{N}$. It is well known that there exists $C = C(\Omega, q) > 0$ such that

$$\|J_m\| + \|m^{-1} A_q^{1/2} J_m\| \leqslant C, \quad m \in \mathbb{N}, \tag{2.12}$$

in the operator norm on $L^q_\sigma(\Omega)$ and that $J_m u \to u$ in $L^q_\sigma(\Omega)$ as $m \to \infty$. See [4], [22], [23], [24], [27], [31], [33], concerning the Stokes operator.

Using (2.11) we get for $f = \operatorname{div} F$, $f \in L^q(\Omega)$, $F \in L^r(\Omega)$, and arbitrary $v \in L^{q'}_\sigma(\Omega)$ the estimate

$$\begin{aligned} |\langle f, A_{q'}^{-1} v\rangle_\Omega| &= |\langle F, \nabla A_{q'}^{-1} v\rangle_\Omega| = |\langle F, \nabla A_{r'}^{-1/2} A_{r'}^{-1/2} v\rangle_\Omega| \\ &\leqslant C_1 \|F\|_{r,\Omega}\ \|A_{r'}^{-1/2} v\|_{r',\Omega} \leqslant C_2 \|F\|_{r,\Omega}\ \|v\|_{q',\Omega} \end{aligned} \tag{2.13}$$

with $C_j = C_j(\Omega, q, r) > 0$, $j = 1, 2$. This proves the existence of a unique $\hat{f} \in L^q_\sigma(\Omega)$ satisfying $\langle f, A_{q'}^{-1} v\rangle_\Omega = \langle \hat{f}, v\rangle_\Omega$ for all $v \in L^{q'}_\sigma(\Omega)$, and the estimate

$$\|\hat{f}\|_{q,\Omega} \leqslant C\|F\|_{r,\Omega}, \quad C = C(\Omega, q, r) > 0. \tag{2.14}$$

Similarly as in the theory of distributions, we set, by definition, $\hat{f} = A_q^{-1} P_q f \in L^q_\sigma(\Omega)$ giving this expression a generalizing meaning. Then $A_q^{-1} P_q f$ is well defined by the relation

$$\langle A_q^{-1} P_q f, v\rangle_\Omega = \langle f, A_{q'}^{-1} v\rangle_\Omega, \quad v \in L^{q'}_\sigma(\Omega). \tag{2.15}$$

More generally, let $f \in C_0^\infty(\Omega)'$ be any distribution such that $\langle f, w\rangle_\Omega$ is well defined (by any continuous extension) for all test functions $w \in D(A_{q'}^\beta)$, $0 \leqslant \beta \leqslant 1$, and satisfies the estimate

$$|\langle f, A_{q'}^{-\beta} v\rangle_\Omega| \leqslant C_f \|v\|_{q',\Omega},\ v \in L^{q'}_\sigma(\Omega). \tag{2.16}$$

Then $A_q^{-\beta}P_q f \in L^q_\sigma(\Omega)$ is well defined by the relation

$$\langle A_q^{-\beta}P_q f, v\rangle_\Omega = \langle f, A_{q'}^{-\beta}v\rangle_\Omega,\ \ v \in L^{q'}_\sigma(\Omega), \tag{2.17}$$

giving $A_q^{-\beta}P_q f$ a generalized meaning, and it holds

$$\|A_q^{-\beta}P_q f\|_q \leqslant C_f. \tag{2.18}$$

As an example we mention the estimate

$$\|A_q^{-1/2}P_q \ \operatorname{div} w\|_q \leqslant C\|w\|_q, \quad w \in L^q(\Omega),\ 1 < q < \infty, \tag{2.19}$$

with $C = C(\Omega, q) > 0$. See [31], III, 2.5, 2.6, for similar definitions.

Let $w \in C^2_{0,\sigma}(\overline{\Omega})$ and $v = A_{q'}w$. Then, using (2.11) and the trace estimates, we obtain that

$$\begin{aligned} |\langle g, N\cdot\nabla A_{q'}^{-1}v\rangle_{\partial\Omega}| &\leqslant C_1\|g\|_{-1/q;q,\partial\Omega}\|\nabla A_{q'}^{-1}v\|_{1/q;q',\partial\Omega} \\ &\leqslant C_2\|g\|_{-1/q;q,\partial\Omega}\|\nabla A_{q'}^{-1}v\|_{1;q',\Omega} \\ &\leqslant C_3\|g\|_{-1/q;q,\partial\Omega}\|v\|_{q',\Omega} \end{aligned} \tag{2.20}$$

with $C_j = C_j(\Omega, q) > 0$, $j = 1, 2, 3$. Since $L^q_\sigma(\Omega) = (L^{q'}_\sigma(\Omega))'$, there is a unique $G \in L^q_\sigma(\Omega)$ satisfying

$$\begin{aligned} &\langle G, v\rangle_\Omega = \langle g, N\cdot\nabla A_{q'}^{-1}v\rangle_{\partial\Omega} \quad \text{for all } v \in L^{q'}_\sigma(\Omega), \\ &\|G\|_{q,\Omega} \leqslant C\|g\|_{-1/q;q,\partial\Omega} \end{aligned} \tag{2.21}$$

with $C = C(\Omega, q) > 0$.

Finally we need the density property

$$\overline{A_q C^2_{0,\sigma}(\overline{\Omega})}^{\|\cdot\|_{q,\Omega}} = L^q_\sigma(\Omega). \tag{2.22}$$

Indeed, consider $f \in L^q_\sigma(\Omega)$, choose $f_j \in C^\infty_{0,\sigma}(\Omega)$, $j \in \mathbb{N}$, with $\lim\limits_{j\to\infty}\|f - f_j\|_{q,\Omega} = 0$ and let $u_j = A_q^{-1}f_j$. The regularity property in [30], p. 518, (9.13) shows that $u_j \in C^2_{0,\sigma}(\overline{\Omega})$ for $j \in \mathbb{N}$, and we see that $A_q u_j = f_j \to f$ in $L^q_\sigma(\Omega)$ as $j \to \infty$. This proves (2.22). Moreover, this proof shows that $C^2_{0,\sigma}(\overline{\Omega}) \subseteq D(A_q)$ is a core of $D(A_q)$.

3. Proof of theorems

Given data $f = \operatorname{div} F$, k, g as in (1.9) we derive a representation formula for the solution $u \in L^q(\Omega)$ of the linearized system (1.8).

Consider the solution $\nabla H \in L^q(\Omega)$ of the system (2.4). From (2.8) we know that $\hat{g} := \nabla H_{|\partial\Omega} \in W^{-1/q,q}(\partial\Omega)$ is well defined, and from (2.9) we conclude that $-\langle \nabla H, \Delta w\rangle_\Omega + \langle \hat{g}, N \cdot \nabla w\rangle_{\partial\Omega} = 0$ for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, $v = A_{q'} w$, $w = A_{q'}^{-1} v$. This shows, see (1.10), that $u_1 := \nabla H$ is a very weak solution of the linear system

$$-\Delta u_1 + \nabla p_1 = 0, \quad \operatorname{div} u_1 = k, \quad u_{1|\partial\Omega} = \hat{g}. \tag{3.1}$$

Next set $\tilde{g} := g - \hat{g} \in W^{-1/q,q}(\partial\Omega)$ and choose $\widetilde{G} \in L^q_\sigma(\Omega)$, using (2.21) with g replaced by $\tilde{g}$, such that $\langle \tilde{g}, N \cdot \nabla A_{q'}^{-1} v\rangle_{\partial\Omega} = \langle \widetilde{G}, v\rangle_\Omega$, $v \in L^{q'}_\sigma(\Omega)$. Setting $w = A_{q'}^{-1} v$ we get

$$\langle \widetilde{G}, \Delta w\rangle_\Omega = -\langle \widetilde{G}, -P_{q'} \Delta w\rangle_\Omega = -\langle \widetilde{G}, v\rangle_\Omega = -\langle \tilde{g}, N \cdot \nabla w\rangle_{\partial\Omega}$$

which shows that $u_2 := -\widetilde{G}$ is a very weak solution of the linear system

$$-\Delta u_2 + \nabla p_2 = 0, \quad \operatorname{div} u_2 = 0, \quad u_{2|\partial\Omega} = \tilde{g}. \tag{3.2}$$

Finally, we set $u_3 := A_q^{-1} P_q f$, see (2.15), and conclude that u_3 is a very weak solution of the linear system

$$-\Delta u_3 + \nabla p_3 = f, \quad \operatorname{div} u_3 = 0, \quad u_{3|\partial\Omega} = 0. \tag{3.3}$$

Combining (3.1), (3.2), (3.3) and using $\operatorname{div}(u_1+u_2+u_3) = k$ and $N \cdot (u_1+u_2+u_3)_{|\partial\Omega} = N \cdot g$ we see that $u \in L^q(\Omega)$ defined by

$$u := u_1 + u_2 + u_3 = \nabla H - \widetilde{G} + A_q^{-1} P_q f \tag{3.4}$$

is a very weak solution of the linearized system (1.8). Using (2.7), (2.14) and (2.21) with G, g replaced by $\widetilde{G}, \tilde{g}$, we obtain the estimate

$$\|u\|_{q,\Omega} \leqslant C(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}) \tag{3.5}$$

with $C = C(\Omega, q, r) > 0$.

To prove uniqueness let $v \in L^q(\Omega)$ be another very weak solution of (1.8) for the same data (1.9). Then $u - v$ is a very weak solution of (1.8) with data $f = 0$, $k = 0$, $g = 0$. From (1.10) we obtain that $-\langle u - v, \Delta w\rangle_\Omega = \langle u - v, A_{q'} w\rangle_\Omega$ for all $w \in C^2_{0,\sigma}(\overline{\Omega})$, and using (2.22) we get that $u - v = 0$, $u = v$. Therefore, each very weak solution of (1.8) with data (1.9) has the representation (3.4).

Observe that in the proof of (3.4) we only used that $A_q^{-1}P_q f \in L^q_\sigma(\Omega)$ is well defined in the sense of (2.17) with $\beta = 1$. Thus instead of $f = \operatorname{div} F$ with $F \in L^r(\Omega)$ we only need to assume that f is a distribution such that $A_q^{-1}P_q f \in L^q_\sigma(\Omega)$ is well defined with (2.16)–(2.18) for $\beta = 1$. In this case we may define a very weak solution $u \in L^q(\Omega)$ of (1.8) replacing the term $-\langle F, \nabla w\rangle_\Omega$ in (1.10) by $\langle f, w\rangle_\Omega$, and obtaining for u the formula (3.4) and the estimate

$$\|u\|_{q,\Omega} \leqslant C(\|A_q^{-1}P_q f\|_{q,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}) \tag{3.6}$$

with $C = C(\Omega, q, r) > 0$. This generalizes slightly the notion of a very weak solution u in Definition 1.2. The same extension is allowed in Definition 1.1.

Proof of Theorem 1.3. Considering the nonlinear case we suppose that the data $f = \operatorname{div} F$, k, g satisfy the conditions (1.2). First assume that $u \in L^q(\Omega)$ is a given very weak solution of (1.1). Setting $\hat{f} := f - \operatorname{div}(uu) + ku$ we obtain that $A_q^{-1}P_q\hat{f} \in L^q_\sigma(\Omega)$ is well defined in the general sense (2.17), see (3.9), (3.10) below.

Therefore, u is a very weak solution of the linear system

$$-\Delta u + \nabla p = \hat{f}, \quad \operatorname{div} u = k, \quad u_{|\partial\Omega} = g, \tag{3.7}$$

and, using (3.4), we get the representation

$$u = \mathscr{F}(u) := \nabla H - \widetilde{G} + A_q^{-1}P_q f - A_q^{-1}P_q \operatorname{div}(uu) + A_q^{-1}P_q(ku). \tag{3.8}$$

Next we show that $u = \mathscr{F}(u)$ has a solution $u \in L^q(\Omega)$ using Banach's fixed point principle in a standard way.

Indeed, using (2.15) and (2.11) we obtain similarly as in (2.13) that

$$\begin{aligned} |\langle A_q^{-1}P_q \operatorname{div}(uu), v\rangle_\Omega| &= |\langle uu, \nabla A_{q'}^{-1}v\rangle_\Omega| \\ &\leqslant C_1\|uu\|_{q/2,\Omega}\|\nabla A_{q'}^{-1}v\|_{(q/2)',\Omega} \\ &\leqslant C_2\|u\|_q^2\|A_{(q/2)'}^{-1/2}v\|_{(q/2)',\Omega} \\ &\leqslant C_3\|u\|_{q,\Omega}^2\|v\|_{q',\Omega} \end{aligned} \tag{3.9}$$

and that

$$\begin{aligned} |\langle A_q^{-1}P_q(ku), v\rangle_\Omega| &= |\langle ku, A_{q'}^{-1}v\rangle_\Omega| \\ &\leqslant C_1\|ku\|_{(1/r+1/q)^{-1},\Omega}\|A_{q'}^{-1}v\|_{(1-1/r-1/q)^{-1},\Omega} \\ &\leqslant C_2\|k\|_{r,\Omega}\|u\|_{q,\Omega}\|v\|_{q',\Omega} \end{aligned} \tag{3.10}$$

for $v \in L^{q'}_\sigma(\Omega)$ and with C_1, C_2, C_3 depending on Ω, q, r. Here we need that $q' < r \leqslant q$ yielding $1/r + 1/q < 1$, and $q \geqslant n$, $1/n + 1/q \geqslant 1/r$. This shows that $-A_q^{-1} P_q \operatorname{div}(uu) + A_q^{-1} P_q(ku) \in L^q_\sigma(\Omega)$ is well defined for $u \in L^q(\Omega)$; moreover, we get from (3.6), (3.9), (3.10) and (2.14) the estimate

$$\|\mathscr{F}(u)\|_{q,\Omega} \leqslant C(\|u\|^2_{q,\Omega} + \|k\|_{r,\Omega}\|u\|_{q,\Omega} + \|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega}), \tag{3.11}$$

with $C = C(\Omega, q, r) > 0$, which can be written in the form

$$\|\mathscr{F}(u)\|_{q,\Omega} \leqslant a\|u\|^2_{q,\Omega} + b\|u\|_{q,\Omega} + c$$

with $a := C$, $b := C\|k\|_{r,\Omega}$, $c := C(\|F\|_{r,\Omega} + \|k\|_{r,\Omega} + \|g\|_{-1/q;q,\partial\Omega})$. In the same way we obtain that

$$\|\mathscr{F}(u) - \mathscr{F}(v)\|_{q,\Omega} \leqslant (a(\|u\|_{q,\Omega} + \|v\|_{q,\Omega}) + b)\|u - v\|_{q,\Omega} \tag{3.12}$$

for $u, v \in L^q(\Omega)$.

Up to now $u \in L^q(\Omega)$ was a given very weak solution of (1.1). To prove existence, we have to solve the fixed point problem $u = \mathscr{F}(u)$. For this purpose assume that

$$4ac + 2b < 1, \tag{3.13}$$

and consider the closed ball $\mathscr{B} := \{u \in L^q(\Omega);\ \|u\|_{q,\Omega} \leqslant y_1\}$ where $y_1 = 2c(1 - b + \sqrt{1 + b^2 - (4ac + 2b)})^{-1} > 0$ is the smallest root of the equation $y = ay^2 + by + c$. Setting $K = K(\Omega, q, r) := (4C^2 + 3C)^{-1}$ with C from (3.11) we see that (1.11) is sufficient for (3.13) to be satisfied. If $u \in \mathscr{B}$, we obtain that $\|\mathscr{F}(u)\|_{q,\Omega} \leqslant ay_1^2 + by_1 + c = y_1 \leqslant 2c$ and that $\mathscr{F}(u) \in \mathscr{B}$. Thus Banach's fixed point principle yields a unique $u \in \mathscr{B}$ with $u = \mathscr{F}(u)$. This u is a very weak solution of (3.7) and therefore also of (1.1). Further we see that $\|u\|_{q,\Omega} \leqslant y_1 \leqslant 2c$ which proves (1.12).

This completes the existence proof. The uniqueness of the solution u is a consequence of Theorem 1.4 when we use the estimate (1.12). Note that the constant $K = (4C^2 + 3C)^{-1}$ with C from (3.11) is only sufficient for the existence; in general, the uniqueness requires another constant. The assertion concerning p follows by de Rham's argument. Now Theorem 1.3 is completely proved. □

Proof of Theorem 1.4. Given very weak solutions $u, v \in L^q(\Omega)$ where u satisfies (1.13) a calculation shows that $w = u - v \in L^q_\sigma(\Omega)$ is a very weak solution of the linear system

$$-\Delta w + \nabla p = \hat{f}, \quad \operatorname{div} w = 0 \text{ in } \Omega, \quad w_{|\partial\Omega} = 0,$$

with $\hat{f} = -\operatorname{div}(vw + wu) + kw$. Then the representation formula (3.4) yields the well defined relation

$$w = -A_q^{-1} P_q \operatorname{div}(vw + wu) + A_q^{-1} P_q(kw). \tag{3.14}$$

First let $q > n$. Then we conclude using estimates as in the previous proof that

$$-A_q^{-1/2} P_q \operatorname{div}(vw + wu) + A_q^{-1/2} P_q(kw) \in L_\sigma^{q/2}(\Omega). \tag{3.15}$$

In view of (3.14) we see that $w \in D(A_{q/2}^{1/2})$, yielding $w \in L^{q_1}(\Omega)$ where $1/n + 1/q_1 = 2/q$, see (2.11). Since $q > n$ and consequently $q_1 > q$, we may repeat this argument and obtain in a finite number of steps that $w \in D(A_2^{1/2})$. Then take in (3.14) the scalar product with $A_2 w$, write $vw = uw - ww$ and use that $\langle \operatorname{div}(ww), w\rangle = 0$. Now the smallness assumption (1.13) and an absorption argument show that $\|A_2^{1/2} w\|_2 \leqslant 0$ yielding $w = 0$ and $u = v$.

If $q = n$ we need an additional smoothing step using the Yosida operators $J_m = (I + m^{-1} A_q^{1/2})^{-1}$, $m \in \mathbb{N}$, see [31], p. 298, concerning a similar procedure. Furthermore, we choose C_0^∞-functions k_j, v_j and u_j, $j \in \mathbb{N}$, satisfying $\|k - k_j\|_r \to 0$, and $\|v - v_j\|_n + \|u - u_j\|_n \to 0$ as $j \to \infty$. Then (3.14) will be rewritten, using $w = J_m w + m^{-1} A_q^{1/2} J_m w$ on the right-hand side, in the form

$$\begin{aligned} A_q^{1/2} J_m w = & - J_m A_q^{-1/2} P_q \ \operatorname{div}((v - v_j) J_m w + (J_m w)(u - u_j)) \\ & - \frac{1}{m} J_m A_q^{-1/2} P_q \ \operatorname{div}((v - v_j) A_q^{1/2} J_m w + (A_q^{1/2} J_m w)(u - u_j)) \\ & - J_m A_q^{-1/2} P_q \ \operatorname{div}(v_j w + w u_j) + J_m A_q^{-1/2} P_q((k - k_j) J_m w) \\ & + \frac{1}{m} J_m A_q^{-1/2} P_q((k - k_j) A_q^{1/2} J_m w) + J_m A_q^{-1/2} P_q(k_j w) \\ =: & \ h_1 + h_2 + h_3 + h_4 + h_5 + h_6; \end{aligned} \tag{3.16}$$

see [31], V.1.8, p. 298 concerning this smoothing procedure.

Next choose $q_1 > q = n$ and $\alpha \in [0, 1]$ such that $(2 + \alpha)/n + 1/q_1 < 1$ and $(1 + \alpha)/n \geqslant 1/r$. If $n > 3$, then $\alpha = 1$ is possible. In the case $q = n = 3$ and consequently $r > q' = \frac{3}{2}$ we find $\alpha \in [0, 1)$ to fulfill both conditions. Further observe that $q_1 > n$ can be chosen so large that $\varrho := \big(1/n + 1/q_1\big)^{-1} \geqslant 2$. Using (2.12), (2.13), and (2.19), h_1 in (3.16) is estimated by

$$\begin{aligned} \|h_1\|_\varrho &\leqslant C_1 \|(v - v_j) J_m w + (J_m w)(u - u_j)\|_\varrho \\ &\leqslant C_2 (\|v - v_j\|_n + \|u - u_j\|_n) \|J_m w\|_{q_1} \\ &\leqslant C_3 (\|v - v_j\|_n + \|u - u_j\|_n) \|A_\varrho^{1/2} J_m w\|_\varrho. \end{aligned}$$

Concerning h_2 let $\varrho_1 \in (1, n)$ be defined by $1/n + 1/\varrho = 1/\varrho_1$. Then by (2.12), (2.13), (2.19),

$$\begin{aligned}\|h_2\|_\varrho &\leqslant C_1 \|A_\varrho^{1/2} h_2\|_{\varrho_1} \leqslant C_2 \|(v - v_j) A_q^{1/2} J_m w + (A_q^{1/2} J_m w)(u - u_j)\|_{\varrho_1} \\ &\leqslant C_2(\|v - v_j\|_n + \|u - u_j\|_n) \|A_\varrho^{1/2} J_m w\|_\varrho.\end{aligned}$$

Moreover,

$$\|h_3\|_\varrho \leqslant C \|v_j w + w u_j\|_\varrho \leqslant C(\|v_j\|_{q_1} + \|u_j\|_{q_1}) \|w\|_n.$$

Next, since $r \geqslant \frac{1}{2} n$,

$$\begin{aligned}\|h_4\|_\varrho &\leqslant C_1 \|(k - k_j) J_m w\|_{\varrho_1} \leqslant C_1 \|k - k_j\|_{n/2} \|J_m w\|_{q_1} \\ &\leqslant C_2 \|k - k_j\|_r \|A_\varrho^{1/2} J_m w\|_\varrho.\end{aligned}$$

Looking at the estimate of h_2 and (2.13), we get for h_5 with $\varrho_2 > 1$ defined by $1/\varrho_2 = \alpha/n + 1/\varrho_1$, that

$$\begin{aligned}\|h_5\|_\varrho &\leqslant C_1 \|A_q^{-1/2} P_q((k - k_j) A_q^{1/2} J_m w)\|_{\varrho_1} \\ &\leqslant C_2 \|A_q^{\alpha/2 - 1/2} (P_q(k - k_j) A_q^{1/2} J_m w)\|_{\varrho_2} \\ &\leqslant C_3 \|(k - k_j) A_q^{1/2} J_m w\|_{\varrho_2} \\ &\leqslant C_3 \|k - k_j\|_{n/(1+\alpha)} \|A_\varrho^{1/2} J_m w\|_\varrho \\ &\leqslant C_4 \|k - k_j\|_r \|A_\varrho^{1/2} J_m w\|_\varrho.\end{aligned}$$

Finally,

$$\|h_6\|_\varrho \leqslant C_1 \|k_j w\|_{\varrho_1} \leqslant C_1 \|k_j\|_\varrho \|w\|_n \leqslant C_2 \|k_j\|_{q_1} \|w\|_n.$$

Summarizing the L^ϱ-estimates of h_j, $1 \leqslant j \leqslant 6$, we get from (3.16) the estimate

$$\begin{aligned}\|A_\varrho^{1/2} J_m w\|_\varrho &\leqslant C_5(\|v - v_j\|_n + \|u - u_j\|_n + \|k - k_j\|_r) \|A_\varrho^{1/2} J_m w\|_\varrho \\ &\quad + C_6(\|v_j\|_{q_1} + \|u_j\|_{q_1} + \|k_j\|_{q_1}) \|w\|_n\end{aligned} \tag{3.17}$$

with constants $C, C_1, \ldots, C_6 > 0$ independent of $m \in \mathbb{N}$. Now choose $j \in \mathbb{N}$ sufficiently large such that $\|v - v_j\|_n + \|u - u_j\|_n + \|k - k_j\|_r \leqslant 1/(2C_5)$. Hence, for this fixed j and for every $m \in \mathbb{N}$

$$\|A_\varrho^{1/2} J_m w\|_\varrho \leqslant M := 2C_6(\|v_j\|_{q_1} + \|u_j\|_{q_1} + \|k_j\|_{q_1}) \|w\|_n.$$

Since the graph of $A_\varrho^{1/2}$ is weakly closed and since $J_m w \to w$ in $L^\varrho_\sigma(\Omega)$, we conclude that $w \in D(A_\varrho^{1/2})$. Hence $w \in L^{q_1}_\sigma(\Omega)$ where $q_1 > n$. Since $\varrho \geqslant 2$, we conclude that $w \in D(A_2^{1/2})$, and the same argument as in the first part of the proof shows that $w = 0$. This completes the proof. □

Proof of Theorem 1.5. (i) We use the vector-valued version of $E^1(g) \in W^{1,q}(\Omega)$ satisfying $E^1(g)_{|\partial\Omega} = g$ and the solution $b(g) \in W_0^{1,q}(\Omega)$ of the equation $\operatorname{div} b(g) = \operatorname{div}(u - E^1(g)) = k - \operatorname{div} E^1(g)$, see §2; note that $\int_\Omega (k - \operatorname{div} E^1(g))\, \mathrm{d}x = 0$. Setting

$$\hat{u} = u - \widehat{E}, \quad \widehat{E} = E^1(g) + b(g),$$

we see that $\hat{u}$ is a very weak solution of the linear system

$$-\Delta \hat{u} + \nabla p = \hat{f}, \quad \operatorname{div} \hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u}_{|\partial\Omega} = 0,$$

where $\hat{f} = f + \operatorname{div} \nabla \widehat{E} - \operatorname{div}(uu) + ku$. The linear representation formula (3.4) yields

$$\hat{u} = A_q^{-1} P_q \operatorname{div}(F + \nabla \widehat{E} - uu) + A_q^{-1} P_q(ku). \tag{3.18}$$

We argue as in the proof of Theorem 1.4. If $q > n$, we obtain in a finite number of steps that $\hat{u} \in D(A_q^{1/2}) \subset W^{1,q}(\Omega)$ and consequently also $u \in W^{1,q}(\Omega)$.

If $q = n$, we use the same smoothing procedure as in the proof of Theorem 1.4. First write (3.18) in the form

$$\hat{u} = A_q^{-1} P_q \operatorname{div}(F + \nabla \widehat{E}) - A_q^{-1} P_q \operatorname{div}(u(\hat{u} + \widehat{E})) + A_q^{-1} P_q(k(\hat{u} + \widehat{E})) \tag{3.19}$$

and choose $u_j \in C_0^\infty(\Omega)$, $j \in \mathbb{N}$, satisfying $\|u - u_j\|_n \to \infty$ as $j \to \infty$. Then using the Yosida operators $J_m = (I + m^{-1} A_q^{1/2})^{-1}$ we get from (3.19) that

$$\begin{aligned} A_q^{1/2} J_m \hat{u} &= -J_m A_q^{-1/2} P_q \operatorname{div}((u - u_j) J_m \hat{u}) \\ &\quad - \frac{1}{m} J_m A_q^{-1/2} P_q \operatorname{div}((u - u_j) A_q^{1/2} J_m \hat{u}) \\ &\quad - J_m A_q^{-1/2} P_q \operatorname{div}(u_j \hat{u}) \\ &\quad + J_m A_q^{-1/2} P_q \operatorname{div}(F + \nabla \widehat{E}) - J_m A_q^{-1/2} P_q \operatorname{div}(u \widehat{E}) \\ &\quad + J_m A_q^{-1/2} P_q k(\hat{u} + \widehat{E}) \\ &= h_1 + h_2 + h_3 + h_4 + h_5 + h_6. \end{aligned} \tag{3.20}$$

Choose $q_1 > q = n$ and define $\varrho \in (1, n)$ by $1/\varrho = 1/n + 1/q_1$. The functions h_1, h_2 and h_3 are estimated similarly as h_1, h_2, h_3 in the proof of Theorem 1.4; we get that

$$\|h_i\|_\varrho \leqslant C_1 \|u - u_j\|_n \|A_\varrho^{1/2} J_m \hat{u}\|_\varrho + C_2 \|u_j\|_{q_1} \|\hat{u}\|_n, \quad i = 1, 2, 3.$$

The last three functions h_i are easily seen to satisfy the estimate

$$\|h_4\|_\varrho + \|h_5\|_\varrho + \|h_6\|_\varrho \leqslant C((\|\hat{u}\|_n + \|\widehat{E}\|_n)\|k\|_n + \|u\|_n \|\widehat{E}\|_{W^{1,n}} + \|F + \nabla \widehat{E}\|_n).$$

Choosing $j \in \mathbb{N}$ sufficiently large, the absorption principle and (3.20) show that

$$\|A_\varrho^{1/2} J_m \hat{u}\|_\varrho \leqslant M \quad \text{for all } m \in \mathbb{N},$$

where $M = M(\|u_j\|_{q_1}, \|\hat{u}\|_n, \|k\|_n, \|\widehat{E}\|_{W^{1,n}}, \|F\|_n) > 0$ is independent of $m \in \mathbb{N}$. Hence $\hat{u} \in D(A_\varrho^{1/2}) \subset L^{q_1}(\Omega)$ and also $u \in L^{q_1}(\Omega)$ where $q_1 > q = n$. Now we choose $q_1 = 2q$ and obtain from (3.19) that $A_q^{1/2}\tilde{u} \in L^q(\Omega)$ and consequently $u \in W^{1,q}(\Omega)$.

(ii) A functional analytic argument shows the existence of some $F \in L^q(\Omega)$ with $f = \operatorname{div} F$. Then we conclude by part (i) that $u \in W^{1,q}(\Omega)$. Further we use the vector-valued version of the extension operator $E^2(g, h_2) \in W^{2,q}(\Omega)$ with a suitably chosen function $h_2 \in W^{1-1/q,q}(\partial\Omega)$ such that $\operatorname{div} E^2(g, h_2)_{|\partial\Omega} = -k_{|\partial\Omega}$. Since

$$\int_\Omega (k - \operatorname{div} E^2(g, h_2))\,\mathrm{d}x = 0 \quad \text{and} \quad (k - \operatorname{div} E^2(g, h_2))_{|\partial\Omega} = 0,$$

we find a solution $b \in W_0^{2,q}(\Omega)$ of the equation $\operatorname{div} b = \operatorname{div}(u - E^2(g, h_2)) = k - \operatorname{div} E^2(g, h_2)$, see [18], Theorem III, 3.2, with $m = 1$, or [31], II, Lemma 2.3.1, with $k = 1$. Setting $\hat{u} = u - E^2(g, h_2) - b$, we see that $\hat{u}$ is a very weak solution of the linear system

$$-\Delta\hat{u} + \nabla p = \tilde{f}, \quad \operatorname{div}\hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u}_{|\partial\Omega} = 0,$$

where $\tilde{f} = f + \Delta E^2(g, h_2) + \Delta b - \operatorname{div}(uu) + ku$.

If $q > n$, standard estimates directly show that $\operatorname{div}(uu) - ku = u \cdot \nabla u \in L^q(\Omega)$. Hence the solution $\hat{u}$ has the representation

$$\hat{u} = A_q^{-1}P_q f + A_q^{-1}P_q(\Delta E^2(g, h_2) + \Delta b) - A_q^{-1}P_q(\operatorname{div}(uu) - ku) \tag{3.21}$$

yielding $\hat{u} \in D(A_q)$ and consequently $u \in W^{2,q}(\Omega)$.

If $q = n$, we find some $q^* > n$ and $F^* \in L^{q^*}(\Omega)$ with $f = \operatorname{div} F^*$; the exponent $q^* > n$ can be chosen such that $k \in L^{q^*}$, $g \in W^{1-1/q^*,q^*}(\partial\Omega)$. By part (i) we get $u \in W^{1,q^*}(\Omega)$. Now we conclude that $u \cdot \nabla u \in L^q(\Omega)$ which leads to $\hat{u} \in W^{2,q}(\Omega)$ as in the case $q > n$. This proves Theorem 1.5. □

Proof of Remark 1.6. First let $q > n$. Then $\operatorname{div}(uu) - k\,u = u \cdot \nabla u \in L^q(\Omega)$, and using (3.21) with $A_q^{-1}P_q f$ replaced by $A_s^{-1}P_s f$ we see that $\hat{u} \in D(A_s) + W^{2,q}(\Omega)$. If $q = n$ and $s > n/2$, we find—using Sobolev embedding theorems—some $q^* > n$ and $F^* \in L^{q^*}(\Omega)$ such that $f = \operatorname{div} F^*$, $k \in L^{q^*}$, $g \in W^{1-1/q^*,q^*}(\partial\Omega)$. This shows that $u \in W^{1,q^*}(\Omega)$, $u \cdot \nabla u \in L^q(\Omega)$, and therefore that $\hat{u} \in D(A_s) + W^{2,q}(\Omega)$. Finally, in the limit case $q = n$, $s = n/2$, we obtain directly that $u \cdot \nabla u \in L^{q_1}(\Omega)$ for every $1 < q_1 < n$, and (3.21) holds with the last term replaced by $A_{q_1}^{-1}P_{q_1}(\operatorname{div}(uu) - ku)$. Choosing $s < q_1 < n$ we get that $\hat{u} \in D(A_s) + D(A_{q_1}) \subset D(A_s)$. This completes the proof. □

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References

[1] *R. A. Adams*: Sobolev Spaces. Academic Press, New York, 1975.
[2] *H. Amann*: Nonhomogeneous Navier-Stokes equations with integrable low-regularity data. Int. Math. Ser., Kluwer Academic/Plenum Publishing, New York, 2002, pp. 1–28.
[3] *H. Amann*: Navier-Stokes equations with nonhomogeneous Dirichlet data. J. Nonlinear Math. Physics *10* (2003), 1–11.
[4] *W. Borchers and T. Miyakawa*: Algebraic L^2 decay for Navier-Stokes flows in exterior domains. Hiroshima Math. J. *21* (1991), 621–640.
[5] *M. E. Bogovskij*: Solution of the first boundary value problem for the equation of continuity of an incompressible medium. Soviet Math. Dokl. *20* (1979), 1094–1098.
[6] *M. Cannone*: Viscous flows in Besov spaces. Advances in Math. Fluid Mech., Springer, Berlin, 2000, pp. 1–34.
[7] *E. B. Fabes, B. F. Jones and N. M. Rivière*: The initial value problem for the Navier-Stokes equations with data in L^p. Arch. Rational Mech. Anal. *45* (1972), 222–240.
[8] *R. Farwig and H. Sohr*: Generalized resolvent estimates for the Stokes system in bounded and unbounded domains. J. Math. Soc. Japan *46* (1994), 607–643.
[9] *R. Farwig, G. P. Galdi and H. Sohr*: A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data. J. Math. Fluid Mech. *8* (2006), 423–444.
[10] *J. Frehse and M. Růžička*: Weighted estimates for the stationary Navier-Stokes equations. Acta Appl. Math. *37* (1994), 53–66.
[11] *J. Frehse and M. Růžička*: Regularity for the stationary Navier-Stokes equations in bounded domains. Arch. Rational Mech. Anal. *128* (1994), 361–380.
[12] *J. Frehse and M. Růžička*: On the regularity of the stationary Navier-Stokes equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. (IV) *21* (1994), 63–95.
[13] *J. Frehse and M. Růžička*: Existence of regular solutions to the stationary Navier-Stokes equations. Math. Ann. *302* (1995), 669–717.
[14] *J. Frehse and M. Růžička*: Existence of regular solutions to the steady Navier-Stokes equations in bounded six-dimensional domains. Ann. Sc. Norm. Super. Pisa Cl. Sci. (IV) *23* (1996), 701–719.
[15] *J. Frehse and M. Růžička*: Regularity for steady solutions of the Navier-Stokes equations J. G. Heywood, et al. (eds.), Theory of the Navier-Stokes equations. Proc. 3rd Intern. Conf. Navier-Stokes Equations: theory and numerical methods. World Scientific Ser. Adv. Math. Appl. Sci., Singapore *47* (1998), 159–178.
[16] *J. Frehse and M. Růžička*: A new regularity criterion for steady Navier-Stokes equations. Differential Integral Equations *11* (1998), 361–368.
[17] *D. Fujiwara and H. Morimoto*: An L_r-theory of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo (1A) *24* (1977), 685–700.
[18] *G. P. Galdi*: An Introduction to the Mathematical Theory of the Navier-Stokes Equations; Linearized Steady Problems. Springer Tracts in Natural Philosophy, Vol. 38, Springer-Verlag, New York, 1998.
[19] *G. P. Galdi*: An Introduction to the Mathematical Theory of the Navier-Stokes Equations; Nonlinear Steady Problems. Springer Tracts in Natural Philosophy, Vol. 39, New York, 1998.

[20] *G. P. Galdi, C. G. Simader and H. Sohr*: A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}(\partial\Omega)$. Math. Ann. *331* (2005), 41–74.

[21] *C. Gerhardt*: Stationary solutions of the Navier-Stokes equations in dimension four. Math. Z. *165* (1979), 193–197.

[22] *Y. Giga*: Analyticity of the semigroup generated by the Stokes operator in L_r-spaces. Math. Z. *178* (1981), 287–329.

[23] *Y. Giga*: Domains of fractional powers of the Stokes operator in L_r-spaces. Arch. Rational Mech. Anal. *89* (1985), 251–265.

[24] *Y. Giga and H. Sohr*: On the Stokes operator in exterior domains. J. Fac. Sci. Univ. Tokyo, Sec. IA *36* (1989), 103–130.

[25] *Y. Giga and H. Sohr*: Abstract L^q-estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. J. Funct. Anal. *102* (1991), 72–94.

[26] *T. Kato*: Strong L^p-solutions to the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions. Math. Z. *187* (1984), 471–480.

[27] *H. Kozono and M. Yamazaki*: Local and global solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$. Houston J. Math. *21* (1995), 755–799.

[28] *J. Nečas*: Les Méthodes Directes en Théorie des Équations Elliptiques. Academia, Prague, 1967.

[29] *C. G. Simader and H. Sohr*: A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. Adv. Math. Appl. Sci., World Scientific *11* (1992), 1–35.

[30] *V. A. Solonnikov*: Estimates for solutions of nonstationary Navier-Stokes equations. J. Soviet Math. *8* (1977), 467–528.

[31] *H. Sohr*: The Navier-Stokes equations. An elementary functional analytic approach. Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2001.

[32] *R. Temam*: Navier-Stokes Equations. Theory and numerical analysis. North-Holland, Amsterdam, New York, Tokyo, 1984.

[33] *H. Triebel*: Interpolation Theory, Function Spaces. Differential Operators. North-Holland, Amsterdam, 1978.

[34] *W. von Wahl*: Regularity of weak solutions of the Navier-Stokes equations. Proc. Symp. Pure Math. *45* (1986), 497–503.

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