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CLEAN MATRICES OVER COMMUTATIVE RINGS

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Abstract. A matrix $A \in M_n(R)$ is e-clean provided there exists an idempotent $E \in M_n(R)$ such that $A - E \in \mathrm{GL}_n(R)$ and $\det E = e$. We get a general criterion of e-cleanness for the matrix $[[a_1, a_2, \ldots, a_{n+1}]]$. Under the n-stable range condition, it is shown that $[[a_1, a_2, \ldots, a_{n+1}]]$ is 0-clean iff $(a_1, a_2, \ldots, a_{n+1}) = 1$. As an application, we prove that the 0-cleanness and unit-regularity for such $n \times n$ matrix over a Dedekind domain coincide for all $n \geqslant 3$. The analogous for $(s, 2)$ property is also obtained.

Keywords: matrix, clean element, unit-regularity

MSC 2010: 15A23, 16E50

1. Introduction

An element in a ring is clean (unit-regular) provided it is the sum (product) of an idempotent and an invertible element. A ring R is unit-regular provided every element in R is unit-regular. In [1, Theorem 1], Camillo and Khurana proved that every element in a unit-regular ring is clean. In [9, Theorem], Nicholson and Varadarjan proved that every countable linear transformation over a division ring is clean. This shows that clean elements may not be unit-regular even in a regular ring. In fact, the relationship between cleanness and unit-regularity is rather subtle (cf. [4] and [10]).

Recall that $A \in M_n(R)$ is e-clean provided there exists an idempotent $E \in M_n(R)$ such that $A - E \in \mathrm{GL}_n(R)$ and $\det E = e$. We get a general criterion of e-cleanness for the matrix $[[a_1, a_2, \ldots, a_{n+1}]]$. We use $(a_1, \ldots, a_n) = 1$ to stand for the condition $a_1R + \ldots + a_nR = R$. A ring R is said to satisfy the n-stable range condition provided $(a_1, \ldots, a_n, a_{n+1}) = 1$ in R implies that there exist $c_1, \ldots, c_n \in R$ such that $(a_1 + a_{n+1}c_1, \ldots, a_n + a_{n+1}c_n) = 1$ in R (see [8]). Let $a_1, a_2, \ldots, a_{n+1} \in R$ ($n \in \mathbb{N}$). If R satisfies the n-stable range condition, we will prove that $[[a_1, a_2, \ldots, a_{n+1}]]$ is

0-clean iff $(a_1, a_2, \ldots, a_{n+1}) = 1$. In [7], Khurana and Lam proved that there are many matrices $[[a, b]] \in M_2(\mathbb{Z})$ which are unit-regular while they are not 0-clean, e.g., $[[12, 5]], [[13, 5]]$, $[[12, 7]]$, etc. As an application, we prove that the 0-cleanness and unit-regularity for such $n \times n$ matrix over a Dedekind domain coincide for all $n \geqslant 3$. We say that $a \in R$ is $(s, 2)$ provided a is the sum of two units. An analog of the $(s, 2)$ property is also obtained.

Throughout the paper, all rings are commutative rings with an identity. $M_n(R)$ denotes the set of all $n \times n$ matrices over R, $\mathrm{GL}_n(R)$ denotes the n-dimensional general linear group of R and $U(R) = \mathrm{GL}_1(R)$. $\mathbb{N}$ stands for the set of all natural numbers. We write $[[a_1, a_2, \ldots, a_n]]$ for the matrix whose first row is $(a_1, a_2, \ldots, a_n)$ and other entries are zeros.

2. Cleanness

In this section we get a general criterion for an $n \times n$ matrix $[[a_1, a_2, \ldots, a_n]]$ over a commutative ring to be e-clean. This gives a generalization of [7, Theorem 3.2] as well.

Theorem 2.1. *Let $a_1, \ldots, a_n \in R$, and let $e \in R$ be an idempotent. Then $[[a_1, a_2, \ldots, a_n]]$ is e-clean if and only if the following conditions hold:*
(1) *There exist $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \ldots + a_nx_n \in R$ is e-clean.*
(2) $ex_2 = \ldots = ex_n = 0$.
(3) $x_1 \equiv 1 \pmod{x_2R + \ldots + x_nR}$.

P r o o f. Suppose that $[[a_1, a_2, \ldots, a_n]]$ is e-clean. Then we have an idempotent matrix $E = (e_{ij}) \in M_n(R)$ and a $U = (u_{ij}) \in \mathrm{GL}_n(R)$ such that $[[a_1, a_2, \ldots, a_n]] = E + U$ and $\det E = e$. Thus,

$$\begin{pmatrix} a_1 - e_{11} & a_2 - e_{12} & \ldots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \ldots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \ldots & -e_{nn} \end{pmatrix} = U.$$

This implies that

$$\begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ -e_{21} & -e_{22} & \ldots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \ldots & -e_{nn} \end{vmatrix} + (-1)^n \det E = \det U.$$

Hence, $a_1A_{11}+a_2A_{12}+\ldots+a_nA_{1n} = (-1)^{n+1}e+u$, where $u=\det U\in U(R)$ and each A_{1i} is the algebraic complement corresponding to a_i $(1\leqslant i\leqslant n)$. Let $x_1=(-1)^{n+1}A_{11}, x_2=(-1)^{n+1}A_{12},\ldots,x_n=(-1)^{n+1}A_{1n}$. Then $a_1x_1+a_2x_2+\ldots+a_nx_n=e+(-1)^{n+1}u$ is e-clean. As $E\in M_n(R)$ is an idempotent with $\det E=e$, in view of [7, Proposition 2.7] we get $ee_{ii}=e, ee_{ij}=0$ $(1\leqslant i\neq j\leqslant n)$. This implies that $eA_{12}=\ldots=eA_{1n}=0$; hence, $ex_2=\ldots=ex_n=0$.

Clearly, we have $(-e_{21})A_{11}+(-e_{22})A_{12}+\ldots+(-e_{2n})A_{1n}=0$, and thus,

$$(-e_{21})A_{11}\equiv 0 \pmod{x_2R+\ldots+x_nR}.$$

On the other hand, $u_{11}A_{11}+u_{12}A_{12}+\ldots+u_{1n}A_{1n}=u$, and thus,

$$u_{11}A_{11}\equiv u \pmod{x_2R+\ldots+x_nR}.$$

As $u\in U(R)$, we deduce that

$$-e_{21}\equiv 0 \pmod{x_2R+\ldots+x_nR}.$$

Similarly, we show that

$$-e_{31},\ldots,-e_{n1}\equiv 0 \pmod{x_2R+\ldots+x_nR}.$$

Since $(-E)(-E)=E$, we see that

$$\begin{pmatrix} -e_{11} & -e_{12} & \ldots & -e_{1n}\\ 0 & -e_{22} & \ldots & -e_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & -e_{n2} & \ldots & -e_{nn}\end{pmatrix}\begin{pmatrix} -e_{11} & -e_{12} & \ldots & -e_{1n}\\ 0 & -e_{22} & \ldots & -e_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & -e_{n2} & \ldots & -e_{nn}\end{pmatrix}$$
$$\equiv \begin{pmatrix} e_{11} & e_{12} & \ldots & e_{1n}\\ 0 & e_{22} & \ldots & e_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & e_{n2} & \ldots & e_{nn}\end{pmatrix} \pmod{x_2R+\ldots+x_nR}.$$

This implies that

$$\begin{pmatrix} -e_{22} & \ldots & -e_{2n}\\ \vdots & \ddots & \vdots\\ -e_{n2} & \ldots & -e_{nn}\end{pmatrix}\begin{pmatrix} -e_{22} & \ldots & -e_{2n}\\ \vdots & \ddots & \vdots\\ -e_{n2} & \ldots & -e_{nn}\end{pmatrix}$$
$$\equiv \begin{pmatrix} e_{22} & \ldots & e_{2n}\\ \vdots & \ddots & \vdots\\ e_{n2} & \ldots & e_{nn}\end{pmatrix} \pmod{x_2R+\ldots+x_nR}.$$

As a result we have

$$A_{11}^2 = (-1)^{n+1} A_{11} (\bmod x_2R + \ldots + x_nR).$$

Hence,

$$u_{11}A_{11}^2 \equiv (-1)^{n+1} u_{11} A_{11} (\bmod x_2R + \ldots + x_nR).$$

Therefore we get

$$A_{11} \equiv (-1)^{n+1} \pmod{x_2R + \ldots + x_nR},$$

that is,

$$x_1 \equiv 1 (\bmod x_2R + \ldots + x_nR).$$

Conversely, assume that (1), (2) and (3) hold. By (1), we can find $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \ldots + a_ix_i + \ldots + a_nx_n$ is e-clean. Let $c_1 = x_1$ and $c_i = -x_i$ $(2 \leqslant i \leqslant n)$. Then $a_1c_1 - \ldots - a_ic_i - \ldots - a_nc_n$ is e-clean. By (3), we can find $k_2, \ldots, k_n \in R$ such that $c_1 = 1 + k_2c_2 + \ldots + k_nc_n$. Let

$$E = (e_{ij}) = \begin{pmatrix} 1 & -k_2 & \ldots & -k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} e & & & \\ c_2 & 1 & & \\ \vdots & & \ddots & \\ c_n & & & 1 \end{pmatrix} \begin{pmatrix} 1 & k_2 & \ldots & k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

By (2), it is easy to verify that $E = E^2 \in M_n(R)$ and $\det E = e$. Let

$$U = (u_{ij}) = \begin{pmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} - E.$$

Then

$$\begin{aligned} \det U &= \begin{vmatrix} a_1 - e_{11} & a_2 - e_{12} & \ldots & a_n - e_{1n} \\ -e_{21} & -e_{22} & \ldots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \ldots & -e_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ -e_{21} & -e_{22} & \ldots & -e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & -e_{n2} & \ldots & -e_{nn} \end{vmatrix} + (-1)^n \det E \\ &= (-1)^{n-1}(a_1A_{11} + \ldots + a_nA_{1n}) + (-1)^n e, \end{aligned}$$

where $A_{11},\ldots,A_{1n}$ are algebraic complements of E corresponding to $e_{11},\ldots,e_{1n}$ respectively. Obviously,

$$E=\begin{pmatrix} 1+(e-c_1) & (e-c_1)k_2 & (e-c_1)k_3 & \ldots & (e-c_1)k_n \\ c_2 & 1+k_2c_2 & k_3c_2 & \ldots & k_nc_2 \\ c_3 & k_2c_3 & 1+k_3c_3 & \ldots & k_nc_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & k_2c_n & k_3c_n & \ldots & 1+k_nc_n. \end{pmatrix}$$

It is easy to see that $A_{11}=1+k_2c_2+\ldots+k_nc_n=c_1$. Furthermore, we see that each $A_{1i}=-c_i$ $(2\leqslant i\leqslant n)$. Clearly, there is a $u\in U(R)$ such that $a_1A_{11}+\ldots+a_{1n}A_{1n}=a_1c_1-\ldots-a_ic_i-\ldots-a_nc_n=e+u$. Thus, $\det U=(-1)^{n-1}(e+u)+(-1)^ne=(-1)^{n-1}u\in U(R)$, and then $U\in \mathrm{GL}_n(R)$. Therefore A is e-clean, as asserted. □

Corollary 2.2. *Let $a_1,\ldots,a_n\in R$ $(n\in\mathbb{N})$. If $[[a_1,a_2,\ldots,a_n]]$ is 0-clean, then so is $[[a_1u_1,a_2u_2,\ldots,a_nu_n]]$ for any $u_1,\ldots,u_n\in U(R)$.*

P r o o f. Assume that $[[a_1,a_2,\ldots,a_n]]$ is 0-clean. According to Theorem 2.1, there exist $x_1,x_2,\ldots,x_n\in R$ such that $a_1x_1+\ldots+a_nx_n=u\in U(R)$ and $x_1\equiv 1$ $(\mathrm{mod}\ x_2R+\ldots+x_nR)$. Thus, we deduce that $(a_1u_1)x_1+a_2(u_1x_2)+\ldots+a_n(u_1x_n)=u_1u\in U(R)$. In addition,

$$x_1\equiv 1\ (\mathrm{mod}\ (u_1x_2)R+\ldots+(u_1x_n)R).$$

In view of Theorem 2.1, we have an idempotent $E\in M_n(R)$ and a $U\in\mathrm{GL}_n(R)$ such that

$$\begin{pmatrix} a_1u_1 & a_2 & \ldots & a_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}=E+U \quad\text{and}\quad \det E=0.$$

Therefore we conclude that

$$[[a_1u_1,a_2u_2,\ldots,a_nu_n]]=\begin{pmatrix} 1 & & & \\ & u_2^{-1} & & \\ & & \ddots & \\ & & & u_n^{-1} \end{pmatrix} E \begin{pmatrix} 1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix}$$
$$+\begin{pmatrix} 1 & & & \\ & u_2^{-1} & & \\ & & \ddots & \\ & & & u_n^{-1} \end{pmatrix} U \begin{pmatrix} 1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix},$$

as desired. □

Example 2.3. Let us show that $[[12,5,3]] \in M_3(\mathbb{Z})$ is clean, while $[[12,5]] \in M_2(\mathbb{Z})$ is not. In view of [7, Example 4.5], $[[12,5]] \in M_2(\mathbb{Z})$ is not clean. Since $12\times(-2)+5\times 2+3\times 5 = 1$ and $-2 \equiv 1 \pmod{2R+5R}$, it follows by Theorem 2.1 that $[[12,5,3]] \in M_3(\mathbb{Z})$ is 0-clean. In fact, we have the decomposition: $[[12,5,3]] = E+U$, where

$$E = \begin{pmatrix} 3 & 8 & -2 \\ -2 & -7 & 2 \\ -5 & -20 & 6 \end{pmatrix}, \quad U = \begin{pmatrix} 9 & -3 & 5 \\ 2 & 7 & -2 \\ 5 & 20 & -6 \end{pmatrix}$$

with $E = E^2, \det E = 0$ and $\det U = 1$. □

Note that Theorem 2.1 illustrates the process of computing "clean decompositions" of numerical examples. Let $a_1,\dots,a_n,a_{n+1} \in R$ $(n \in \mathbb{N})$. If $[[a_1,a_2,\dots,a_n]] \in M_n(R)$ is e-clean, then so is $[[a_1,a_2,\dots,a_{n+1}]] \in M_{n+1}(R)$. Example 2.3 shows that the converse is not true.

3. Stable ranges

Lemma 3.1. *Let $a_1,a_2,\dots,a_{n+1} \in R$ $(n \in \mathbb{N})$. If $(a_2,\dots,a_{n+1}) = 1$, then $[[a_1,a_2,\dots,a_{n+1}]] \in M_{n+1}(R)$ is 0-clean.*

Proof. Since $(a_2,\dots,a_{n+1}) = 1$, there are $x_2,\dots,x_{n+1} \in R$ such that $a_2x_2 + \dots + a_{n+1}x_{n+1} = 1$. Thus, $a_1 \times 0 + a_2x_2 + \dots + a_{n+1}x_{n+1} = 1$. It is easy to see that

$$0 \equiv 1 \pmod{x_2R + \dots + x_{n+1}R}.$$

Applying to Theorem 2.1, we complete the proof. □

Theorem 3.2. *Let $a_1,a_2,\dots,a_{n+1} \in R$ $(n \in \mathbb{N})$. If R satisfies the n-stable range condition, then the following conditions are equivalent:*
(1) *$[[a_1,a_2,\dots,a_{n+1}]]$ is 0-clean.*
(2) *$(a_1,a_2,\dots,a_{n+1}) = 1$.*

Proof. (1) ⇒ (2) By virtue of Theorem 2.1, there exist $x_1,\dots,x_{n+1} \in R$ such that $a_1x_1 + \dots + a_{n+1}x_{n+1} = u \in U(R)$; hence, $a_1x_1u^{-1} + \dots + a_{n+1}x_{n+1}u^{-1} = 1$. That is, $(a_1,a_2,\dots,a_{n+1}) = 1$.

(2) ⇒ (1) Since $(a_1,a_2,\dots,a_{n+1}) = 1$ in R, there exist $c_2,\dots,c_{n+1} \in R$ such that $(a_2 + a_1c_2,\dots,a_{n+1} + a_1c_{n+1}) = 1$. In view of Lemma 3.1, $[[a_1, a_2 + a_1c_2,\dots, a_{n+1} + a_1c_{n+1}]] \in M_{n+1}(R)$ is 0-clean. Thus, we have an idempotent $E \in M_{n+1}(R)$

and a $U \in \mathrm{GL}_{n+1}(R)$ such that $[[a_1, a_2 + a_1c_2, \ldots, a_{n+1} + a_1c_{n+1}]] = E + U$ and $\det E = 0$. Let

$$Q = \begin{pmatrix} 1 & c_2 & \ldots & c_{n+1} \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R).$$

Then, $Q^{-1}[[a_1, a_2, \ldots, a_{n+1}]]Q = [[a_1, a_2 + a_1c_2, \ldots, a_{n+1} + a_1c_{n+1}]] = E + U$. Therefore $[[a_1, a_2, \ldots, a_{n+1}]] = QEQ^{-1} + QUQ^{-1}$. In addition, $QEQ^{-1} \in M_{n+1}(R)$ is an idempotent matrix, $\det QEQ^{-1} = 0$ and $QUQ^{-1} \in \mathrm{GL}_{n+1}(R)$. Thus we complete the proof. □

Recall that a domain ring R is a Dedekind domain provided every ideal of R is a projective R-module. The class of Dedekind domains is very large. It includes all principal ideal domains. The ring $\mathbb{Z}[\sqrt{-d}]$ is a Dedekind domain provided d is square-free and $d \neq 3 \pmod 4$. Also we note that $\mathbb{R}[x,y]/(x^2+y^2-1)$, the ring of polynomial functions on a circle, is a Dedekind domain. It is well known that every Dedekind domain satisfies the 2-stable range condition.

Corollary 3.3. *Let R be a Dedekind domain and let $a_1, \ldots, a_n \in R$ ($n \geqslant 3$). Then the following conditions are equivalent:*

(1) $[[a_1, a_2, \ldots, a_n]]$ *is 0-clean.*

(2) $[[a_1, a_2, \ldots, a_n]] \neq 0$ *is unit-regular.*

(3) $(a_1, \ldots, a_n) = 1$.

Proof. (1) ⇔ (3) Since R is a Dedekind domain, it satisfies the 2-stable range condition, and so this is clear by virtue of Theorem 3.2.

(2) ⇒ (3) Let $[[a_1, a_2, \ldots, a_n]] \neq 0$ be unit-regular. Then there exist an idempotent $E = (e_{ij}) \in M_n(R)$ and a $U = (u_{ij}) \in \mathrm{GL}_n(R)$ such that $[[a_1, a_2, \ldots, a_n]] = EU$, i.e., $[[a_1, a_2, \ldots, a_n]]U^{-1} = E$. This implies that $e_{ij} = 0$ for $i = 2, \ldots, n$. Thus, $[[a_1, a_2, \ldots, a_n]] = [[e_{11}, e_{12}, \ldots, e_{1n}]]U$; hence, $(a_1, \ldots, a_n) = e_{11}(u_{11}, \ldots, u_{1n})$. Clearly, $e_{11} = e_{11}^2 \in R$, and then, $e_{11} = 1$. Thus we get $(a_1, \ldots, a_n) = (u_{11}, \ldots, u_{1n}) = 1$.

(3) ⇒ (2) Since $(a_1, \ldots, a_{n-1}, a_n) = 1$, there are $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \ldots + a_nx_n = 1$. As R satisfies the 2-stable range condition, we have b_i, c_i ($3 \leqslant i \leqslant n$) such that $(a_1 + a_3b_3 + \ldots + a_nb_n, a_2 + a_3c_3 + \ldots + a_nc_n) = 1$. Thus, $(a_1 + a_3b_3 + \ldots + a_nb_n)x + (a_2 + a_3c_3 + \ldots + a_nc_n)y = 1$ for some $x, y \in R$. One

easily checks that

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -y & x & 0 & \dots & 0 & 0 \\ -b_3 & -c_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{n-1} & -c_{n-1} & 0 & \dots & 1 & 0 \\ -b_n & -c_n & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(R).$$

Therefore

$$[[a_1, a_2, \dots, a_n]] = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ -y & x & \dots & 0 \\ -b_3 & -c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -b_n & -c_n & \dots & 1 \end{pmatrix},$$

as desired. □

The following result should be compared to the fact that the problem of deciding the cleanness of $[[a, b]] \in M_2(\mathbb{Z})$ is considerably harder (cf. [7]).

Corollary 3.4. *Let* $a_1, \dots, a_n \in \mathbb{Z}$ $(n \geqslant 3)$. *Then* $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ *is clean iff* $a_1 = 0$ *or* $a_1 = 2$ *or* $(a_1, \dots, a_n) = 1$.

Proof. If $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ is 1-clean, then we can find an idempotent $E \in M_n(\mathbb{Z})$ and a $U = (u_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ such that $[[a_1, a_2, \dots, a_n]] = E + U$ and $\det E = 1$. Thus, $E = diag(1, \dots, 1) \in M_n(\mathbb{Z})$. This implies that $u_{ij} = 0$ $(i \neq 1, j)$, $u_{ii} = -1$ $(2 \leqslant i \leqslant n)$. Hence, $a_1 - 1 \in U(\mathbb{Z})$, i.e., $a_1 = 0, 2$. Thus we conclude that $[[a_1, a_2, \dots, a_n]] \in M_n(\mathbb{Z})$ is 1-clean if and only if either $a_1 = 0$ or $a_1 = 2$. Consequently, the result follows from Corollary 3.3. □

We say that $0 \neq A \in M_n(R)$ has rank 1 provided there exist $P, Q \in \mathrm{GL}_n(R)$ such that $PAQ = [[a_1, \dots, a_n]]$ for some $a_1, \dots, a_n \in R$.

Corollary 3.5. *Let R be a Dedekind domain and let* $A \in M_n(R)$ $(n \geqslant 3)$. *If A has rank* 1, *then the following conditions are equivalent:*
(1) *A is* 0*-clean.*
(2) *A is unit-regular.*

Proof. (1) ⇒ (2) As A has rank 1, there exist $P, Q \in \mathrm{GL}_n(R)$ such that $PAQ = [[a_1, \dots, a_n]]$ for some $a_1, \dots, a_n \in R$. Thus,

$$PAP^{-1} = [[a_1, \dots, a_n]]Q^{-1}P^{-1} = [[b_1, \dots, b_n]]$$

for some $b_1, \ldots, b_n \in R$. This implies that $[[b_1, \ldots, b_n]]$ is 0-clean. According to Corollary 3.3, $[[b_1, \ldots, b_n]]$ is unit-regular. Therefore A is unit-regular.

(2) $\Rightarrow$ (1) As in the preceding discussion, $PAP^{-1} = [[b_1, \ldots, b_n]] \neq 0$ for some $b_1, \ldots, b_n \in R$. Thus, $[[b_1, \ldots, b_n]]$ is unit-regular. In view of Corollary 3.3, $[[b_1, \ldots, b_n]]$ is 0-clean, and therefore so is A. □

It is clear that no polynomial in the polynomial ring over a field is clean. Furthermore, [1, Example 3.3] shows that no polynomial in the polynomial ring over a commutative ring is semiclean. We end this section by noting that Theorem 2.1 provides an explicit program to represent such kind of a matrix as the sum of an idempotent matrix and an invertible matrix.

Example 3.6. Let $[[1 + xy, x^2, y]] \in M_3(\mathbb{Z}[x, y])$. Obviously, we have $(1 + xy) \cdot (1-xy)+x^2 \cdot y+y \cdot x^2(-1+y) = 1$. In addition, $1-xy = 1+y \cdot (-x)+x^2(-1+y) \cdot 0$. Thus, we have

$$\begin{aligned} E &= \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -y & 1 & 0 \\ x^2(1-y) & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} xy & -x(1-xy) & 0 \\ -y & 1-xy & 0 \\ x^2(1-y) & x^3(1-y) & 1 \end{pmatrix}. \end{aligned}$$

Then $E = E^2 \in M_3(\mathbb{Z}[x, y])$ and $\det E = 0$. Let

$$U = \begin{pmatrix} 1+xy & x^2 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - E = \begin{pmatrix} 1 & x(1+x-xy) & y \\ y & -1+xy & 0 \\ -x^2(1-y) & -x^3(1-y) & -1 \end{pmatrix}.$$

Then $U \in \mathrm{GL}_3(\mathbb{Z}[x, y])$ and $\det U = 1$. This proves that

$$\begin{aligned} &[[1 + xy, x^2, y]] \\ &= \begin{pmatrix} xy & -x(1-xy) & 0 \\ -y & 1-xy & 0 \\ x^2(1-y) & x^3(1-y) & 1 \end{pmatrix} + \begin{pmatrix} 1 & x(1+x-xy) & y \\ y & -1+xy & 0 \\ -x^2(1-y) & -x^3(1-y) & -1 \end{pmatrix} \end{aligned}$$

is clean. □

4. Extensions

In [2], Camillo and Yu proved that every element of a clean ring in which 2 is invertible is $(s, 2)$. In this section, we investigate some sufficient conditions under which an $n \times n$ matrix $[[a_1, a_2, \ldots, a_n]]$ over a commutative ring is $(s, 2)$.

Theorem 4.1. *Let $a_1, \dots, a_n \in R$. Then $[[a_1, a_2, \dots, a_n]]$ is $(s, 2)$ provided the following conditionshold:*

(1) *There exist $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_nx_n \in R$ is $(s, 2)$.*

(2) $x_1 \equiv 1 \pmod{x_2R + \dots + x_nR}$.

P r o o f. By (1), we can find $x_1, \dots, x_n \in R$ such that $a_1x_1 + \dots + a_ix_i + \dots + a_nx_n$ is $(s, 2)$. Let $c_1 = x_1$ and $c_i = -x_i$ $(2 \leqslant i \leqslant n)$. Then $a_1c_1 - \dots - a_ic_i - \dots - a_nc_n = u + v$ for some $u, v \in U(R)$. Let

$$U = (e_{ij}) = \begin{pmatrix} 1 & -k_2 & \dots & -k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} u & & & \\ c_2 & 1 & & \\ \vdots & & \ddots & \\ c_n & & & 1 \end{pmatrix} \begin{pmatrix} 1 & k_2 & \dots & k_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Obviously, $U \in \mathrm{GL}_n(R)$ and $\det U = u$. Let

$$V = (v_{ij}) = \begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} - U.$$

Then

$$\begin{aligned} \det V &= \begin{vmatrix} a_1 - u_{11} & a_2 - u_{12} & \dots & a_n - u_{1n} \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ -u_{21} & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \dots & -u_{nn} \end{vmatrix} + (-1)^n \det U \\ &= (-1)^{n-1}(a_1A_{11} + \dots + a_nA_{1n}) + (-1)^n u, \end{aligned}$$

where $A_{11}, \dots, A_{1n}$ are algebraic complements of U corresponding to $u_{11}, \dots, u_{1n}$, respectively. Clearly,

$$U = \begin{pmatrix} 1 + (u - c_1) & (u - c_1)k_2 & (u - c_1)k_3 & \dots & (u - c_1)k_n \\ c_2 & 1 + k_2c_2 & k_3c_2 & \dots & k_nc_2 \\ c_3 & k_2c_3 & 1 + k_3c_3 & \dots & k_nc_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & k_2c_n & k_3c_n & \dots & 1 + k_nc_n. \end{pmatrix}$$

It is easy to see that $A_{11} = 1 + k_2c_2 + \ldots + k_nc_n = c_1$. As in the proof of Theorem 2.1, we see that $A_{1i} = -c_i$ $(2 \leqslant i \leqslant n)$. Thus, $a_1A_{11} + \ldots + a_{1n}A_{1n} = a_1c_1 - \ldots - a_ic_i - \ldots - a_nc_n = u + v$. Hence, $\det U = (-1)^{n-1}(u+v) + (-1)^n u = (-1)^{n-1}v \in U(R)$, and so $U \in \mathrm{GL}_n(R)$. Consequently, we conclude that A is $(s,2)$, as desired. □

Corollary 4.2. *Let $a_1, \ldots, a_n \in R$. Then $[[a_1, a_2, \ldots, a_{n+1}]] \in M_{n+1}(R)$ is $(s,2)$ provided the following conditions hold:*

(1) *There exist $u, v \in U(R)$ such that $1 = u + v$.*

(2) *R satisfies the n-stable range condition.*

(3) $(a_1, \ldots, a_{n+1}) = 1$.

Proof. By (2) and (3), there exist $c_2, \ldots, c_{n+1} \in R$ such that $(a_2 + a_1c_2, \ldots, a_{n+1} + a_1c_{n+1}) = 1$. Let $b_i = a_i + a_1c_i (2 \leqslant i \leqslant n)$. Then there are $x_2, \ldots, x_{n+1} \in R$ such that $b_2x_2 + \ldots + b_{n+1}x_{n+1} = 1$. By (1), $a_1 \times 0 + b_2x_2 + \ldots + b_{n+1}x_{n+1} = u + v$. Clearly,

$$0 \equiv 1 \pmod{x_2R + \ldots + x_{n+1}R}.$$

By Theorem 4.1, there are $U, V \in \mathrm{GL}_{n+1}(R)$ such that $[[a_1, b_2, \ldots, b_{n+1}]] = U + V$. Let

$$Q = \begin{pmatrix} 1 & c_2 & \ldots & c_{n+1} \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R).$$

Then $[[a_1, a_2, \ldots, a_{n+1}]]Q = [[a_1, a_2 + a_1c_2, \ldots, a_{n+1} + a_1c_{n+1}]] = U + V$. This implies that $[[a_1, a_2, \ldots, a_{n+1}]] = UQ^{-1} + VQ^{-1}$, as required. □

Example 4.3. Let $R = \{0, e, a, b\}$ be a set. Define operations by the following tables:

$+$	0	e	a	b
0	0	e	a	b
e	e	0	b	a
a	a	b	0	e
b	b	a	e	0

$\times$	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	a	b	e
b	0	b	e	a

Then R is a field with four elements. In this case, $2 \notin U(R)$ and the identity $e \in R$ is the sum $a + b$ of two units $a, b \in R$. Let $[[e + x, x^2, e - x]] \in M_3(R[x])$. Then $(e + x)(e - x) + x^2 \times 1 + (e - x) \times 0 = e$. Clearly, $R[x]$ satisfies the 2-stable range condition. According to Corollary 4.2, $[[e + x, x^2, e - x]] \in M_3(R[x])$ is $(s,2)$. □

Theorem 4.4. *Let $a_1, \ldots, a_n \in R$. If R satisfies the 1-stable range condition, then $[[a_1, a_2, \ldots, a_n]]$ is $(s, 2)$ iff the following conditions hold:*
(1) *There exist $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \ldots + a_nx_n$ is $(s, 2)$.*
(2) $x_1 \equiv 1 \pmod{x_2R + \ldots + x_nR}$.

Proof. "$\Leftarrow$" is clear by Theorem 4.1.

"$\Rightarrow$" Suppose that $[[a_1, a_2, \ldots, a_n]]$ is $(s, 2)$. Then we have two matrices $U = (u_{ij}), V = (v_{ij}) \in \mathrm{GL}_n(R)$ such that $[[a_1, a_2, \ldots, a_n]] = U + V$. Thus,

$$\begin{pmatrix} a_1 - u_{11} & a_2 - u_{12} & \ldots & a_n - u_{1n} \\ -u_{21} & -u_{22} & \ldots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \ldots & -u_{nn} \end{pmatrix} = V.$$

Hence, we get

$$\begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ -u_{21} & -u_{22} & \ldots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n1} & -u_{n2} & \ldots & -u_{nn} \end{vmatrix} + (-1)^n \det U = \det V.$$

It follows that $a_1A_{11} + a_2A_{12} + \ldots + a_nA_{1n} = (-1)^{n+1}u + v$, where $u = \det U$, $v = \det V$ and A_{1i} $(1 \leqslant i \leqslant n)$ is the algebraic complement corresponding to a_i $(1 \leqslant i \leqslant n)$. Let each $x_i = A_{1i}$. Then $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is $(s, 2)$. Obviously, $(-u_{21})A_{11} + (-u_{22})A_{12} + \ldots + (-u_{2n})A_{1n} = 0$, and thus,

$$(-u_{21})A_{11} \equiv 0 \pmod{x_2R + \ldots + x_nR}.$$

Furthermore, $u_{11}A_{11} + u_{12}A_{12} + \ldots + u_{1n}A_{1n} = (-1)^{n+1}u$, and then

$$u_{11}A_{11} \equiv (-1)^{n+1}u \pmod{x_2R + \ldots + x_nR}.$$

Since $u \in U(R)$, we see that

$$-u_{21} \equiv 0 \pmod{x_2R + \ldots + x_nR}.$$

Likewise, we show that

$$-u_{31}, \ldots, -u_{n1} \equiv 0 \pmod{x_2R + \ldots + x_nR}.$$

Therefore

$$\begin{pmatrix} -u_{11} & -u_{12} & \dots & -u_{1n} \\ 0 & -u_{22} & \dots & -u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -u_{n2} & \dots & -u_{nn} \end{pmatrix} \quad \text{is invertible (mod } x_2R + \ldots + x_nR).$$

This yields that $A_{11} \in R$ is invertible modulo $x_2R + \ldots + x_nR$. That is, there exists a $v \in R$ such that $r := A_{11}v - 1 \in x_2R + \ldots + x_nR$. Since R satisfies the 1-stable range condition, it follows from $A_{11}v - r = 1$ that $w := A_{11} - rz \in U(R)$ for some $z \in R$. Let $x'_i = A_{1i}w^{-1}$ $(1 \leqslant i \leqslant n)$. Then $a_1x'_1 + \ldots + a_nx'_n = (-1)^{n+1}uw^{-1} + vw^{-1} \in R$ is $(s, 2)$. In addition, $x'_1 \equiv 1 \pmod{x_2R + \ldots + x_nR}$, and we are done. □

Corollary 4.5. *Let R be a strongly π-regular ring. Then $[[a_1, a_2, \ldots, a_n]]$ is $(s, 2)$ iff the following conditions hold:*
(1) *There exist $x_1, \ldots, x_n \in R$ such that $a_1x_1 + \ldots + a_nx_n$ is $(s, 2)$.*
(2) *$x_1 \equiv 1 \pmod{x_2R + \ldots + x_nR}$.*

P r o o f. Since every strongly π-regular ring satisfies the 1-stable range condition, we complete the proof by Theorem 4.4. □

Example 4.6. Let $R = \{a + bt \colon a, b \in \mathbb{Z}/2\mathbb{Z},\ t^2 = 0\}$. Then neither 1 nor $1 + t$ is $(s, 2)$. For any $a + bt \in R$, $(a + bt)^2 = (a + bt)^4$; hence, R is strongly π-regular. As $1 \times 1 + (1 + t) \times 1$ is $(s, 2)$ and $1 \equiv 1 \pmod{R}$, it follows by Corollary 4.5 that $[[1, 1 + t]]$ is $(s, 2)$. In fact, we have the decomposition:

$$[[1, 1 + t]] = \begin{pmatrix} 0 & 1 + t \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In this case, $2 \notin U(R)$. □

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