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BOUNDEDLY EXPRESSIBLE SETS

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Abstract. For a given sequence a boundedly expressible set is introduced. Three criteria concerning the Hausdorff dimension of such sets are proved.

Keywords: expressible set, Hausdorff dimension.

MSC 2010: 11K55

1. Introduction

Following Erdős [2] we say that for a given sequence $\{a_n\}_{n=1}^{\infty}$ the set

$$X_B\{a_n\}_{n=1}^{\infty} := \left\{x \in \mathbb{R}\colon\ \exists\, K \in \mathbb{N}\ \exists\, \{c_n\}_{n=1}^{\infty},\ c_n \in \{1,\dots,K\},\ x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n}\right\}$$

is its *boundedly expressible set.* In [2] it is shown that if $\lim\limits_{n\to\infty} a_n^{1/2^n} = \infty$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then $X_B\{a_n\}_{n=1}^{\infty}$ does not contain any rational number. It appears to be the case that in general evaluating the Lebesgue measure or Hausdorff dimension of the set $X_B\{a_n\}_{n=1}^{\infty}$ is not easy. In this paper we give conditions on $\{a_n\}_{n=1}^{\infty}$ to ensure that the Hausdorff dimension of the set $X_B\{a_n\}_{n=1}^{\infty}$ is zero. We prove the following.

Theorem 1. *Let $\{b_n]\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $b_n = \mathrm{O}\big(2^{2^{n/2}}\big)$. Then $\dim X_B\Big\{b_n + 2^{2^{2^{[\log_2 n]}}}\Big\}_{n=1}^{\infty} = 0$.*

It is unknown to the authors if there exists a sequence of real numbers such that its boundedly expressible set has Hausdorff dimension greater than zero and Lebesgue

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measure zero. For sequences which converge to infinity slower than $\{2^{2^n}\}_{n=1}^{\infty}$ Hančl in [3] proved the following theorem.

Theorem 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers and let K be a positive integer. Assume that $F\colon \mathbb{N} \to \mathbb{R}^+$ is a function such that $F(n) < n$,*

$$a_n < K2^{2^{n-F(n)}} \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{-F(n)} < \infty.$$

Put

$$B := \frac{1}{8K[a_1+1]\Big[2^{4\sum\limits_{n=2}^{\infty} 2^{-F(n)}} + 2\Big]}.$$

Then for every number $x \in (0, B]$ there is a sequence $\{c_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ such that $x = \sum\limits_{n=1}^{\infty} 1/a_n c_n$.

2. Main results

We start with the theorem which is the basis for the other results.

Theorem 3. *Let ε be a positive real number. Assume that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that*

$$L := \liminf_{n\to\infty} \frac{2(1+\varepsilon)\log T_n}{\varepsilon \log a_n} \in [0,1), \tag{1}$$

where $T_n = \operatorname{lcm}(a_1, \dots, a_{n-1})$, and that for every sufficiently large n

$$a_n \geqslant n^{1+\varepsilon}. \tag{2}$$

Then $\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant L$.

Theorem 1 follows immediately from the following more general result.

Theorem 4. *Let ε be a positive real number. Assume that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that*

$$\liminf_{n\to\infty} \frac{n \log a_{n-1}}{\log a_n} = 0 \tag{3}$$

and that for every sufficiently large n

$$a_n \geqslant n^{1+\varepsilon}. \tag{4}$$

Then $\dim X_B\{a_n\}_{n=1}^{\infty} = 0$.

Example 1. Set $A := 1.1$ and $a_n := \left[2^{A^{2^{[\log_2 n]}}} + n\right]$. From $n < 1.82 \times 2^{A^n}$ we obtain for $K = 3$ and $F(n) = n \log_2 2/A$ that $a_n < K2^{2^{n-F(n)}}$. Theorem 2 implies that for every $x \in \left(0, \frac{1}{768}\right]$ there is a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $x = \sum_{n=1}^{\infty} 1/a_n c_n$. Theorem 4 implies that for almost every x such sequence $\{c_n\}_{n=1}^{\infty}$ must be unbounded.

In the case that the sequence $\{a_n\}_{n=1}^{\infty}$ is of the Cantor type we can use the following criterion.

Theorem 5. *Let A, B, S and ε be a positive real numbers such that $S > 1$ and $(2/S)(A/B)((1+\varepsilon)/\varepsilon) < 1$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that a_n divides a_{n+1} for every $n \in \mathbb{N}$. Suppose that for every sufficiently large n*

$$a_n \geqslant n^{1+\varepsilon}. \tag{5}$$

Assume that for infinitely many N

$$a_{N-1} \leqslant 2^{AS^{N-1}} \quad \text{and} \quad a_N \geqslant 2^{BS^N}. \tag{6}$$

Then $\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant (2/S)(A/B)((1+\varepsilon)/\varepsilon)$.

Let us note that Theorem 5 also holds in the case when $A > B$. The authors do not know how to find non-trivial lower bounds for X_B.

Corollary 1. *Let A, B and S be positive real numbers such that $S > 1$ and $(2/S)(A/B) < 1$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that a_n divides a_{n+1} for every n. Assume that $a_n \geqslant 2^n$ for every sufficiently large n. Suppose that $a_{N-1} \leqslant 2^{AS^{N-1}}$ and $a_N \geqslant 2^{BS^N}$ for infinitely many N. Then* $\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant (2/S)(A/B)$.

Example 2. Let $S > 2$. As an immediate consequence of Corollary 1 we obtain that $\dim X_B\left\{2^{[(4+(-1)^n)S^n]}\right\}_{n=1}^{\infty} \leqslant \frac{6}{5}/S$.

Corollary 2. *Let $A > 0$ and $S > 2$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that a_n divides a_{n+1} for every $n \in \mathbb{N}$. Assume that $a_k \geqslant 2^k$ for every sufficiently large k. Suppose that $a_N \leqslant 2^{AS^N}$ for infinitely many N and $a_M \geqslant 2^{AS^M}$ for infinitely many M. Then $\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant 2/S$.*

Corollary 3. *Let $0 < B < A$ and $S > 2B/A$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that a_n divides a_{n+1} for every n and $2^{BS^n} \leqslant a_n \leqslant 2^{AS^n}$ for every sufficiently large n. Then $\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant (2/S)(A/B)$.*

Example 3. Let $S > 2$. Corollary 3 implies that $\dim X_B\{2^{[S^n]}\}_{n=1}^{\infty} \leqslant 2/S$.

3. Proofs

We need the following classical Jarník-Besicovitch Theorem which can be found for example in [1]. See also [4].

Theorem 6. *Let $\alpha > 2$. Then the Hausdorff dimension of the set of all positive real numbers x such that for infinitely many pairs $(p, q) \in \mathbb{N}$*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{\alpha}}$$

is equal to $2/\alpha$. In other words,

$$\dim \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=1}^{\infty} \Big(\frac{p}{q} - \frac{1}{q^{\alpha}}, \frac{p}{q} + \frac{1}{q^{\alpha}}\Big) = \frac{2}{\alpha}.$$

P r o o f of Theorem 3. Let $\delta > 0$ be a sufficiently small real number. Let $K \in \mathbb{N}$ and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of integers such that $c_n \in \{1, \ldots, K\}$ for each $n \in \mathbb{N}$. Let N be a sufficiently large integer. Then from (2) and from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is non-decreasing we obtain that

$$(7)\qquad \sum_{n=N}^{\infty} \frac{1}{a_n c_n} \leqslant \sum_{n=N}^{\infty} \frac{1}{a_n} \leqslant \sum_{n \leqslant a_N^{1/(1+\varepsilon)}} \frac{1}{a_n} + \sum_{n \geqslant a_N^{1/(1+\varepsilon)}} \frac{1}{a_n} \leqslant \frac{a_N^{1/(1+\varepsilon)}}{a_N} + \sum_{n \geqslant a_N^{1/(1+\varepsilon)}} \frac{1}{n^{1+\varepsilon}} \leqslant \frac{1}{a_N^{(1-\delta)\varepsilon/(1+\varepsilon)}}.$$

Set $S_K := \operatorname{lcm}(1, \ldots, K)$,

$$q_N := \operatorname{lcm}(a_1 c_1, \ldots, a_{N-1} c_{N-1}) \quad \text{and} \quad p_N := q_N \sum_{n=1}^{N-1} \frac{1}{a_n c_n}.$$

We have

$$q_N \leqslant S_K T_N \leqslant T_N^{1+\delta}.$$

This, (1) and (7) imply that for infinitely many N

$$q_N^{2(1-\delta)/(1+\delta)(L+\delta)} \sum_{n=N}^{\infty} \frac{1}{a_n c_n} \leqslant \frac{T_N^{2(1-\delta)/(L+\delta)}}{a_N^{(1-\delta)\varepsilon/(1+\varepsilon)}} = \left(\frac{T_N^{2(1+\varepsilon)/(L+\delta)\varepsilon}}{a_N}\right)^{(1-\delta)\varepsilon/(1+\varepsilon)} < 1.$$

Hence

$$0 < \sum_{n=1}^{\infty} \frac{1}{a_n c_n} - \frac{p_N}{q_N} = \sum_{n=N}^{\infty} \frac{1}{a_n c_n} < \frac{1}{q_N^{2(1-\delta)/(1+\delta)(L+\delta)}}$$

for infinitely many N.

Set

$$Y_K := \left\{ x \in \mathbb{R} \colon \exists \{c_n\}_{n=1}^{\infty} \subseteq \{1, \ldots, K\} \text{ s.t. } x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n} \right\}.$$

Then

$$X_B\{a_n\}_{n=1}^{\infty} = \bigcup_{K=1}^{\infty} Y_K \subseteq \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=1}^{\infty} \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{q_N^{2(1-\delta)/(1+\delta)(L+\delta)}} \right).$$

The Jarník-Besicovitch Theorem implies that

$$\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant \frac{(1+\delta)(L+\delta)}{1-\delta}.$$

This holds for every small $\delta > 0$. So the result follows. □

Proof of Theorem 4. Use Theorem 3 with $L = 0$ and the fact that $T_n \leqslant a_{n-1}^{n-1}$. □

Proof of Theorem 5. For sequences of the Cantor type we have $T_n = a_{n-1}$. Again use Theorem 3. □

Proof of Theorem 1. Set $n = 2^m$ where $m \in \mathbb{N}$ and m is sufficiently large. Then $a_{n-1} = O(2^{2^{n/2}})$ and $a_n = 2^{2^n}$. So (3) follows. Condition (4) is clear. Now we can apply Theorem 4. □

Proof of Corollary 1. For every sufficiently large $\varepsilon > 0$ we have $(2/S)(A/B)$ $((1+\varepsilon)/\varepsilon) < 1$. The inequality (5) is obviously fulfilled. Theorem 5 implies that

$$\dim X_B\{a_n\}_{n=1}^{\infty} \leqslant \lim_{\varepsilon \to \infty} \frac{2}{S} \frac{A}{B} \frac{1+\varepsilon}{\varepsilon} = \frac{2}{S} \frac{A}{B}.$$

□

P r o o f of Corollary 2. Set $B := A$ and use Corollary 1. □

P r o o f of Corollary 3. This is an immediate consequence of Corollary 1. □

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