

Wiesław Kubiś; Matatyahu Rubin

Extension and reconstruction theorems for the Urysohn universal metric space

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 1–29

Persistent URL: <http://dml.cz/dmlcz/140544>

Terms of use:

© Institute of Mathematics AS CR, 2010

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

EXTENSION AND RECONSTRUCTION THEOREMS FOR THE
URYSOHN UNIVERSAL METRIC SPACE

WIESŁAW KUBIŚ, Praha, MATATYAHU RUBIN, Be'er Sheva

(Received May 21, 2008)

Abstract. We prove some extension theorems involving uniformly continuous maps of the universal Urysohn space. As an application, we prove reconstruction theorems for certain groups of autohomeomorphisms of this space and of its open subsets.

Keywords: Urysohn space, bilipschitz homeomorphism, modulus of continuity, reconstruction theorem, extension theorem

MSC 2010: 22F50, 54E40, 51F99, 20E36, 54H11

1. INTRODUCTION

This work deals with the *Urysohn space* [6], which we denote by \mathbb{U} . This is the unique, up to isometry, complete separable metric space with the following properties.

- (1) Every separable metric space is isometrically embeddable in \mathbb{U} .
- (2) For every $A, B \subseteq \mathbb{U}$ and $f: A \rightarrow B$: if A is finite and f is an isometry between A and B , then there is an isometric bijection g from \mathbb{U} to \mathbb{U} such that $f \subseteq g$.

We investigate the group $\text{LIP}(\mathbb{U})$ of bilipschitz homeomorphisms of \mathbb{U} and some related groups. Indeed, the group $\text{H}(\mathbb{U})$ of all homeomorphisms of \mathbb{U} comes to mind first. However, by the result of V. Uspenskiy [5], \mathbb{U} is homeomorphic to ℓ_2 . So $\text{H}(\mathbb{U})$ is in fact the group of homeomorphisms of a Banach space, and can be better understood as such. For $\text{LIP}(\mathbb{U})$ and for other groups defined via the metric of \mathbb{U} , the fact that $\mathbb{U} \cong \ell_2$ does not seem to help.

This paper was supported by the Grant IAA 100 190 901 and by the Institutional Research Plan of the Academy of Sciences of Czech Republic No. AVOZ 101 905 03.

The main tool and also the main result in this work is an extension theorem for finite bilipschitz functions defined on subsets of \mathbb{U} (Theorem 2.1). Suppose that A is a finite subset of \mathbb{U} and $f: A \rightarrow \mathbb{U}$ is K -bilipschitz. We shall show that there is $g \in \text{LIP}(\mathbb{U})$ such that $g \supseteq f$, g is K -bilipschitz and $g \upharpoonright (\mathbb{U} \setminus B) = \text{id}$ for some open ball B , where id denotes the identity map.

We now turn to the description of the other groups considered in this work. Our interest in these groups is two-fold: extension theorems and reconstruction theorems. By a reconstruction theorem we mean a statement of the following form: If $\varphi: \text{LIP}(X) \cong \text{LIP}(Y)$, that is, if φ is an isomorphism between the groups $\text{LIP}(X)$ and $\text{LIP}(Y)$, then there is a bilipschitz homeomorphism τ between X and Y such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \text{LIP}(X)$.

Recall that a *modulus of continuity* is a concave homeomorphism of $[0, \infty)$, i.e. a homeomorphism α of $[0, \infty)$ satisfying $\alpha(\lambda x + (1 - \lambda)y) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y)$ for every $x, y \in [0, \infty)$ and $\lambda \in (0, 1)$. Denote by MC the set of all moduli of continuity. Given $\alpha \in \text{MC}$, we say that a function f from a metric space (X, d^X) to a metric space (Y, d^Y) is α -continuous if $d^Y(f(u), f(v)) \leq \alpha(d^X(u, v))$ for every $u, v \in X$.

We do not know how to generalize the extension theorem for finite bilipschitz functions (Theorem 2.1) to a general $\alpha \in \text{MC}$. Whereas for bilipschitz functions we prove that every finite K -bilipschitz function can be extended to a K -bilipschitz homeomorphism of \mathbb{U} which is the identity outside a ball, for a general $\alpha \in \text{MC}$ we are only able to prove that every finite α -bicontinuous function is extendible to an α -bicontinuous homeomorphism of \mathbb{U} (Corollary 3.3).

The fact that in bilipschitz case the extending homeomorphism can be constructed in such a way that it is the identity outside a ball, means that we also get an ‘‘Extension theorem for finite bilipschitz functions’’ for open subsets of \mathbb{U} and not just for \mathbb{U} .

However, both Theorem 2.1 and Corollary 3.3 can be strengthened by proving the extension theorem not just for functions with a finite domain, but also for functions whose domain is a totally bounded set. For isometries, this fact is due to Huhunaišvili [2]. The argument we use is similar to Huhunaišvili’s. These results appear in Theorems 3.5 and 3.4.

In order to state the next result, let us give necessary definitions.

Fix $\alpha, \beta \in \text{MC}$. We shall write $\alpha \preceq \beta$ if there is $a > 0$ such that $\alpha \upharpoonright [0, a] \leq \beta \upharpoonright [0, a]$, where $f \leq g$ means that $f(x) \leq g(x)$ for every x in the common domain of f, g . Note that when $K > 0$ then the function $x \mapsto Kx$ belongs to MC. Also, if $\alpha, \beta \in \text{MC}$ then $\alpha + \beta \in \text{MC}$ and $\alpha \circ \beta \in \text{MC}$.

A subset Γ of MC will be called a *measure of continuity semigroup* (briefly: MC-semigroup) if the following holds:

- (1) Γ contains the function $y = 2x$.

(2) Γ is closed under compositions, i.e. $\alpha \circ \beta \in \Gamma$ whenever $\alpha, \beta \in \Gamma$.

(3) For every $\alpha \in \Gamma$, $\{\beta \in \text{MC}: \beta \preceq \alpha\} \subseteq \Gamma$.

Further, we say that Γ is *countably generated* if there exists a countable set $\Gamma_0 \subseteq \Gamma$ such that for every $\alpha \in \Gamma$ there is $\beta \in \Gamma_0$ with $\alpha \preceq \beta$. In other words, $\Gamma = \{\alpha \in \text{MC}: (\exists \beta \in \Gamma_0) \alpha \preceq \beta\}$.

Important examples of countably generated MC-semigroups are the *Lipschitz MC-semigroup* Γ^{LIP} and the *Hölder semigroup* Γ^{HLD} . The former is generated by functions of type $x \mapsto nx$ ($n \in \mathbb{N}$) while the latter is generated by functions of the form $x \mapsto x^{1/n}$ ($n \in \mathbb{N}$). Note that every MC-semigroup contains Γ^{LIP} .

Let Γ be an MC-semigroup and f a function from a metric space X to a metric space Y . Then f is *locally Γ -continuous* if for every $x \in X$ there is a neighborhood U of x and $\alpha \in \Gamma$ such that $f \upharpoonright U$ is α -continuous. The function f is *locally Γ -bicontinuous*, if f is a homeomorphism between X and $\text{rng}(f)$, and both f and f^{-1} are locally Γ -continuous.

From the extension theorem for finite bilipschitz functions we shall deduce the following result (Corollary 6.4 (a)).

Theorem 1.1. *Let X and Y be open subsets of \mathbb{U} , let Γ and Δ be countably generated MC-semigroups and $\varphi: H_{\Gamma}^{\text{LC}}(X) \cong H_{\Delta}^{\text{LC}}(Y)$. Then there is $\tau: X \cong Y$ such that τ is locally Γ -bicontinuous, τ is locally Δ -bicontinuous and $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in H_{\Gamma}^{\text{LC}}(X)$.*

Note that the above theorem does not claim that $\Gamma = \Delta$. However, the theorem does imply the weaker statement that $H_{\Gamma}^{\text{LC}}(X) = H_{\Delta}^{\text{LC}}(X)$. We can conclude that, indeed, $\Gamma = \Delta$ only when $X = Y = \mathbb{U}$. It is Theorem 3.4 which is used in order to deduce this.

In fact, we do not know how to prove the following general statement.

If X is an open subset of \mathbb{U} and Γ and Δ are different MC-semigroups,
then $H_{\Gamma}^{\text{LC}}(X) \neq H_{\Delta}^{\text{LC}}(X)$.

Even the following special case is open. For $\alpha \in \text{MC}$ let Γ_{α} denote the MC-semigroup generated by α . Suppose that Δ is a countably generated MC-semigroup and $\alpha \notin \Delta$. Is it true that $H_{\Gamma_{\alpha}}^{\text{LC}}(X) \not\subseteq H_{\Delta}^{\text{LC}}(X)$? Note that if \mathbb{U} is replaced by \mathbb{R} , that is, we consider an open subset of \mathbb{R} , then different MC-semigroups do define different subgroups of $H(X)$. The same is true for any open subset of a normed linear space.

2. EXTENSIONS OF FINITE BILIPSCHITZ FUNCTIONS

The main result of this section is Theorem 2.1. It is the basis for our reconstruction theorem for open subsets of \mathbb{U} .

Let $B(x, r) \subseteq \mathbb{U}$ be an open ball in \mathbb{U} . We wish to prove the following claim. Let $A \subseteq B(x, r)$ be a finite set and $K > 1$. Assume that $f: A \rightarrow B(x, r)$ is such that $f \cup \text{id}_{\mathbb{U} \setminus B(x, r)}$ is K -bilipschitz, where id_S denotes the identity map on the set S . Then there is $g \in \text{H}(\mathbb{U})$ such that $g \supseteq f \cup \text{id}_{\mathbb{U} \setminus B(x, r)}$ and g is K -bilipschitz. It turns out that we need some extra assumptions in order to prove such a claim. These assumptions are somewhat technical, although good enough for applications.

Fix $r > 0$ and let f be a function whose domain and range are subsets of $B(x, r)$. Also, let $N > 1$. We shall say that f is N -good, if

$$d(y, f(y)) \leq \frac{1}{N} \cdot (r - d(y, x)) \quad \text{for every } y \in \text{dom}(f).$$

The function f is N -bigood, if both f and f^{-1} are N -good.

Theorem 2.1 (Bilipschitz Extension Theorem for the Urysohn space). *Let $B(x, r)$ be an open ball in \mathbb{U} , $N \geq 4$, $\frac{1}{2}(N - \sqrt{N^2 - 4N}) \leq K \leq \frac{1}{2}(N + \sqrt{N^2 - 4N})$, let $A \subseteq B(x, r)$ be finite and $x \in A$. Suppose that $f: A \rightarrow B(x, r)$, f is K -bilipschitz, f is N -bigood and $f(x) = x$.*

Then there exists $g: B(x, r) \rightarrow B(x, r)$ such that

- (1) g is a bijection and $g \supseteq f$,
- (2) $g \cup \text{id}_{\mathbb{U} \setminus B(x, r)}$ is K -bilipschitz,
- (3) g is N -bigood.

Let us first see that the assumptions in the above statement are meaningful. Note that

$$\lim_{N \rightarrow \infty} \frac{N - \sqrt{N^2 - 4N}}{2} = 1 \quad \text{and} \quad \lim_{N \rightarrow \infty} \frac{N + \sqrt{N^2 - 4N}}{2} = \infty.$$

It follows that for every $K > 1$ there is $N \geq 4$ such that K, N fulfil the assumptions in Theorem 2.1. Thus, Theorem 2.1 can be applied to any $K > 1$ provided the distance between any point $u \in A$ and its image $f(u)$ is sufficiently small in comparison with the distance of u and $f(u)$ from the boundary of $B(x, r)$.

The following simple facts related to the assumptions of the above theorem will be used later.

Proposition 2.2. (a) Assume $K > 1$ and $N \geq K^2/(K-1)$. Then $N \geq 4$ and

$$(*) \quad \frac{N - \sqrt{N^2 - 4N}}{2} \leq K \leq \frac{N + \sqrt{N^2 - 4N}}{2}.$$

(b) If $N \geq 4$ and $(*)$ holds then $K > 1$ and $N \geq K^2/(K-1)$.

(c) $1/N + 1/K \leq 1$ iff $1 + 1/(N-1) \leq K$.

(d) The inequalities in (c) hold whenever K, N satisfy the requirements of Theorem 2.1.

A function f whose domain and range are subsets of $B(x, r)$ will be called briefly (K, N) -compliant if it is both K -bilipschitz and N -bigood. Note that f is (K, N) -compliant iff f^{-1} is (K, N) -compliant.

Proposition 2.3. Let X be a metric space, $x \in X$, $r > 0$. Suppose that h is a function whose domain and range are subsets of $B(x, r)$. Let $K, N > 0$ and assume that $1 + 1/N \leq K$. If h is (K, N) -compliant then $h \cup \text{id}_{X \setminus B(x, r)}$ is K -bilipschitz.

Proof. Fix $u \in \text{dom}(h)$ and $w \in X \setminus B(x, r)$. Then $r - d(x, u) \leq d(w, x) - d(x, u) \leq d(w, u)$. Hence $r - d(x, u) \leq d(u, w)$. Thus

$$\begin{aligned} d(h(u), h(w)) &= d(h(u), w) \leq d(h(u), u) + d(u, w) \\ &\leq \frac{1}{N} \cdot (r - d(x, u)) + d(u, w) \\ &\leq \frac{1}{N} \cdot d(u, w) + d(u, w) = \left(1 + \frac{1}{N}\right) \cdot d(u, w) \leq K d(u, w). \end{aligned}$$

□

Before we prove Theorem 2.1, we recall an important property of (isometric embeddings of) metric spaces, called *amalgamation*. Namely, given finite metric spaces $\langle X_0, d_0 \rangle$ and $\langle X_1, d_1 \rangle$ such that their metrics coincide on the intersection $Z = X_0 \cap X_1$, there exist a metric space (Y, d) and isometric embeddings $f_i: \langle X_i, d_i \rangle \rightarrow \langle Y, d \rangle$, $i = 0, 1$, such that $f_0 \upharpoonright Z = f_1 \upharpoonright Z$. In other words, there is a metric on $X := X_0 \cup X_1$ which extends both d_0 and d_1 . The space $\langle X, d \rangle$ will be called the *amalgamation* of $\langle X_0, d_0 \rangle$ and $\langle X_1, d_1 \rangle$. The amalgamation property is essentially used for constructing the Urysohn space. We shall need the following more specific statement, which actually gives inductive argument for amalgamating two finite metric spaces.

Proposition 2.4. Let $\langle X_i, d_i \rangle$ for $i = 0, 1$ be finite metric spaces such that $Z = X_0 \cap X_1$ is nonempty and $d_0 \upharpoonright Z^2 = d_1 \upharpoonright Z^2$. Assume further that $X_i = Z \cup \{p_i\}$ for $i < 2$. Then

$$l := \max_{z \in Z} |d_0(p_0, z) - d_1(z, p_1)| \leq \min_{z \in Z} (d_0(p_0, z) + d_1(z, p_1)) =: r$$

and every extension d of $d_0 \cup d_1$ satisfying $l \leq d(p_0, p_1) = d(p_1, p_0) \leq r$ and $0 < d(p_0, p_1)$ is a metric on $X_0 \cup X_1$.

It may happen that $l = 0$ in the above statement and then setting $d(p_0, p_1) = d(p_1, p_0) = l$ we also obtain amalgamation of X_1, X_2 in which the points p_0, p_1 are identified. Amalgamation satisfying $d(p_0, p_1) = l$ will be called *minimal*.

The properties of the Urysohn space imply the following: given a finite metric space $X \subseteq \mathbb{U}$ and its finite metric extension $Y \supseteq X$, there exists an isometric embedding $h: Y \rightarrow \mathbb{U}$ such that $h \upharpoonright X = \text{id}_X$.

The crucial argument in constructing the homeomorphism g promised in Theorem 2.1 is showing how to add a single point to the domain or range of a (K, N) -compliant function so that the resulting new function remains compliant. Note that the problem of adding a point to the domain or to the range of f is the same, since f is compliant iff f^{-1} is compliant.

Lemma 2.5. Assume that $N \geq 4$, $r > 0$ and $\frac{1}{2}(N - \sqrt{N^2 - 4N}) \leq K \leq \frac{1}{2}(N + \sqrt{N^2 - 4N})$. Suppose that f is such that $\text{dom}(f), \text{rng}(f)$ are finite subsets of $B(x_1, r)$, $f(x_1) = x_1$ and f is (K, N) -compliant. Let $x \in B(x_1, r)$. Then there is $y \in B(x_1, r)$ such that $f \cup \{\langle x, y \rangle\}$ is (K, N) -compliant.

Proof. Let $\text{dom}(f) = \{x_1, \dots, x_n\}$, where x_1, \dots, x_n are pairwise distinct. Let $y_m = f(x_m)$. Denote $d_{i,j} = d(x_i, x_j)$, $e_{i,j} = d(y_i, y_j)$ and $s_{i,j} = d(x_i, y_j)$. The assumptions concerning N -bigoodness say that for every $m = 1, \dots, n$,

$$(B) \quad s_{m,m} \leq \frac{1}{N} \cdot (r - d_{m,1}) \quad \text{and} \quad s_{m,m} \leq \frac{1}{N} \cdot (r - e_{m,1}).$$

Assume $x \in B(x_1, r) \setminus \{x_1, \dots, x_n\}$ and define $d_m = d(x, x_m)$, $s_m = d(x, y_m)$. We take an imaginary point y which we shall later identify, using amalgamation, with a suitable element of $B(x_1, r)$. The following step is crucial.

Claim 2.6. There exist $e_1, \dots, e_n > 0$ such that, defining $d(y, y_i) = e_i$ for $i = 1, \dots, n$, the set $\{y, y_1, \dots, y_n\}$ becomes a metric space, the function $f \cup \{\langle x, y \rangle\}$ is K -bilipschitz and for every $m = 1, \dots, n$ the following inequality holds:

$$(G) \quad |e_m - s_m| \leq \min\left(\frac{1}{N}(r - d_1), \frac{1}{N}(r - e_1)\right).$$

Condition (G) is necessary for N -bigoodness of the extension: given $m \leq n$, the distance between x and y must be at least $|e_m - s_m|$ and, on the other hand, it must not exceed the right-hand side of (G).

Suppose we have proved the above claim.

We amalgamate the two metric spaces $X' = Y \cup \{x\}$ and $Y' = Y \cup \{y\}$, where $Y = \{y_1, \dots, y_n\}$ and the metric d on Y' is given by Claim 2.6, that is, d extends the metric of Y and $d(y_m, y) = e_m$ for $m \leq n$. Let

$$s = \min\left(\min_{1 \leq i \leq n} (e_i + s_i), \frac{1}{N}(r - d_1), \frac{1}{N}(r - e_1)\right).$$

By (G), we have $\max_{1 \leq i \leq n} |e_i - s_i| \leq s$. Clearly, $s \leq \min_{1 \leq i \leq n} (e_i + s_i)$, therefore by Proposition 2.4, there exists a metric on $X' \cup Y'$ (which we still denote by d) that extends the metrics of X' and Y' and satisfies $d(x, y) = s$. Now, amalgamate \mathbb{U} with the finite space $X' \cup Y'$. In this way, we may assume that $y \in \mathbb{U}$. In fact $y \in B(x_1, r)$, because $d(x_1, y) \leq d(x_1, x) + d(x, y) = d_1 + s \leq d_1 + N^{-1}(r - d_1) < r$. Finally, $f \cup \{\langle x, y \rangle\}$ is (K, N) -compliant.

It remains to prove Claim 2.6.

The distances e_1, \dots, e_n will be defined by induction on $m = 1, \dots, n$. Fix $m \leq n$ and suppose that e_1, \dots, e_{m-1} have been defined. We assume by induction that for every $l < i < m$ and $j \geq m$ the following inequalities hold:

$$(IH1) \quad e_{l,j} - Kd_j \leq e_l \leq e_{l,j} + Kd_j,$$

$$(IH2) \quad |e_{i,l} - e_l| \leq e_i \leq e_{i,l} + e_l,$$

$$(IH3) \quad \frac{1}{K} \cdot d_l \leq e_l \leq Kd_l,$$

$$(IH4) \quad |e_l - s_l| \leq \frac{1}{N} \cdot (r - d_1),$$

$$(IH5) \quad |e_l - s_l| \leq \frac{1}{N} \cdot (r - e_1).$$

Condition (IH1) consists of two of the three triangle inequalities in the triangle whose vertices are y_l, y_j and the future point y . One may think of Kd_j as a “temporary“ distance between y and y_j . The inequality $Kd_j \leq e_{l,j} + e_l$ is not assumed, since Kd_j will be replaced by a smaller final distance. (IH2) consists of the three inequalities in the triangle whose vertices are y_l, y_i and the future point y . (IH3) is the bilipschitz condition for the pairs x_l, x and y_l, y . Finally, (IH4) and (IH5) are just condition (G) for $l < m$. It is clear that e_1, \dots, e_n constructed by this inductive procedure fulfil the requirements of Claim 2.6.

Consider the following system of inequalities in the unknown e_m :

$$\begin{aligned}
(\text{IE1}_m) \quad & e_{m,j} - Kd_j \leq e_m \leq e_{m,j} + Kd_j \quad (j > m), \\
(\text{IE2}_m) \quad & |e_{m,l} - e_l| \leq e_m \leq e_{m,l} + e_l \quad (l < m), \\
(\text{IE3}_m) \quad & \frac{1}{K} \cdot d_m \leq e_m \leq Kd_m, \\
(\text{IE4}_m) \quad & s_m - \frac{1}{N} \cdot (r - d_1) \leq e_m \leq s_m + \frac{1}{N} \cdot (r - d_1), \\
(\text{IE5}_m) \quad & s_m - \frac{1}{N} \cdot (r - e_1) \leq e_m \leq s_m + \frac{1}{N} \cdot (r - e_1), \\
(\text{IE6}_1) \quad & e_1 \leq N \cdot \left(s_i - \frac{d_i}{K} \right) + r \quad (i > 1), \\
(\text{IE7}_1) \quad & e_1 \leq N \cdot (Kd_i - s_i) + r \quad (i > 1).
\end{aligned}$$

Observe that a solution e_m to the above system satisfies inequalities (IH1)–(IH5) with $l = m$, i.e. the inductive step can be accomplished. It remains to show that the above system is solvable.

Given an inequality with label (e), we shall denote by $\text{lhs}(\text{e})$ and $\text{rhs}(\text{e})$ the expression on the left-hand side and on the right-hand side, respectively.

Inequalities (IE6₁), (IE7₁) are required for solving some of the inequalities with $m > 1$. Namely, inequality (IE6₁) is needed in the proof of $\text{lhs}(\text{IE3}_m) \leq \text{rhs}(\text{IE5}_m)$. Inequality (IE7₁) is needed for the proof of $\text{lhs}(\text{IE1}_m) \leq \text{rhs}(\text{IE5}_m)$, $\text{lhs}(\text{IE5}_m) \leq \text{rhs}(\text{IE1}_m)$ and $\text{lhs}(\text{IE5}_m) \leq \text{rhs}(\text{IE3}_m)$.

It has to be shown that each expression appearing on the left is less than or equal to every expression appearing on the right.

It is worthwhile to note that the original assumptions in the lemma together with hypotheses (IH4) and (IH5) will be used only in proving the inequalities involving (IE4_m) and (IE5_m). The verification that the inequalities arising from (IE1_m)–(IE4_m) hold does not depend on m . For the inequalities involving (IE5_m) we need to distinguish between the cases $m > 1$ and $m = 1$.

Subclaim 2.7. $\text{lhs}(\text{IE1}_m) \leq \text{rhs}(\text{IE2}_m)$, i.e. $e_{m,j} - Kd_j \leq e_{m,l} + e_l$.

Proof. By the triangle inequality, $e_{m,j} \leq e_{m,l} + e_{l,j}$. Thus $e_{m,j} - Kd_j \leq e_{m,l} + e_{l,j} - Kd_j$. Finally, the first inequality in (IH1) says that $e_{l,j} - Kd_j \leq e_l$. \square

Subclaim 2.8. $\text{lhs}(\text{IE2}_m) \leq \text{rhs}(\text{IE1}_m)$, i.e. $|e_{m,l} - e_l| \leq e_{m,j} + Kd_j$.

Proof. Using the fact that $e_{m,l} \leq e_{m,j} + e_{l,j}$, we get $e_{m,l} - e_l \leq e_{m,j} + e_{l,j} - e_l$. The first inequality in (IH1) gives $e_{l,j} \leq e_l + Kd_j$, therefore $e_{m,l} - e_l \leq e_{m,j} + Kd_j$.

On the other hand, the second inequality in (IH1) says that $e_l \leq e_{l,j} + Kd_j$, therefore $e_l - e_{m,l} \leq e_{l,j} + Kd_j - e_{m,l} \leq e_{m,j} + Kd_j$. \square

Subclaim 2.9. $\text{lhs}(\text{IE1}_m) \leq \text{rhs}(\text{IE3}_m)$, i.e. $e_{m,j} - Kd_j \leq Kd_m$.

Proof. This follows from $d_{m,j} \leq d_j + d_m$ and from the fact that f is K -Lipschitz. \square

Subclaim 2.10. $\text{lhs}(\text{IE3}_m) \leq \text{rhs}(\text{IE1}_m)$, i.e. $K^{-1} \cdot d_m \leq e_{m,j} + Kd_j$.

Proof. Since f^{-1} is K -Lipschitz, $K^{-1} \cdot d_{m,j} \leq e_{m,j}$. Thus $K^{-1} \cdot d_m \leq K^{-1}(d_{m,j} + d_j) \leq e_{m,j} + K^{-1} \cdot d_j \leq e_{m,j} + Kd_j$. The last inequality holds because $K > 1$. \square

Subclaim 2.11. $\text{lhs}(\text{IE2}_m) \leq \text{rhs}(\text{IE3}_m)$, i.e. $|e_{m,l} - e_l| \leq Kd_m$.

Proof. This is a particular case of (IH1), where $j := m$. \square

Subclaim 2.12. $\text{lhs}(\text{IE3}_m) \leq \text{rhs}(\text{IE2}_m)$, i.e. $K^{-1} \cdot d_m \leq e_{m,l} + e_l$.

Proof. Notice that $K^{-1} \cdot d_{m,l} \leq e_{m,l}$, because f is K -Lipschitz and $K^{-1} \cdot d_l \leq e_l$, by the first inequality in (IH3). Thus $K^{-1} \cdot d_m \leq K^{-1} \cdot d_{m,l} + K^{-1} \cdot d_l \leq e_{m,l} + e_l$. \square

Note that until now we have used neither assumptions (B) nor hypotheses (IH4) and (IH5).

Subclaim 2.13. $\text{lhs}(\text{IE1}_m) \leq \text{rhs}(\text{IE4}_m)$, i.e. $e_{m,j} - Kd_j \leq s_m + N^{-1} \cdot (r - d_1)$.

Proof. Triangle inequalities give $e_{m,j} \leq s_m + s_j$ and $s_j \leq d_j + s_{j,j}$, so $e_{m,j} - Kd_j \leq s_m + d_j + s_{j,j} - Kd_j$. Thus, it suffices to show that $d_j + s_{j,j} \leq Kd_j + N^{-1} \cdot (r - d_1)$. By the first inequality in (B), $s_{j,j} \leq N^{-1} \cdot (r - d_{j,1})$. So it suffices to show that

$$d_j + \frac{1}{N} \cdot (r - d_{j,1}) \leq Kd_j + \frac{1}{N} \cdot (r - d_1),$$

which reduces to $N^{-1} \cdot (d_1 - d_{j,1}) \leq (K - 1)d_j$. The last inequality holds, because $d_1 - d_{j,1} \leq d_j$, by triangle inequality, and $N^{-1} \leq K - 1$, by the assumptions on N , K (see Proposition 2.2). \square

Subclaim 2.14. $\text{lhs}(\text{IE4}_m) \leq \text{rhs}(\text{IE1}_m)$, i.e. $s_m - N^{-1} \cdot (r - d_1) \leq e_{m,j} + Kd_j$.

Proof. Note that $s_m \leq e_{m,j} + s_j$, so it suffices to show that $s_j \leq Kd_j + N^{-1} \cdot (r - d_1)$. This is the same as in the proof of Subclaim 2.13. \square

Subclaim 2.15. $\text{lhs}(\text{IE2}_m) \leq \text{rhs}(\text{IE4}_m)$, i.e. $|e_{m,l} - e_l| \leq s_m + N^{-1} \cdot (r - d_1)$.

Proof. We have $e_{m,l} \leq s_m + s_l$, so $e_{m,l} - e_l \leq s_m + |e_l - s_l|$. Similarly, $e_{m,l} + s_m \geq s_l$ implies that $e_l - e_{m,l} \leq s_m - s_l + e_l \leq s_m + |e_l - s_l|$. Finally, $|e_l - s_l| \leq N^{-1} \cdot (r - d_1)$, by (IH4). \square

Subclaim 2.16. $\text{lhs}(\text{IE}4_m) \leq \text{rhs}(\text{IE}2_m)$, i.e. $s_m - N^{-1} \cdot (r - d_1) \leq e_{m,l} + e_l$.

Proof. We have $s_m \leq e_{m,l} + s_l$, so the above inequality follows from $s_l - N^{-1} \cdot (r - d_1) \leq e_l$, which is part of (IH4). \square

Subclaim 2.17. $\text{lhs}(\text{IE}3_m) \leq \text{rhs}(\text{IE}4_m)$, i.e. $K^{-1} \cdot d_m \leq s_m + N^{-1} \cdot (r - d_1)$.

Proof. Recall that $K^{-1} \leq 1 - N^{-1}$ (see Proposition 2.2). Also, $d_m \leq s_m + s_{m,m}$ and $s_{m,m} \leq N^{-1} \cdot (r - d_{m,1})$, by (B). Thus

$$K^{-1} \cdot d_m \leq d_m - \frac{1}{N} \cdot d_m \leq s_m + s_{m,m} - \frac{1}{N} \cdot d_m \leq s_m + \frac{1}{N} \cdot (r - d_{m,1} - d_m).$$

Finally, $r - d_{m,1} - d_m \leq r - d_1$ holds by the triangle inequality. \square

Subclaim 2.18. $\text{lhs}(\text{IE}4_m) \leq \text{rhs}(\text{IE}3_m)$, i.e. $s_m - N^{-1} \cdot (r - d_1) \leq K d_m$.

Proof. Using (B) and triangle inequalities, we have

$$\begin{aligned} s_m &\leq d_m + s_{m,m} \leq d_m + \frac{1}{N} \cdot (r - d_{m,1}) \\ &= d_m + \frac{1}{N} \cdot (r - d_1) + \frac{1}{N} \cdot (d_1 - d_{m,1}) \\ &\leq d_m + \frac{1}{N} \cdot (r - d_1) + \frac{1}{N} \cdot d_m \\ &\leq \left(1 + \frac{1}{N}\right) \cdot d_m + \frac{1}{N} \cdot (r - d_1). \end{aligned}$$

Finally, $(1 + 1/N) \cdot d_m \leq K d_m$, by Proposition 2.2. \square

We now deal with the inequalities involving (IE5_m). Here we have to distinguish between the cases $m = 1$ and $m > 1$. We start with the case $m > 1$.

Subclaim 2.19. $m > 1 \implies \text{lhs}(\text{IE}1_m) \leq \text{rhs}(\text{IE}5_m)$, i.e. $e_{m,j} - K d_j \leq s_m + N^{-1} \cdot (r - e_1)$.

Proof. Since $e_{m,j} - s_m \leq s_j$, the above inequality follows from $s_j \leq N^{-1} \times (r - e_1) + K d_j$. That is, $e_1 \leq N \cdot (K d_j - s_j) + r$. This is just (IE7₁). \square

Subclaim 2.20. $m > 1 \implies \text{lhs}(\text{IE}5_m) \leq \text{rhs}(\text{IE}1_m)$, i.e. $s_m - N^{-1} \cdot (r - e_1) \leq e_{m,j} + K d_j$.

Proof. Like in the previous subclaim, using the inequality $s_m - e_{m,j} \leq s_j$ instead. \square

Subclaim 2.21. $m > 1 \implies \text{lhs}(\text{IE}2_m) \leq \text{rhs}(\text{IE}5_m)$, i.e. $|e_{m,l} - e_l| \leq s_m + N^{-1} \times (r - e_1)$.

Proof. Using the inequalities $e_{m,l} - s_m \leq s_l \leq e_{m,l} + s_m$, we obtain that $e_{m,l} - e_l - s_m \leq s_l - e_l$ and $e_l - e_{m,l} - s_m \leq e_l - s_l$. Thus $|e_{m,l} - e_l| - s_m \leq |s_l - e_l|$. Finally, $|s_l - e_l| \leq N^{-1} \cdot (r - e_1)$ is hypothesis (IH5). \square

Subclaim 2.22. $m > 1 \implies \text{lhs}(\text{IE}5_m) \leq \text{rhs}(\text{IE}2_m)$, i.e. $s_m - N^{-1} \cdot (r - e_1) \leq e_{m,l} + e_l$.

Proof. Note that $s_m - e_{m,l} \leq s_l$, therefore the above inequality follows from (IH5). \square

Subclaim 2.23. $m > 1 \implies \text{lhs}(\text{IE}3_m) \leq \text{rhs}(\text{IE}5_m)$, i.e. $K^{-1} \cdot d_m \leq s_m + N^{-1} \times (r - e_1)$.

Proof. This is equivalent to $e_1 \leq N \cdot (s_m - \frac{d_m}{K}) + r$, which is (IE6₁). \square

Subclaim 2.24. $m > 1 \implies \text{lhs}(\text{IE}5_m) \leq \text{rhs}(\text{IE}3_m)$, i.e. $s_m - N^{-1} \cdot (r - e_1) \leq Kd_m$.

Proof. That is, $e_1 \leq N \cdot (Kd_m - s_m) + r$. This is (IE7₁). \square

Subclaim 2.25. $m > 1 \implies \text{lhs}(\text{IE}4_m) \leq \text{rhs}(\text{IE}5_m)$, i.e. $s_m - N^{-1} \cdot (r - d_1) \leq s_m + N^{-1} \cdot (r - e_1)$.

Proof. This follows from $d_1 + e_1 \leq 2r$, since these are the distances between points in the ball $B(x_1, r)$. \square

Subclaim 2.26. $m > 1 \implies \text{lhs}(\text{IE}5_m) \leq \text{rhs}(\text{IE}4_m)$, i.e. $s_m - N^{-1} \cdot (r - e_1) \leq s_m + N^{-1} \cdot (r - d_1)$.

Proof. The same as in the previous subclaim. \square

We are left with the case $m = 1$. Note that (IE2_m) is vacuous in case $m = 1$. Inequalities (IE5_m) with $m = 1$ need to be rewritten in such a way that e_1 occurs only in the middle. Further, notice that $s_1 = d_1$, because $x_1 = y_1$. Thus (IE4_m) can be simplified. Summarizing, we now have to deal with the following system of inequalities, together with (IE6₁), (IE7₁).

$$(IE1_1) \quad e_{1,j} - Kd_j \leq e_1 \leq e_{1,j} + Kd_j \quad (j > 1),$$

$$(IE3_1) \quad \frac{1}{K} \cdot d_1 \leq e_1 \leq Kd_1,$$

$$(IE4_1) \quad \frac{N+1}{N} \cdot d_1 - \frac{1}{N} \cdot r \leq e_1 \leq \frac{N-1}{N} \cdot d_1 + \frac{1}{N} \cdot r,$$

$$(IE5_1) \quad \frac{N}{N-1} \cdot d_1 - \frac{1}{N-1} \cdot r \leq e_1 \leq \frac{N}{N+1} \cdot d_1 + \frac{1}{N+1} \cdot r,$$

$$(IE6_1) \quad e_1 \leq N \cdot \left(s_i - \frac{d_i}{K} \right) + r \quad (i > 1),$$

$$(IE7_1) \quad e_1 \leq N \cdot (Kd_i - s_i) + r \quad (i > 1).$$

In the above system, only (IE5₁) is different from (IE5_{*m*}) with $m > 1$. Inequalities (IE1₁)–(IE4₁) are special cases of (IE1_{*m*})–(IE4_{*m*}). Observe that $\text{lhs}(IE5_1) < \text{lhs}(IE4_1) \leq \text{rhs}(IE4_1) < \text{rhs}(IE5_1)$, because $d_1 < r$. It follows that Subclaims 2.19–2.26 remain true also in case $m = 1$. That is, $\text{lhs}(IE1_1) \leq \text{rhs}(IE5_1)$, $\text{lhs}(IE5_1) \leq \text{rhs}(IE1_1)$, $\text{lhs}(IE3_1) \leq \text{rhs}(IE5_1)$, $\text{lhs}(IE5_1) \leq \text{rhs}(IE3_1)$, $\text{lhs}(IE4_1) \leq \text{rhs}(IE5_1)$ and $\text{lhs}(IE5_1) \leq \text{rhs}(IE4_1)$. It remains to check the inequalities involving $\text{rhs}(IE6_1)$ and $\text{rhs}(IE7_1)$.

Subclaim 2.27. $\text{lhs}(IE1_1) \leq \text{rhs}(IE6_1)$, i.e. $e_{1,j} - Kd_j \leq N \cdot (s_i - d_i/K) + r$ for $i, j > 1$.

Proof. Knowing that $s_i \geq d_i - s_{i,i}$ and $N \cdot s_{i,i} \leq r - e_{i,1}$ (see (B)), observe that

$$(1) \quad N \cdot \left(s_i - \frac{d_i}{K} \right) + r \geq N \left(1 - \frac{1}{K} \right) \cdot d_i + e_{i,1}.$$

Thus, using the fact that $e_{1,j} - e_{i,1} \leq e_{i,j} \leq Kd_{i,j}$, inequality $\text{lhs}(IE1_1) \leq \text{rhs}(IE6_1)$ is implied by

$$(2) \quad Kd_{i,j} - Kd_j \leq N \left(1 - \frac{1}{K} \right) \cdot d_i.$$

Further, $Kd_{i,j} - Kd_j \leq Kd_i$, therefore (2) is implied by

$$(3) \quad K \leq N \left(1 - \frac{1}{K} \right).$$

Finally, (3) is satisfied if and only if $\frac{1}{2}(N - \sqrt{N^2 - 4N}) \leq K \leq \frac{1}{2}(N + \sqrt{N^2 - 4N})$, which is one of our assumptions. \square

Subclaim 2.28. $\text{lhs}(IE3_1) \leq \text{rhs}(IE6_1)$, i.e. $K^{-1} \cdot d_1 \leq N \cdot (s_i - d_i/K) + r$.

Proof. Using the inequality (1) from the proof of Subclaim 2.27 together with $1 - 1/K \geq 1/N$ (Proposition 2.2), we see that our inequality is implied by $K^{-1} \cdot d_1 \leq d_i + e_{i,1}$. This, by the fact that f is K -bilipschitz, follows from $K^{-1}d_1 \leq d_i + K^{-1} \cdot d_{i,1}$. The last inequality is true, because $d_1 - d_{i,1} \leq d_i$ and $K^{-1} < 1$. \square

Subclaim 2.29. $\text{lhs}(\text{IE4}_1) \leq \text{rhs}(\text{IE6}_1)$, i.e. $(1 + N^{-1}) \cdot d_1 - N^{-1} \cdot r \leq N \cdot (s_i - d_i/K) + r$.

Proof. Knowing the inequalities $s_i \geq d_i - s_{i,i}$, $-Ns_{i,i} \geq d_{i,1} - r$ (see (B)) and $N(1 - 1/K) \geq 1$ (see Proposition 2.2), we obtain

$$N \cdot \left(s_i - \frac{d_i}{K} \right) + r \geq N \left(1 - \frac{1}{K} \right) \cdot d_i + d_{i,1} \geq d_i + d_{i,1} \geq d_1.$$

Finally, $d_1 \geq (1 + 1/N) \cdot d_1 - 1/N \cdot r$, because $d_1 < r$. \square

Subclaim 2.30. $\text{lhs}(\text{IE1}_1) \leq \text{rhs}(\text{IE7}_1)$, i.e. $e_{1,j} - Kd_j \leq N \cdot (Kd_i - s_i) + r$, $j, i > 1$.

Proof. Using (B), some triangle inequalities and $e_{i,j} \leq Kd_{i,j}$, we get

$$\begin{aligned} N \cdot (Kd_i - s_i) + r &= (NK - N) \cdot d_i - N \cdot (s_i - d_i) + r \\ &\geq (NK - N) \cdot d_i - N \cdot s_{i,i} + r \\ &\geq (NK - N) \cdot d_i - (r - e_{i,1}) + r \\ &= (NK - N) \cdot d_i + e_{i,1} \\ &\geq (NK - N) \cdot d_i + e_{1,j} - e_{i,j} \\ &\geq (NK - N) \cdot d_i + e_{1,j} - Kd_{i,j} \\ &\geq N(K - 1) \cdot d_i + e_{1,j} - Kd_j - Kd_i. \end{aligned}$$

It remains to check that $N(K - 1) \cdot d_i - Kd_i \geq 0$. This follows from the fact that $N \geq 1 + 1/(K - 1)$ (see Proposition 2.2). \square

Subclaim 2.31. $\text{lhs}(\text{IE3}_1) \leq \text{rhs}(\text{IE7}_1)$, i.e. $K^{-1} \cdot d_1 \leq N \cdot (Kd_i - s_i) + r$.

Proof. Repeating the beginning of the proof of Subclaim 2.30 and using the fact that $d_{i,1} \leq Ke_{i,1}$, we get

$$N \cdot (Kd_i - s_i) + r \geq (NK - N) \cdot d_i + e_{i,1} \geq (NK - N) \cdot d_i + K^{-1} \cdot d_{i,1}.$$

By Proposition 2.2, $NK - N \geq 1 \geq K^{-1}$, therefore finally $\text{rhs}(\text{IE7}_1) \geq K^{-1} \cdot d_i + K^{-1} \cdot d_{i,1} \geq K^{-1} \cdot d_i = \text{lhs}(\text{IE3}_1)$. \square

Subclaim 2.32. $\text{lhs}(\text{IE4}_1) \leq \text{rhs}(\text{IE7}_1)$, i.e. $(1 + \frac{1}{N}) \cdot d_1 - N^{-1} \cdot r \leq N \cdot (Kd_i - s_i) + r$.

Proof. Using (B) and some triangle inequalities, we have

$$\begin{aligned} \text{rhs}(\text{IE7}_1) - \text{lhs}(\text{IE4}_1) &= NKd_i + N(d_i - s_i) - N \cdot d_i + \left(1 + \frac{1}{N}\right)(r - d_1) \\ &\geq N(K - 1)d_i - Ns_{i,i} + (r - d_1) \\ &\geq N(K - 1)d_i - (r - d_{i,1}) + r - d_1 \\ &\geq N(K - 1)d_i - d_i = (NK - N - 1)d_i. \end{aligned}$$

Finally, $NK - N - 1 \geq 0$, because $K \geq 1 + 1/N$. \square

We have checked all the required inequalities, thus showing that the system of inequalities (IE1_m)–(IE5_m) plus (IE6₁), (IE7₁) has a solution. Thus, the inductive procedure of finding distances e_m can be carried out. This completes the proof of Claim 2.6. \square

Proof of Theorem 2.1. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of $B(x_1, r)$. We define by induction a sequence of finite functions $\{f_n : n \in \mathbb{N}\}$. Let $f_0 = f$. Suppose that f_n has been defined. Assume that f_n is (K, N) -compliant. By Lemma 2.5, there is a (K, N) -compliant function $g_n \supseteq f_n$ such that $x_n \in \text{dom}(g_n)$. And there is a (K, N) -compliant function $f_{n+1} \supseteq g_n$ such that $x_n \in \text{rng}(f_{n+1})$. Let $h = \bigcup_{n \in \mathbb{N}} f_n$. Since $\text{dom}(h)$, $\text{rng}(h)$ are dense subsets of $B(x_1, r)$ and h is K -bilipschitz, there is $g \in \text{H}(B(x, r))$ such that $g \supseteq h$. It is also obvious that g is (K, N) -compliant. By Proposition 2.3, $g \cup \text{id}_{\cup \setminus B(x_1, r)}$ is K -bilipschitz. \square

3. EXTENDING UNIFORMLY CONTINUOUS FUNCTIONS

We say that $f: X \rightarrow Y$ is (β, α) -bicontinuous if f is a β -continuous bijection such that f^{-1} is α -continuous. In other words, a bijection $f: X \rightarrow Y$ is (β, α) -bicontinuous if

$$\alpha^{-1}(d(x, x')) \leq d(f(x), f(x')) \leq \beta(d(x, x'))$$

holds for every $x, x' \in X$. Note that this makes sense only if $\alpha^{-1} \leq \beta$, at least on the range of the metric d of X . It turns out that this is not sufficient for the existence of (β, α) -bicontinuous extensions.

Recall that every modulus of continuity α satisfies

$$\alpha(s + t) \leq \alpha(s) + \alpha(t) \quad \text{for every } s, t \geq 0.$$

Lemma 3.1. Let $\alpha, \beta \in \text{MC}$. Assume $X \cup \{p\}$, Y are finite metric spaces and $f: X \rightarrow Y$ is (β, α) -bicontinuous, where $\alpha, \beta \in \text{MC}$ are such that

$$(*) \quad \alpha^{-1}(s) + \beta(t) \geq \alpha^{-1}(s+t) \quad \text{for every } s, t \geq 0.$$

Assume $q \notin Y$. Then the formula

$$(**) \quad d(q, y) = \min\{d(f(z), y) + \beta(d(z, p)) : z \in X\}$$

defines a metric extension $Y \cup \{q\}$ of Y such that $f \cup \{p, q\}$ is (β, α) -bicontinuous.

Proof. In order to justify that $(**)$ defines a metric on $Y \cup \{q\}$, we use an argument from [3]. Define a two-place symmetric function φ on $Y \cup \{q\}$ by setting $\varphi(y_0, y_1) = d(y_0, y_1)$ for $y_0, y_1 \in Y$ and $\varphi(q, y) = \beta(d(p, f^{-1}(y)))$. Then the formula

$$\bar{\varphi}(y, z) = \min \left\{ \sum_{i=0}^{k-1} \varphi(y_i, y_{i+1}) : y = y_0, y_1, \dots, y_k = z \text{ and } k \in \omega \right\}$$

clearly defines a metric, called the *shortest path metric* measured by φ . It is straightforward to see that $\bar{\varphi}(q, y) = d(f(z), y) + \beta(d(z, p))$, i.e. the shortest path from q to y is of the form $(q, f(z), y)$ for some $z \in X$. Note that

$$\varphi(f(x_0), q) + \varphi(q, f(x_1)) \geq \beta(d(x_0, p) + d(p, x_1)) \geq \beta(d(x_0, x_1)) \geq d(f(x_0), f(x_1)),$$

therefore $\bar{\varphi}(y_0, y_1) = d(y_0, y_1)$ for $y_0, y_1 \in Y$. This shows that $(**)$ indeed defines a metric on $Y \cup \{q\}$ which extends the metric of Y .

Now observe that $d(q, f(x)) \leq d(f(x), f(x)) + \beta(d(p, x)) = \beta(d(p, x))$, i.e. $f \cup \{p, q\}$ is β -continuous. Fix $x \in X$ and $z \in X$ such that $d(q, f(x)) = d(f(x), f(z)) + \beta(d(p, z))$. Knowing that $d(f(x), f(z)) \geq \alpha^{-1}(d(x, z))$ and using $(*)$, we get

$$d(q, f(x)) \geq \alpha^{-1}(d(x, z)) + \beta(d(p, z)) \geq \alpha^{-1}(d(x, z) + d(p, z)) \geq \alpha^{-1}(d(p, x)).$$

Thus f^{-1} is α -continuous. □

It is easy to see that the above lemma holds for arbitrary (not necessarily finite) metric spaces. The main change in the proof is replacing “min” by “inf” in the definition of $\bar{\varphi}$. For applications, we need the finite version only.

Remark 3.2. Let us see that the assumption $(*)$ on α, β is necessary for the existence of extensions. For, fix $\alpha, \beta \in \text{MC}$ such that $\alpha^{-1} \leq \beta$ and let $X = \{x_0, x_1\}$, $Y = \{y_0, y_1\}$, where $d(x_0, x_1) = s > 0$ and $d(y_0, y_1) = \alpha^{-1}(s)$. Define $f: X \rightarrow Y$ by $f(x_i) = y_i$. Then f is (β, α) -bicontinuous, because $\alpha^{-1} \leq \beta$. Now fix $p \notin X$ and

define $d(p, x_0) = t$ and $d(p, x_1) = s + t$. This defines a metric on $X \cup \{p\}$. Suppose $Y \cup \{q\}$ is a metric extension of Y such that $f \cup \{\langle p, q \rangle\}$ is (β, α) -bicontinuous. Then $d(q, y_0) \leq \beta(t)$ and $d(q, y_1) \geq \alpha^{-1}(s + t)$. Thus

$$\alpha^{-1}(s + t) \leq d(q, y_1) \leq d(q, y_0) + d(y_0, y_1) \leq \beta(t) + \alpha^{-1}(s),$$

which shows that $(*)$ holds.

We shall say that $\beta, \alpha \in \text{MC}$ are *compatible* if

$$\alpha^{-1}(s) + \beta(t) \geq \alpha^{-1}(s + t) \quad \text{and} \quad \beta^{-1}(s) + \alpha(t) \geq \beta^{-1}(s + t)$$

hold for every $s, t \geq 0$. The above lemma clearly implies the extension property for finite (β, α) -bicontinuous maps, where $\alpha, \beta \in \text{MC}$ are compatible. We state this result below.

Corollary 3.3. *Assume $\alpha, \beta \in \text{MC}$ are compatible moduli of continuity. Then every finite (β, α) -bicontinuous bijection between subsets of the Urysohn space \mathbb{U} can be extended to a (β, α) -bicontinuous homeomorphism of \mathbb{U} .*

We now prove a more general version, which involves totally bounded sets. The version for isometries was proved by Huhunaišvili [2] in 1955.

Theorem 3.4. *Assume $X, Y \subseteq \mathbb{U}$ are totally bounded sets and $f: X \rightarrow Y$ is a (β, α) -bicontinuous map, where α, β are compatible moduli of continuity. Then there is a (β, α) -bicontinuous map $F: \mathbb{U} \rightarrow \mathbb{U}$ which extends f .*

Proof. We may assume that both X, Y are closed, since f has a unique continuous extension onto the closure of X . Since the assumptions are symmetric, it suffices to show that f can be extended by adding one point to its domain. Then, by the separability of \mathbb{U} , a standard back-and-forth argument will complete the proof.

Fix $p \in \mathbb{U} \setminus X$. Fix a sequence $\varepsilon_0 > \varepsilon_1 > \dots > 0$ such that $\beta(\varepsilon_n) \leq 2^{-n}$ for every $n \in \omega$. For each $n \in \omega$ choose an ε_n -net $D_n \subseteq X$ in such a way that $D_{n+1} \supseteq D_n$ for every $n \in \omega$. Let $E_n = f[D_n]$ and $f_n = f \upharpoonright D_n$. We construct inductively a sequence $\{q_n\}_{n \in \omega} \subseteq \mathbb{U}$ such that $f_n \cup \{\langle p, q_n \rangle\}$ is (β, α) -bicontinuous and $d(q_n, q_{n+1}) < 2^{-n+1}$.

Start with any q_0 obtained by applying Lemma 3.1 and by the ultrahomogeneity of \mathbb{U} . Note that in the construction we will need to use formula $(**)$ given by this lemma.

Now suppose q_n has been already constructed. Apply Lemma 3.1 to get a metric extension $Z = E_{n+1} \cup \{q\}$ of E_{n+1} such that $f_{n+1} \cup \{\langle p, q \rangle\}$ is (β, α) -bicontinuous and the metric on Z is given by $(**)$. Now Z and $E_{n+1} \cup \{q_n\}$ are two compatible

metric spaces whose intersection is E_{n+1} . We can amalgamate them in the minimal way, i.e. setting

$$d(q, q_n) = \max\{|d(q, y) - d(y, q_n)| : y \in E_{n+1}\}.$$

Let us now estimate $d(q, q_n)$ from above. Fix $x \in D_{n+1}$ such that

$$d(q, q_n) = |d(q, f(x)) - d(f(x), q_n)|.$$

Applying (**) we see immediately that $d(q, f(x)) \leq d(q_n, f(x))$, because $D_n \subseteq D_{n+1}$. Now find $u \in D_{n+1}$ with

$$d(q, f(x)) = d(f(x), f(u)) + \beta(d(u, p))$$

and find $z \in D_n$ such that $d(u, z) < \varepsilon_n$. Then

$$d(q_n, f(x)) \leq d(f(x), f(z)) + \beta(d(z, p))$$

and hence

$$\begin{aligned} d(q_n, f(x)) - d(q, f(x)) &\leq d(f(x), f(z)) + \beta(d(z, p)) - d(f(x), f(u)) - \beta(d(u, p)) \\ &\leq d(f(z), f(u)) + (\beta(d(z, p)) - \beta(d(u, p))) \\ &\leq \beta(d(z, u)) + (\beta(d(z, u) + d(u, p)) - \beta(d(u, p))) \\ &< \beta(\varepsilon_n) + \beta(d(z, u)) + \beta(d(u, p)) - \beta(d(u, p)) \\ &< 2\beta(\varepsilon_n) < 2^{-n+1}. \end{aligned}$$

Thus we have proved that $d(q, q_n) = |d(q_n, f(x)) - d(q, f(x))| < 2^{-n+1}$.

Finally, we find $q_{n+1} \in \mathbb{U}$ which realizes our amalgamation, i.e. $d(q_{n+1}, q_n) = d(q, q_n)$. This completes the description of the inductive construction.

Clearly, $\{q_n\}_{n \in \omega}$ is a Cauchy sequence in \mathbb{U} . Let $q = \lim_{n \rightarrow \infty} q_n$. We claim that $f \cup \{ \langle p, q \rangle \}$ is (β, α) -bicontinuous. Indeed, given $x \in D_k$ and $n \geq k$ we have

$$d(q, f(x)) \leq d(q, q_n) + d(q_n, f(x)) < 2^{-n+1} + \beta(d(p, x))$$

and

$$d(q, f(x)) \geq d(q_n, f(x)) - d(q_n, q) > \alpha^{-1}(d(p, x)) - 2^{-n+1}.$$

Passing to the limit, we get

$$\alpha^{-1}(d(p, x)) \leq d(q, f(x)) \leq \beta(d(p, x)).$$

Since $\bigcup_{n \in \omega} D_n$ is dense in X , the above inequalities hold for every $x \in X$. This completes the proof. \square

It turns out that the Bilipschitz Extension Theorem can be generalized to the case of totally bounded subsets of a ball, using ideas from the above proof and elaborating arguments from the proof of Lemma 2.5. The precise statement reads as follows.

Theorem 3.5. *Let $B(x, r)$ be an open ball in \mathbb{U} , let $N \geq 4$, $\frac{1}{2}(N - \sqrt{N^2 - 4N}) < K < \frac{1}{2}(N + \sqrt{N^2 - 4N})$ and let $A \subseteq B(x, r)$ be a totally bounded set. Assume further that $f: A \rightarrow B(x, r)$ is K -bilipschitz and N -bigood and $x \in A$ is such that $f(x) = x$.*

Then there exists a bijection $g: B(x, r) \rightarrow B(x, r)$ such that

- (1) $g \supseteq f$,
- (2) $g \cup \text{id}_{\mathbb{U} \setminus B(x, r)}$ is K -bilipschitz,
- (3) g is N -bigood.

4. A METRIC ON THE BILIPSCHITZ GROUP

In order to prove our main result, we need to know that the group of bilipschitz auto-homeomorphisms of a metric space can be endowed with a suitable metrizable topology, compatible with the group structure. This is the contents of the current section. The results are rather standard, however we were unable to find any bibliographic references, therefore we give all the details.

Let $\langle X, d \rangle$ be a metric space. Recall that $\text{LIP}(\langle X, d \rangle)$ denotes the group of the bilipschitz auto-homeomorphisms of $\langle X, d \rangle$. For $g \in \text{LIP}(X)$ let

$$\text{lip}(g) = \min\{K : g \text{ is } K\text{-bilipschitz}\}.$$

We define on $\text{LIP}(X)$ the semimetrics

$$d_L(f, g) = \log(\text{lip}(f^{-1} \circ g)).$$

Let $x_0 \in X$. For $n \in \mathbb{N}$ define

$$d_n(f, g) = \sup\{d(f(x), g(x)) : x \in B(x_0, n)\}.$$

Further, define

$$d_S(f, g) = \sum_{n \in \mathbb{N}} \frac{d_n(f, g)}{2^n}$$

and

$$\hat{d}(f, g) = \max(d_L(f, g), d_S(f, g)).$$

Let τ_L the topology of the semimetric d_L and τ_n the topology of the semimetric d_n . Also, τ will denote the topology of d . Finally, $\hat{\tau}$ will be the topology on $\text{LIP}(X)$ induced by \hat{d} .

Proposition 4.1. Let $\langle X, d \rangle$ be as above.

- (a) For every $f, g \in \text{LIP}(X)$, $d_S(f, g) < \infty$.
- (b) \hat{d} is a metric on $\text{LIP}(X)$.
- (c) The topology $\hat{\tau}$ is generated by $\tau_L \cup \bigcup_{n \in \mathbb{N}} \tau_n$.

Proof. Let $G = \text{LIP}(X)$.

(a) Let $f, g \in G$ be K -Lipschitz and fix $n \in \mathbb{N}$ and $x \in B(x_0, n)$. Denote $a = d(f(x_0), g(x_0))$. Then

$$d(f(x), g(x)) \leq d(f(x), f(x_0)) + d(f(x_0), g(x_0)) + d(g(x_0), g(x)) \leq 2Kn + a.$$

So $d_S(f, g) \leq \sum_{n \in \mathbb{N}} 2^{-n}(2Kn + a)$ and the series is convergent.

Part (b) is trivial.

(c) It is obvious that $\tau_n \subseteq \hat{\tau}$ and $\tau_L \subseteq \hat{\tau}$ for every $n \in \mathbb{N}$. Let $f \in G$ and $r > 0$. Denote $B = B^{d_S}(f, r)$. Let $B_L = B^{d_L}(f, \log(2))$ and $B_1 = B^{d_1}(f, \frac{1}{2}r)$. Denote $K = \text{lip}(f)$. Let $g \in B_L \cap B_1$. Then $\text{lip}(g) \leq \text{lip}(f) \cdot \text{lip}(f^{-1} \circ g) < 2K$.

Suppose that $x \in B(x_0, n)$. Then

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), f(x_0)) + d(f(x_0), g(x_0)) + d(g(x_0), g(x)) < \\ &< Kn + \frac{r}{2} + 2Kn = 3Kn + \frac{1}{2}r. \end{aligned}$$

That is,

$$(1) \quad d(f(x), g(x)) < 3Kn + \frac{1}{2}r.$$

There is n_0 such that

$$(2) \quad \sum_{n > n_0} \frac{3Kn + \frac{1}{2}r}{2^n} < \frac{r}{4}.$$

Let $s > 0$ be such that $\sum_{n=2}^{n_0} s/2^n < \frac{1}{4}r$. Let $C = B_L \cap B_1 \cap \bigcap_{n=2}^{n_0} B^{d_n}(f, s)$. We show that $C \subseteq B$. Let $g \in C$. Since $g \in B_1$, we have

$$(3) \quad d_1(f, g) < \frac{1}{2}r.$$

Since $g \in \bigcap_{n=2}^{n_0} B^{d_n}(f, s)$, we have

$$(4) \quad \sum_{n=2}^{n_0} \frac{d_n(f, g)}{2^n} < \frac{1}{4}r.$$

Let $n > n_0$ and $x \in B(x_0, n)$. By (1), $d(f(x), g(x)) < 3Kn + \frac{1}{2}r$. Hence $d_n(f, g) \leq 3Kn + \frac{1}{2}r$. By (2), we get

$$(5) \quad \sum_{n > n_0} \frac{d_n(f, g)}{2^n} < \frac{1}{4}r.$$

It follows from (3)–(5) that $d_S(f, g) < r$. That is, $g \in B$.

We have shown that every ball B in the metric d_S with center at f contains a finite intersection of balls with center f in the semimetrics $d_L, d_1, \dots, d_n, \dots$. It follows that B is open in the topology generated by $\tau_L \cup \bigcup_{n \in \mathbb{N}} \tau_n$. \square

Theorem 4.2. *Let $\langle X, d \rangle$ be a metric space. Then*

- (a) $\langle \text{LIP}(X), \hat{\tau} \rangle$ is a topological group;
- (b) the action of $\langle \text{LIP}(X), \hat{\tau} \rangle$ on $\langle X, \tau \rangle$ is continuous;
- (c) if $\langle X, d \rangle$ is complete, then so is $\langle \text{LIP}(X), \hat{d} \rangle$.

Proof. Let $G = \text{LIP}(X)$.

(a) Let $n \in \mathbb{N}$. Let $f, g \in G$ and let $\varepsilon > 0$. Let $K = \text{lip}(f)$. Let m be such that $g(B(x_0, n)) \subseteq B(x_0, m)$ and let $l > 2nK$. Take any

$$f_1 \in B^{d_m}\left(f, \frac{1}{2}\varepsilon\right) \cap B^{d_L}(f, \log(2)) \quad \text{and} \quad g_1 \in B^{d_n}\left(g, \frac{\varepsilon}{4K}\right).$$

So $\text{lip}(f_1) < 2K$. Let $x \in B(x_0, n)$. So $g(x) \in B(x_0, m)$. We have

$$\begin{aligned} d(f_1 g_1(x), f g(x)) &\leq d(f_1 g_1(x), f_1 g(x)) + d(f_1 g(x), f g(x)) \\ &< 2K \cdot d(g_1(x), g(x)) + \frac{1}{2}\varepsilon < 2K \frac{\varepsilon}{4K} + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

This shows that the inverse image of a d_S -open subset of G under multiplication is \hat{d} -open in $G \times G$.

We now show that the inverse image of a d_L -open subset of G under multiplication is \hat{d} -open in $G \times G$. Let $f, g \in G$ and $M > 1$. We shall show that there are d_L -neighborhoods U, V of f and g , respectively, such that if $f_1 \in U$ and $g_1 \in V$, then $\text{lip}((f \circ g)^{-1} f_1 \circ g_1) < M$. Let $K = \text{lip}(g)$ and let

$$U = \left\{ h \in G : \text{lip}(f^{-1} \circ h) < \frac{\sqrt{M}}{K^2} \right\}, \quad V = \left\{ h \in G : \text{lip}(g^{-1} \circ h) < \sqrt{M} \right\}.$$

Fix $f_1 \in U$ and $g_1 \in V$. Then

$$\begin{aligned} \text{lip}((f \circ g)^{-1} f_1 \circ g_1) &= \text{lip}(g^{-1} \circ f^{-1} f_1 \circ g_1) \\ &= \text{lip}((g^{-1} \circ f^{-1} \circ f_1 \circ g) \circ (g^{-1} \circ g_1)) \\ &< K \frac{\sqrt{M}}{K^2} K \cdot \sqrt{M} = M. \end{aligned}$$

So $U \cdot V \subseteq B^{d_L}(f \circ g, \log(M))$.

The function $f \mapsto f^{-1}$ is an isometry of d_L . It thus remains to show that for every $n \in \mathbb{N}$, $f \in G$ and $V \in \tau^{d_n}$ such that $f^{-1} \in V$, there exists a neighborhood U of f satisfying $U^{-1} \subseteq V$.

Suppose that $V = B^{d_n}(f^{-1}, \varepsilon)$. Let $K = \text{lip}(f)$. Let m be such that

$$f^{-1}(B(x_0, n)) \subseteq B(x_0, m).$$

Let $l > m + 2\varepsilon$. We shall show that

$$U = B^{d_l}\left(f, \frac{\varepsilon}{K}\right) \cap B^{d_L}(f, \log(2)).$$

Let $f_1 \in B^{d_l}\left(f, \frac{1}{K}\varepsilon\right) \cap B^{d_L}(f, \log(2))$. So $\text{lip}(f_1) < 2K$. Let $x \in B(x_0, n)$. Denote $y = f^{-1}(x)$ and $z = f_1^{-1}(x)$. We prove that $z \in B(x_0, l)$. Note that $y \in B(x_0, m)$, so $d(f(y), f_1(y)) < K^{-1}\varepsilon$. That is,

$$d(f_1(z), f_1(y)) < \frac{\varepsilon}{K}.$$

Hence $d(y, z) < 2\varepsilon$ and therefore $d(x_0, z) \leq d(x_0, y) + d(y, z) < m + 2\varepsilon < l$. Thus $z \in B(x_0, l)$.

It follows that $d(f(z), f_1(z)) < K^{-1}\varepsilon$. That is, $d(f(z), f(y)) < K^{-1}\varepsilon$. Since f is K -bilipschitz, $d(y, z) < \varepsilon$. Finally, $f_1 \in B^{d_n}(f, \varepsilon)$.

(b) Suppose that $f(x) = y$ and let $\varepsilon > 0$.

Let δ be such that if $d(x_1, x) < \delta$, then $d(f(x_1), f(x)) < \frac{1}{2}\varepsilon$. Let n be such that $B(x, \delta) \subseteq B(x_0, n)$. Let $x_1 \in B(x, \delta)$ and $f_1 \in B^{d_n}\left(f, \frac{1}{2}\varepsilon\right)$. Then $d(f_1(x_1), f(x)) \leq d(f_1(x_1), f(x_1)) + d(f(x_1), f(x)) \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. So the action of G on X is continuous.

(c) Let $\{f_n : n \in \mathbb{N}\}$ be a Cauchy sequence. Then for every $m \in \mathbb{N}$, $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to d_m . So there is a continuous function g_m such that g_m is the uniform limit of $\{f_n \upharpoonright B(x_0, m) : n \in \mathbb{N}\}$. Let $g = \bigcup_{m \in \mathbb{N}} g_m$. Since $\{f_n : n \in \mathbb{N}\}$ is a Cauchy sequence, it is bounded. That is, there is r such that $\{f_n : n \in \mathbb{N}\} \subseteq B^{d_L}(\text{id}, r)$. Hence there is K such that for every $n \in \mathbb{N}$, f_n is K -bilipschitz. g is the pointwise limit of $\{f_n : n \in \mathbb{N}\}$. So g is bilipschitz.

For every $m \in \mathbb{N}$, g is a τ_m limit of $\{f_n : n \in \mathbb{N}\}$. It remains to show that g is a τ_L limit of $\{f_n\}_{n \in \mathbb{N}}$. Let $k > 1$. There is n_0 such that $\text{lip}(f_n^{-1} \circ f_m) < k$ for every $n, m \geq n_0$. Let $n \geq n_0$. $f_n^{-1} \circ g$ is the pointwise limit of $\{f_n^{-1} \circ f_m\}_{m \in \mathbb{N}}$. Since for every m , $f_n^{-1} \circ f_m$ is k -bilipschitz, $f_n^{-1} \circ g$ is k -bilipschitz. It follows that $\text{lip}(f_n^{-1} \circ g) \leq k$ for every $n \geq n_0$. So g is a τ_L limit of the sequence $\{f_n\}_{n \in \mathbb{N}}$. \square

5. OBTAINING A HOMEOMORPHISM FROM A GROUP ISOMORPHISM

Suppose that X and Y are open subsets of \mathbb{U} and $\varphi: H_{\Gamma}^{\text{LC}}(X) \cong H_{\Delta}^{\text{LC}}(Y)$. We shall show that there is $\tau: X \cong Y$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in H_{\Gamma}^{\text{LC}}(X)$. The proof relies on a theorem from [1]. In order to state it, we introduce some new notions.

Let X be a topological space and G a subgroup of the group $H(X)$ of all auto-homeomorphisms of X . The pair $\langle X, G \rangle$ is then called a *space-group pair*. Let K be a class of space-group pairs. K is called a *faithful class* if for every $\langle X, G \rangle, \langle Y, H \rangle \in K$ and an isomorphism φ between the groups G and H there is a homeomorphism τ between X and Y such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$. Let $\langle X, G \rangle$ be a space-group pair and let $S \subseteq X$ be open. S is *strongly flexible*, if for every infinite $A \subseteq S$ without accumulation points in X there is a nonempty open set $V \subseteq X$ such that for every nonempty open set $W \subseteq V$ there is $g \in G$ such that the sets $\{a \in A: g(a) \in W\}$ and $\{a \in A: \text{for some neighborhood } U \text{ of } a, g \upharpoonright U = \text{id}\}$ are infinite.

Theorem 5.1 ([1, Theorem B]). *Let K_F be the class of all space-group pairs $\langle X, G \rangle$ such that*

- (1) *X is regular, first countable and has no isolated points;*
- (2) *for every $x \in X$ and an open neighborhood U of x the set*

$$\{g(x): g \in G \text{ and } g \upharpoonright (X \setminus U) = \text{id}\}$$

is somewhere dense;

- (3) *The family of strongly flexible sets is a cover of X .*

Then K_F is faithful.

We wish to show that if X is an open subset of \mathbb{U} then $\langle X, H_{\Gamma}^{\text{LC}}(X) \rangle \in K_F$. Note that if $\langle X, G \rangle \in K_F$ and $G \leq H \leq H(X)$, then $\langle X, H \rangle \in K_F$. So it suffices to show that if X is an open subset of \mathbb{U} , then $\langle X, H_{\Gamma}^{\text{LC}}(X) \rangle \in K_F$.

Clause (1) in the definition of K_F certainly holds for open subsets of \mathbb{U} , and Clause (2) follows trivially from Theorem 2.1. So it remains to show that open subsets of \mathbb{U} have a cover consisting of strongly flexible sets. The proof of this fact is the contents of this section.

Suppose that $\langle X, G \rangle$ is a space-group pair and $A \subseteq X$ is infinite. We say that A is *dissectable* with respect to $\langle X, G \rangle$, if there is a nonempty open set $V \subseteq X$ such that for every nonempty open set $W \subseteq V$ there is $g \in G$ such that the sets

$$\{a \in A: g(a) \in W\} \quad \text{and} \quad \{a \in A: g \upharpoonright U = \text{id for some neighborhood } U \text{ of } a\}$$

are infinite.

Let X be a topological space and $A \subseteq X$. We say that A is *completely discrete*, if A has no accumulation points.

Suppose that X is a metric space, $A \subseteq X$ and $r > 0$. We say that A is *r -spaced*, if $d(a, b) > r$ for every distinct $a, b \in A$. We say that A is *spaced*, if for some $r > 0$, A is r -spaced.

It follows immediately from the definition that if A is dissectable and $B \supseteq A$, then B is dissectable. Since \mathbb{U} is a complete metric space, every completely discrete set contains a spaced subset. For spaces which are open subsets of \mathbb{U} we shall prove that every spaced set contained in a small ball is dissectable.

Suppose that $a < b$ are real numbers, and $h: [a, b] \rightarrow \mathbb{U}$ is an isometry into \mathbb{U} . Then $L := h([a, b])$ is called a *line segment* in \mathbb{U} , and $h(a)$, $h(b)$ are the *endpoints* of L .

Suppose that X is a metric space and for every $n \in \mathbb{N}$, $f_n: [a, b] \rightarrow X$. We say that $\{f_n(t): n \in \mathbb{N}\}$ is *equicontinuous*, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $t, s \in [a, b]$ and $n \in \mathbb{N}$: if $|t - s| < \delta$, then $d(f_n(s), f_n(t)) < \varepsilon$. Let $r > 0$ and let \mathcal{A} be a set of subsets of X . We say that \mathcal{A} is *r -spaced* if $d(A, B) > r$ for every distinct $A, B \in \mathcal{A}$. We say that \mathcal{A} is *spaced* if for some $r > 0$, \mathcal{A} is r -spaced. We say that \mathcal{A} is *almost r -spaced* if for some finite $\mathcal{A}_0 \subseteq \mathcal{A}$, $\mathcal{A} \setminus \mathcal{A}_0$ is r -spaced. We say that \mathcal{A} is *almost spaced* if for some $r > 0$, \mathcal{A} is almost r -spaced. Let $A \subseteq^* B$ mean that $A \setminus B$ is finite.

Given a set $B \subseteq X$ and a family \mathcal{F} of self-maps of a space X , we shall denote by $S(\mathcal{F}, B)$ the set of all $f \in \mathcal{F}$ such that $f \upharpoonright (X \setminus B) = \text{id}$.

Proposition 5.2. (a) Let $x, u, v \in \mathbb{U}$ and let $r > 0$ be such that $u, v \in B(x, \frac{1}{15}r)$. Then there is $g \in S(\text{H}(\mathbb{U}), B(x, r))$ such that $g(u) = v$ and g is 2-bilipschitz.

(b) There is an increasing function $\hat{K}: (0, \infty) \rightarrow \mathbb{N}$ such that the following holds. If L is a line segment in \mathbb{U} with endpoints x and y and $r > 0$, then there is $g \in S(\text{H}(\mathbb{U}), B(L, r))$ such that $g(x) = y$ and g is $\hat{K}(r^{-1} \text{length}(L))$ -bilipschitz.

Proof. (a) Set $s = \frac{1}{15}r$, $t = 12s$ and let $y \in \mathbb{U}$ be such that $d(x, y) = 3s$. Then $B(y, 12s) \subseteq B(x, 15s) = B(x, r)$. Let $f = \{\langle y, y \rangle, \langle u, v \rangle\}$. Then $d(u, y), d(v, y) \in (2s, 4s)$ and hence $\text{lip}(f) < 2$. Also, $t - d(u, y), t - d(v, y) > 12s - 4s = 8s$. So $d(u, v)/(t - d(u, y)), d(u, v)/(t - d(v, y)) < 2s/8s = \frac{1}{4}$. By Theorem 2.1, there is $g \in S(\text{H}(\mathbb{U}), B(y, 12t))$ such that $g \supseteq f$. Hence $g(u) = v$ and $\text{supp}(g) \subseteq B(x, r)$.

(b) Let $\hat{K}(t) = 2^{\lceil 16t \rceil + 1}$. Choose a sequence of points $x = x_0, x_1, \dots, x_n = y$ in L such that $d(x_{i-1}, x_i) = \frac{1}{16}r$ for $i < n$ and $d(x_{n-1}, x_n) \leq \frac{1}{16}r$. Then $n \leq \lceil 16 \cdot \text{length}(L)/r \rceil + 1$. Let $s = \frac{1}{15}r$. Then for every $i = 1, \dots, n$, $x_{i-1} \in B(x_i, s)$ and $B(x_i, 15s) \subseteq B(L, r)$. By (a), there are $g_1, \dots, g_n \in S(\text{H}(\mathbb{U}), B(L, r))$ such that $g_i(x_{i-1}) = x_i$ and g_i is 2-bilipschitz. Let $g = g_n \circ \dots \circ g_1$. Then $g(x) = y$,

$g \in \text{S}(\text{H}(\mathbb{U}), B(L, r))$ and g is 2^n -bilipschitz. That is, g is $\hat{K}(\text{length}(L)/r)$ -bilipschitz. \square

Proposition 5.3. *Let X be a metric space and $\{\gamma_n : n \in \mathbb{N}\}$ a sequence of arcs in X such that $\gamma_n : [0, 1] \rightarrow X$. Suppose that $\{\gamma_n(0) : n \in \mathbb{N}\}$ is spaced and $\{\gamma_n(1) : n \in \mathbb{N}\}$ is a Cauchy sequence. Also assume that $\{\gamma_n : n \in \mathbb{N}\}$ is equicontinuous.*

Then there are $s \in (0, 1]$ and an infinite $\sigma \subseteq \mathbb{N}$ such that $\{\gamma_n(s) : n \in \sigma\}$ is a Cauchy sequence, and for every $t \in [0, s)$, $\{\gamma_n([0, t]) : n \in \sigma\}$ is almost spaced.

Proof. For every infinite $\eta \subseteq \mathbb{N}$ define

$$s_\eta = \sup(\{s \in [0, 1] : \{\gamma_n([0, s]) : n \in \eta\} \text{ is almost spaced}\}).$$

Clearly, if $\eta \subseteq^* \zeta$, then $s_\eta \geq s_\zeta$. This implies that there is an infinite $\sigma \subseteq \mathbb{N}$ such that $s_\eta = s_\sigma$ for every infinite $\eta \subseteq \sigma$. Denote s_σ by s . We show that there is an infinite $\sigma_0 \subseteq \sigma$ such that $\{\gamma_n(s) : s \in \sigma_0\}$ is a Cauchy sequence. If $s = 1$, then $\{\gamma_n(s) : s \in \sigma\}$ is a Cauchy sequence. Suppose that $s < 1$. If there is no σ_0 as required, then there are $r > 0$ and an infinite $\sigma_0 \subseteq \sigma$ such that $\{\gamma_n(s) : s \in \sigma_0\}$ is r -spaced. Let $\delta > 0$ be such that for every $n \in \mathbb{N}$ and $t, u \in [0, 1]$: if $|t - u| < \delta$, then $d(\gamma_n(t), \gamma_n(u)) < \frac{1}{3}r$. We may assume that $s + \delta \leq 1$. Then $\{\gamma_n([0, s + \delta]) : n \in \sigma_0\}$ is $\frac{1}{3}r$ -spaced. So $s_{\sigma_0} \geq s + \delta$, a contradiction. Hence, there is an infinite $\sigma_0 \subseteq \sigma$ such that $\{\gamma_n(s) : n \in \sigma_0\}$ is a Cauchy sequence. We may thus assume that $\{\gamma_n(s) : n \in \sigma\}$ is a Cauchy sequence. By the definition of s , for every $t \in [0, s)$, $\{\gamma_n([0, t]) : n \in \sigma\}$ is almost spaced. \square

Lemma 5.4. *Let $X \subseteq \mathbb{U}$ be open. Suppose that $r, \varepsilon > 0$, $B^\mathbb{U}(x, r + \varepsilon) \subseteq X$, and $A \subseteq B^\mathbb{U}(x, r)$ is an infinite spaced set. Then A is dissectable with respect to $\text{S}(\text{LIP}(\mathbb{U}), X)$.*

Proof. Suppose that $A = \{a_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ let L_n be the line segment connecting a_n with x , and let $\gamma_n : [0, 1] \rightarrow X$ be the parametrization of L_n such that $d(\gamma_n(t), \gamma_n(0)) = t \cdot d(a_n, x)$. Then $\{\gamma_n : n \in \mathbb{N}\}$ is equicontinuous. Let $s \in [0, 1]$ and let $\sigma \subseteq \mathbb{N}$ be such that σ is infinite, $\{\gamma_n(s) : n \in \sigma\}$ is a Cauchy sequence and for every $t \in [0, s)$, $\{\gamma_n([0, t]) : n \in \sigma\}$ is almost spaced. We may assume that $\sigma = \mathbb{N}$. Let $y = \lim_{n \in \mathbb{N}} \gamma_n(s)$. Clearly, $d(\bigcup_{n \in \mathbb{N}} L_n, \mathbb{U} \setminus X) \geq \varepsilon$. Hence $d(y, \mathbb{U} \setminus X) \geq \varepsilon$. We may assume that $y \notin \{a_n : n \in \mathbb{N}\}$. So $\delta := d(y, \{a_n : n \in \mathbb{N}\}) > 0$. Set $s = \frac{1}{16} \min(\varepsilon, \delta)$. We show that for every nonempty open $W \subseteq B(y, s)$ there is $g \in \text{S}(\text{LIP}(\mathbb{U}), X)$ such that the sets

$$\{i \in \mathbb{N} : g(a_i) \in W\} \quad \text{and} \quad \{i \in \mathbb{N} : \text{there is } V \in \text{nbd}(a_i) \text{ such that } g \upharpoonright V = \text{id}\}$$

are infinite. We may assume that $W = B(z, q)$. There is $t < s$ such that $\gamma_n(t) \in B(y, \frac{1}{2}q)$ for all but finitely many n 's. There are a finite set $\eta \subseteq \mathbb{N}$ and $e > 0$ such that $\{\gamma_n([0, t]) : n \in \mathbb{N} \setminus \eta\}$ is e -spaced. Let $p = \min(\varepsilon, \frac{1}{2}e)$. Then for every $m \neq n$ in $\mathbb{N} \setminus \eta$, $B(\gamma_m([0, t]), p) \cap B(\gamma_n([0, t]), p) = \emptyset$ and for every n , $B(\gamma_n([0, t]), p) \subseteq X$. Let $l = \sup_{n \in \mathbb{N}} \text{length}(L_n)$. For every $n \in \mathbb{N} \setminus \eta$ there is $h_n \in S(\mathbb{H}(\mathbb{U}), B(\gamma_n([0, t]), p))$ such that

$h_n(a_n) = \gamma_n(t)$ and h_n is $\hat{K}(l/p)$ -bilipschitz. This follows from Proposition 5.2 (b). Let $h = \circ\{h_{2n} : n \in \mathbb{N} \text{ and } 2n \notin \eta\}$. It follows trivially from the above that $h \in S(\mathbb{H}(\mathbb{U}), X)$ and h is $(\hat{K}(l/p))^2$ -bilipschitz. Also for every $n \in (2\mathbb{N}) \setminus \eta$, $h(a_n) \in B(y, \frac{1}{2}q)$, and for every $n \in (2\mathbb{N} + 1) \setminus \eta$ there is $V \in \text{nbd}(a_n)$ such that $h \upharpoonright V = \text{id}$. Note that $d(y, z) < \frac{1}{16} \min(\varepsilon, \delta)$. By Proposition 5.2 (a), there is $f \in S(\mathbb{H}(\mathbb{U}), B(y, \frac{15}{16} \cdot \min(\varepsilon, \delta)))$ such that $f(y) = z$ and f is 2-bilipschitz. Since $d(y, \{a_n : n \in \mathbb{N}\}) = \delta$, for every $n \in \mathbb{N}$ there is $V \in \text{nbd}(a_n)$ such that $f \upharpoonright V = \text{id}$. It also follows that $f \in S(\mathbb{H}(\mathbb{U}), X)$. Let $g = f \circ h$. Then $g \in S(\mathbb{H}(\mathbb{U}), X)$ and g is bilipschitz. So

(1) $g \in S(\text{LIP}(\mathbb{U}), X)$.

Let $n \in (2\mathbb{N}) \setminus \eta$. Then $h(a_n) \in B(y, \frac{1}{2}q)$. Since f is 2-bilipschitz and $f(y) = z$, it follows that $f(h(a_n)) \in B(z, q)$. That is,

(2) $g(a_n) \in B(z, q)$ for every $n \in (2\mathbb{N}) \setminus \eta$.

Finally,

(3) For every $n \in (2\mathbb{N} + 1) \setminus \eta$ there is $V \in \text{nbd}(a_n)$ such that $g \upharpoonright V = \text{id}$.

We have shown that A is dissectable. □

Corollary 5.5. *Let X be a nonempty open subset of \mathbb{U} and $S(\text{LIP}(\mathbb{U}), X) \leq G \leq \mathbb{H}(X)$. Then $\langle X, G \rangle \in K_F$.*

Proof. (a) Note that X is a first countable regular space without isolated points. That is, Clause 1 in the definition of K_F holds. By Proposition 5.2 (a), for every $x \in X$ and $U \in \text{nbd}^X(x)$ the set $\{g(x) : g \in S(\text{LIP}(\mathbb{U}), X)\}$ is somewhere dense. So Clause 2 in the definition of K_F holds.

Note that in a complete metric space every completely discrete infinite set contains an infinite spaced subset. It thus follows from Lemma 5.4 (a) that if $B^{\mathbb{U}}(x, r + \varepsilon) \subseteq X$, then $B^{\mathbb{U}}(x, r)$ is strongly flexible with respect to $S(\text{LIP}(\mathbb{U}), X)$, and thus it is strongly flexible with respect to G . So X has a cover consisting of strongly flexible sets. That is, Clause 3 holds. □

6. LOCAL Γ -BICONTINUITY OF THE CONJUGATING HOMEOMORPHISM

Let $X \subseteq \mathbb{U}$ be open. Define

$$\text{LIP}^{\text{id}}(X) = \{g \upharpoonright X : g \in \text{S}(\text{LIP}(\mathbb{U}), X)\}.$$

We equip $\text{LIP}^{\text{id}}(X)$ with the topology it inherits from $\text{LIP}(\mathbb{U})$. We shall apply Theorem 3.41 from [4] to the group $\text{LIP}^{\text{id}}(X)$. This requires the following definitions.

Definition 6.1. (a) Let X be a metric space, $G \leq \text{H}(X)$ and let σ be a topology on G such that the action of G on X is continuous with respect to σ . Let $x \in X$. We say that $\langle G, \sigma \rangle$ is *affine-like at x* if the following holds. For every $V \in \text{nbd}(\text{id})$ and $W \in \text{nbd}(x)$ there are $n = n(x, V, W) \in \mathbb{N}$, $U = U(x, V, W) \in \text{nbd}(x)$ such that for every $x_1, y_1, x_2, y_2 \in U$: if $d(x_1, y_1) = d(x_2, y_2)$, then there are $h_1, \dots, h_n \in V$ such that $h_1 \circ \dots \circ h_n(x_1) = x_2$, $h_1 \circ \dots \circ h_n(y_1) = y_2$ and $h_i \circ h_{i+1} \circ \dots \circ h_n(\{x_1, y_1\}) \subseteq W$ for every $1 \leq i \leq n$.

If $\langle G, \sigma \rangle$ is affine-like at every $x \in X$, then $\langle G, \sigma \rangle$ is said to be an *affine-like topological group*.

(b) Let X be a metric space and $x \in X$. We say that X has the *discrete path property at x* (briefly: *X is DPT at x*), if the following holds. There is $U \in \text{nbd}(x)$ and $K \geq 1$ such that

- (*) for every $y, z \in U$ and $d \in (0, d(y, z))$ there are $n \in \mathbb{N}$ and $u_0, \dots, u_n \in X$ such that $n \leq K \cdot d(y, z)/d$, $d(y, u_0), d(u_n, z) < d$ and $d(u_{i-1}, u_i) = d$ for every $i = 1, \dots, n$.

If X is DPT at every $x \in X$, then X is called a *DPT space*.

(c) Let X be a metric space and $x \in X$. We shall say that X has the *connectivity property 1 at x* , (briefly: *X is CP1 at x*), if for every $r > 0$ there is $r^* \in (0, r)$ such that for every $x' \in X$ and $r' > 0$: if $B(x', r') \subseteq B(x, r^*)$ and C is a connected component of $B(x, r) \setminus B(x', r')$, then $C \cap (B(x, r) \setminus B(x, r^*)) \neq \emptyset$.

If X is CP1 at every $x \in X$, then X is called a *CP1 space*.

Theorem 6.2 ([4, Theorem 3.41]). *Assume that the following facts hold.*

- (i) X is a metric space, $G \leq \text{H}(X)$, $\langle G, \sigma \rangle$ is a topological group. The action of G on X is continuous with respect to σ , and $\langle G, \sigma \rangle$ is of the second category;
- (ii) $x \in X$ and $\langle G, \sigma \rangle$ is affine-like at x ;
- (iii) Γ is a countably generated modulus of continuity;
- (iv) Y is a metric space and $\tau: X \cong Y$;
- (v) for every $g \in G$, g^τ is Γ -bicontinuous at $\tau(x)$;
- (vi) X is DPT at x and Y is DPT and CP1 at $\tau(x)$.

Then τ is Γ -bicontinuous at x .

We need to know that X is DPT, CP1 and that $\text{LIP}^{\text{id}}(X)$ is affine-like. The verification of the first two properties is trivial, and is left to the reader. We only prove the affine-likeness of $\text{LIP}^{\text{id}}(X)$.

Lemma 6.3. *Let X be a nonempty open subset of \mathbb{U} . Then*

- (a) X is DPT and CP1,
- (b) $\text{LIP}^{\text{id}}(X)$ is affine-like.

Proof. (b) Let $u \in X$ and $V \in \text{nb}\text{d}^{\text{LIP}^{\text{id}}(X)}(\text{id})$. We prove that there exists $U \in \text{nb}\text{d}^X(u)$ such that for every $x_1, y_1, x_2, y_2 \in U$: if $d(x_1, y_1) = d(x_2, y_2)$, then there is $g \in V$ such that $g(x_1) = x_2$ and $g(y_1) = y_2$. Note that this implies the affine-likeness of $\text{LIP}^{\text{id}}(X)$ with $n(x, V, W) = 1$ for every x, V and W (see Definition 6.1).

We may assume that $V = B(\text{id}, s)$. Choose $K \in (1, e^s)$. Note that for every $g \in \text{LIP}^{\text{id}}(X)$, if g is K -bilipschitz and $d(x, g(x)) < s$ for every $x \in X$, then $g \in V$. Let $r_0 > 0$ be such that $B^{\mathbb{U}}(u, r_0) \subseteq X$. Let $a = \min(\frac{1}{4}r_0, s/(K+1))$ and $b = (K-1) \cdot a / ((2K-1) \cdot (K+1))$. Let u_0 be such that $d(u_0, u) = a$ and $B = B^{\mathbb{U}}(u_0, 2a)$ and $U = B^{\mathbb{U}}(u, b)$.

Note that $B \subseteq B^{\mathbb{U}}(u, r_0) \subseteq X$, because $3a < r_0$. Note also that $U \subseteq B$ and that $d(U, \mathbb{U} \setminus B) = a - b$. This is so, since the radius of B is $2a$ and U is a ball whose center has distance a from the center of B and whose radius is b . Let f be a one-to-one function such that $\text{dom}(f)$ and $\text{rng}(f)$ are finite subsets of U . We estimate from above

$$m(f) := \max\left(\left\{\frac{d(x, f(x))}{2a - d(x, u_0)} : x \in \text{dom}(f)\right\}\right).$$

Clearly, $d(x, u_0) \leq d(u_0, u) + d(u, x) < a + b$. So $2a - d(x, u_0) > a - b$. Hence $m(f) < 2b/(a - b)$. Define $N := (a - b)/2b$. Since $m(f^{-1}) < 2b/(a - b)$, it follows that f is N -bigood. Suppose that f is an isometry and let $g = f \cup \{(u_0, u_0)\}$. We show that g is K -bilipschitz. Since the assumptions about f and f^{-1} are the same, it suffices to check that $d(u_0, f(x))/d(u_0, x) \leq K$ for every $x \in \text{dom}(f)$. Note that $d(u_0, f(x))/d(u_0, x) \leq (a+b)/(a-b)$. Clearly, $b = (K-1) \cdot a / ((2K-1) \cdot (K+1)) \leq (K-1) \cdot a / (K+1)$. So

$$\frac{d(u_0, f(x))}{d(u_0, x)} \leq \frac{a+b}{a-b} \leq \frac{a + (K-1)a/(K+1)}{a - (K-1)a/(K+1)} = \frac{K+1 + K-1}{K+1 - (K-1)} = K.$$

We have shown that g is K -bilipschitz. We have also shown that f is N -bigood and so g is N -bigood.

We shall now apply the Bilipschitz Extension Theorem to g . To this end we still need to show that $N \geq K^2/(K-1)$. Indeed, we have

$$N = \frac{a-b}{2b} = \frac{a}{2b} - \frac{1}{2} = \frac{(2K-1) \cdot (K+1)}{2(K-1)} - \frac{1}{2} = \frac{2K^2 + K - 1 - (K-1)}{2(K-1)} = \frac{K^2}{K-1}.$$

By the Bilipschitz Extension Theorem, there is $\tilde{h} \in S(\mathbb{H}(\mathbb{U}), B)$ such that $g \subseteq \tilde{h}$ and \tilde{h} is K -bilipschitz. Hence $h := \tilde{h} \upharpoonright X \in \text{LIP}^{\text{id}}(X)$. We show that for every $z \in \mathbb{U}$, $d(\tilde{h}(z), z) < s$. If $z \notin B$, then $d(\tilde{h}(z), z) = 0$. Suppose that $z \in B$. Then

$$d(\tilde{h}(z), z) \leq d(z, u_0) + d(u_0, \tilde{h}(z)) < a + Ka \leq (K + 1) \cdot \frac{s}{K + 1} = s.$$

It follows that $d_S(\tilde{h}, \text{id}) < s$. Also, $\text{lip}(\tilde{h}) \leq K$, so $d_L(\tilde{h}, \text{id}) \leq \log(K) < s$. Hence $\hat{d}(\tilde{h}, \text{id}) < s$. That is, $h \in B(\text{id}, s) = V$. \square

Corollary 6.4. (a) *Let X, Y be nonempty open subsets of \mathbb{U} , let Γ and Δ be countably generated MC-semigroups and $\varphi: \mathbb{H}_\Gamma^{\text{LC}}(X) \subseteq \mathbb{H}_\Delta^{\text{LC}}(Y)$. Then there is $\tau: X \cong Y$ such that τ is locally Γ -bicontinuous, τ is locally Δ -bicontinuous and $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \mathbb{H}_\Gamma^{\text{LC}}(X)$.*

(b) *If $X = Y = \mathbb{U}$ in (a), then $\Gamma = \Delta$.*

Proof. (a) By Corollary 5.5 (a), $\langle X, \mathbb{H}_\Gamma^{\text{LC}}(X) \rangle, \langle Y, \mathbb{H}_\Delta^{\text{LC}}(Y) \rangle \in K_F$. So by Theorem 5.1, there is $\tau: X \rightarrow Y$ such that τ induces φ .

Let $G = \text{LIP}^{\text{id}}(X)$. As G is a subgroup of $\text{LIP}(X)$, it inherits the topology defined on $\text{LIP}(X)$ in Theorem 4.2 (c). Denote this topology on G by σ . Since $G \leq \mathbb{H}_\Gamma^{\text{LC}}(X)$, it follows that $G^\tau \subseteq \mathbb{H}_\Delta^{\text{LC}}(Y)$. We shall show that Theorem 6.2 can be applied to X, Y, G, σ, τ and Δ .

We verify that Clause (i) in Theorem 6.2 is fulfilled. By Theorem 4.2 (a) and (b), $\langle G, \sigma \rangle$ is a topological group acting continuously on X . It is easy to see that G is a closed subset of $\text{LIP}(X)$. By Theorem 4.2 (c), $\langle G, \sigma \rangle$ is of the second category.

Clause (ii) follows from Lemma 6.3 (b), and Clause (vi) follows from Lemma 6.3 (a). The remaining requirements of Theorem 6.2 hold automatically. It follows from Theorem 6.2 that τ is locally Δ -bicontinuous. Applying the same argument to τ^{-1} we conclude that τ^{-1} is locally Γ -bicontinuous. So τ is locally Γ -bicontinuous.

(b) It follows from (a) that $\mathbb{H}_\Gamma^{\text{LC}}(\mathbb{U}) = \mathbb{H}_\Delta^{\text{LC}}(\mathbb{U})$. We show that this implies that $\Gamma = \Delta$. Suppose that $\gamma \in \Gamma \setminus \Delta$, and let us show that $\mathbb{H}_\Gamma^{\text{LC}}(\mathbb{U}) \setminus \mathbb{H}_\Delta^{\text{LC}}(\mathbb{U}) \neq \emptyset$. Let $\{\delta_i: i \in \mathbb{N}\}$ be a generating set for Δ . For every $i \in \mathbb{N}$ let $\{t_{n,i}\}_{n \in \mathbb{N}} \subseteq (0, \infty)$ be a sequence converging to 0 so that $\gamma(t_{n,i}) > \delta_i(t_{n,i})$ for every i and n . It is easy to choose one-to-one sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ such that

- (1) $\lim_{j \rightarrow \infty} x_j = x$,
- (2) $d(y_j, x) = \gamma(d(x_j, x))$ for every $j \in \mathbb{N}$,
- (3) for every $i \in \mathbb{N}$ the set $\{n: \text{there is } j \text{ such that } d(x_j, x) = t_{n,i}\}$ is infinite,
- (4) the function f defined by $x \mapsto x$ and $x_j \mapsto y_j, j \in \mathbb{N}$, is (2γ) -bicontinuous.

Since $\text{dom}(f)$ and $\text{rng}(f)$ are convergent sequences, they are totally bounded. Hence by Theorem 3.4 there is $g \in \mathbb{H}(\mathbb{U})$ such that g is (2γ) -bicontinuous and $g \supseteq f$. It

follows that $g \in H_{\Gamma}^{LC}(\mathbb{U})$. However, for every $i \in \mathbb{N}$, f is not δ_i -continuous at x . So g is not δ_i -continuous at x . This means that $g \notin H_{\Delta}^{LC}(\mathbb{U})$. \square

Acknowledgements. The first author would like to thank the Center for Advanced Studies in Mathematics at Ben Gurion University of the Negev, for supporting his visits when this work originated.

References

- [1] *V. P. Fonf, M. Rubin*: Reconstruction theorem for homeomorphism groups without small sets and non-shrinking functions of a normed space. Preprint in Math Arxiv. Available at <http://arxiv.org/abs/math/0510120>.
- [2] *G. E. Huhunaišvili*: On a property of Uryson's universal metric space. Dokl. Akad. Nauk SSSR (N.S.) *101* (1955), 607–610. (In Russian.)
- [3] *M. Kojman, S. Shelah*: Almost isometric embeddings between metric spaces. Israel J. Math. *155* (2006), 309–334.
- [4] *M. Rubin, Y. Yomdin*: Reconstruction of manifolds and subsets of normed spaces from subgroups of their homeomorphism groups. Dissertationes Math. *435* (2005), 1–246.
- [5] *V. Uspenskij*: The Urysohn universal metric space is homeomorphic to a Hilbert space. Topology Appl. *139* (2004), 145–149.
- [6] *P. S. Urysohn*: Sur un espace métrique universel I, II. Bull. Sci. Math. *51* (1927), 43–64, 74–90.

Authors' addresses: W. K u b i š, Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: kubis@math.cas.cz; M. R u b i n, Department of Mathematics, Ben Gurion University of the Negev, Be'er Sheva, Israel.