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Dariusz Zagrodny

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ON MAXIMAL MONOTONE OPERATORS WITH RELATIVELY COMPACT RANGE

Dariusz Zagrodny, Warszawa

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Abstract. It is shown that every maximal monotone operator on a real Banach space with relatively compact range is of type NI. Moreover, if the space has a separable dual space then every maximally monotone operator T can be approximated by a sequence of maximal monotone operators of type NI, which converge to T in a reasonable sense (in the sense of Kuratowski-Painleve convergence).

Keywords: nonlinear operators, maximal monotone operators, range of maximal monotone operator, an approximation method of maximal monotone operators

MSC 2010: 47H05

1. Introduction

Let X be a real Banach space and X^* its dual space (the set of all continuous linear functionals on X); we refer to [3, p. 63] for the definition and several properties of the dual space. We recall the notion of the maximal monotone subset of $X \times X^*$, referring to [11] for classical results concerning maximal monotonicity and to [8], [9], [5], [4] for presentations of several recent achievements, comments and open problems on it.

Definition 1.1. A subset $S \subset X \times X^*$ is said to be *monotone* provided

$$\langle x^* - y^*, x - y\rangle \geqslant 0 \quad \text{whenever} \quad (x, x^*), (y, y^*) \in S,$$

where $(x, x^*) \in X \times X^*$ and $\langle x^*, x\rangle$ stands for the value of x^* at x.

We say that S is *maximally monotone* if S is monotone and has no proper extension (the set S is maximal in the family of monotone subsets of $X \times X^*$, ordered by inclusion).

We denote by $D(S)$ the domain and by $R(S)$ the range of S, i.e.

$$D(S) := \{x \in X \mid \exists\, x^* \in X^* \colon\ (x, x^*) \in S\},$$
$$R(S) := \{x^* \in X^* \mid \exists\, x \in X \colon\ (x, x^*) \in S\}.$$

In the case X is a reflexive Banach space, there are several strong results concerning maximally monotone subsets, for example: the closure of the domain is a convex set (see [6]; this is still an open question in the nonreflexive case), the closure of the range is also convex (see [6]), the range of a coercive maximal monotone operator is the whole space (we refer to [11] for several results of this type and their generalizations, for an example see Corollary 32.35 of [11]), for every maximally monotone subset $S \subset X \times X^*$ there is a pair $(s, s^*) \in S$ such that $\langle s^*, s\rangle = -\|s^*\|\|s\|$, $\|s^*\| = \|s\|$ (this yields several results on maximal monotonicity in a reflexive Banach space set-up, see [9, Theorem 10.3], the perfect square criterion for maximality, or [8] for a more general form of this and some comments on it). There have been several attempts to extend the results beyond the reflexive set-up, we refer to [8] for the most recent repertory on achievements in this direction and to [10] for an extension of the perfect square criterion for the maximality to a nonreflexive Banach set-up. Most of the extensions were done for special classes of maximally monotone subsets of $X \times X^*$, see [5], [9], [8] for a discussion on it. One of them is the class of NI operators, see [9]. Below we recall the notion

Definition 1.2. We say that a maximally monotone subset S of $X \times X^*$ is of type NI if for every $(x^*, x^{**}) \in X^* \times X^{**}$ we have

$$\inf_{(s,s^*)\in S} \langle \varphi(s) - x^{**}, s^* - x^*\rangle \leqslant 0,$$

where X^{**} stands for the dual space of X^* (the bidual of X) and the linear operator $\varphi\colon\ X \to X^{**}$ is such that

$$\forall\, x^* \in X^*,\ \forall\, x \in X,\ \ \langle x^*, x\rangle = \langle \varphi(x), x^*\rangle, \tag{1}$$

i.e. $\varphi(x)$ is the canonical image of x in the bidual, see [9, p. 18–19] for comments.

For the class of NI sets we have a version of the perfect square criterion, see [10], so it is natural to ask how large the class is. Several examples of NI subsets can be found in [9], [8], [10]. Herein it is shown that any maximal monotone subset with its range relatively compact is in the class (the same result holds true whenever the domain of maximally monotone subset is relatively compact, see comments following Property 3.2). What is more, whenever X^* is a separable Banach space then for any maximally monotone operator we are able to construct an approximation by maximally monotone operators of type NI which converge to the operator in "good" sense, namely in the sense of Kuratowski-Paineleve convegence; this corresponds to [2, Theorem 2.3].

Open problem. Is it possible to construct the approximation in Asplund space (maybe in Weakly Compactly Generated Banach space)?

2. Auxiliary results

In this section some notions and their properties are gathered. Let us start with weak topologies. For a real Banach space X and its dual X^* we denote by $w(X^*, X)$ *the weak* topology of* X^*—the topology induced by X, see [7] or [3], [9], and by $w(X^{**}, X^*)$ *the weak* topology of* X^{**}—the topology induced by X^*, see [7] for example. The following two properties of the weak* topology are used in the sequel.

Property 2.1 ([3, p. 75]). *If X is a real separable Banach space then any closed ball in X^* is weak* sequentially compact, i.e. every sequence $\{x_i^*\}_{i=1}^{\infty} \subset X^*$ which is bounded has a weak* converge subsequence.*

Property 2.2 ([7, see Exercise 1.b]). *Let X be a real Banach space and φ the embeding of X into X^{**} (φ is given in* (1)*, see also* [7, 4.5 The second dual of a Banach space, p. 95]*). If*

$$B(0, M) := \{x \in X\colon\ \|x\| \leqslant M\}, \quad M > 0$$

then $\varphi(B(0, M))$ is $w(X^{**}, X^*)$ dense in

$$B^{**}(0, M) := \{x^{**} \in X^{**}\colon\ \|x^{**}\| \leqslant M\},$$

i.e.

$$\forall \varepsilon > 0,\ \forall x^{**} \in B^{**}(0, M),\ \forall\, x^* \in X^*,\ \exists\, x \in B(0, M)\colon\ |\langle x^{**}, x^*\rangle - \langle x^*, x\rangle| \leqslant \varepsilon.$$

Below we recall that if S is a maximally monotone subset with a bounded range then its domain is the whole space, namely, we have

Property 2.3 ([8, see Theorem 25.1]). *Let X be a nonzero real Banach space, let $S \subset X \times X^*$ be maximally monotone and $R(S)$ bounded. Then $D(S) = X$.*

S. Simons gave a useful characterization of maximal monotonicity in reflexive Banach spaces, see [9, Theorem 10.3] and for generalizations [8, Theorem 21.7]. In nonreflexive Banach spaces there is a possibility to give an approximated version of it, namely,

Property 2.4 ([10, Corollary 3.4]). *Let X be a real Banach space and $S \subset X \times X^*$ a nonempty maximally monotone subset of type NI. For every $\varepsilon \in (0,\infty)$ and every $(w, w^*) \in X \times X^*$ there exists $(s_\varepsilon, s_\varepsilon^*) \in S$ such that*

$$\|s_\varepsilon^* - w^*\|^2 + \|s_\varepsilon - w\|^2 + 2\langle s_\varepsilon^* - w^*, s_\varepsilon - w\rangle \leqslant \varepsilon. \tag{2}$$

Moreover, there are $\varepsilon_0 \in (0,\infty)$ and $R > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $(s_\varepsilon, s_\varepsilon^*) \in S$ and (2) is satisfied then $\|s_\varepsilon^*\| + \|s_\varepsilon\| \leqslant R$.

A simple consequence of the above property is

Property 2.5 ([10, Corollary 3.5]). *Let X be a real Banach space and $S \subset X \times X^*$ a nonempty maximal monotone subset of type NI. Then $\operatorname{cl} R(S)$ is a convex set.*

We close this section recalling a property of monotone sets which allows us to construct maximally monotone sets with their range being in a given compact set. The property is a consequence of an extension of the Debrunner-Flor theorem, see [1], [2], [4].

Property 2.6 ([2, Lemma 2.4]). *Suppose that C^* is a compact convex subset of X^* and that $M \subset X \times C^*$ is a monotone set. For any $x_0 \in X$ there exists $x_0^* \in C^*$ such that $\{(x_0, x_0^*)\} \cup M$ is a monotone set.*

3. Results

In this section we prove that any maximally monotone subset $S \subset X \times X^*$ with $R(S)$ included in a compact subset of X^* is of type NI. Next, for a given maximal monotone subset $T \subset X \times X^*$ we construct an approximation of T by maximal monotone subsets of type NI.

Theorem 3.1. *Let X be a real Banach space and let $S \subset X \times X^*$ be a maximally monotone subset such that $R(S) \subset C^*$ for some compact subset $C^* \subset X^*$. Then S is of type* NI.

P r o o f. Let us fix $(a^*, a^{**}) \in X^* \times X^{**}$. We are able to choose a sequence of finite subsets of C^*, say $\{C_i^*\}_{i=1}^\infty$, such that

$$\forall\, i \in \mathbb{N},\ \ C_i^* \subset C_{i+1}^* \subset C^* \subset C_i^* + B^*(0, i^{-1}), \tag{3}$$

where $B^*(0, r) := \{x^* \in X^* \colon \ \|x^*\| \leqslant r\}$ and

$$C_i^* + B^*(0,r) = \{y^* \in X^* \mid \exists\, c^* \in C_i^*,\ z^* \in B^*(0,r) \colon\ y^* = c^* + z^*\}.$$

By Property 2.2 we have

$$(4)\quad \forall\, i \in \mathbb{N},\ \exists\, x_i \in B(0, \|a^{**}\|)\colon\ \forall\, c^* \in C_i^*,\ |\langle a^{**}, a^* - c^*\rangle - \langle a^* - c^*, x_i\rangle| \leqslant i^{-1}.$$

It follows from Property 2.3 that for every $i \in \mathbb{N}$ we are able to choose $x_i^* \in C^*$ such that $(x_i, x_i^*) \in S$, where x_i is given in (4). Thus for every $i \in \mathbb{N}$ and $c^* \in C_i^*$ we get

$$\begin{aligned}\langle a^{**} - \varphi(x_i), a^* - x_i^*\rangle &= \langle a^{**}, a^* - x_i^*\rangle - \langle a^* - x_i^*, x_i\rangle \\ &= \langle a^{**}, a^* - c^*\rangle + \langle a^{**}, c^* - x_i^*\rangle - \langle a^* - c^*, x_i\rangle - \langle c^* - x_i^*, x_i\rangle.\end{aligned}$$

Using (3) we infer that for every $i \in \mathbb{N}$ there is $\overline{c}_i^* \in C_i^*$, $\|\overline{c}_i^* - x_i^*\| \leqslant i^{-1}$, thus by (4) we obtain

$$\langle a^{**} - \varphi(x_i), a^* - x_i^*\rangle \leqslant i^{-1} + (\|x_i\| + \|a^{**}\|)\|\overline{c}_i^* - x_i^*\| \leqslant i^{-1}(1 + 2\|a^{**}\|),$$

hence

$$\inf_{(s,s^*)\in S} \langle a^{**} - \varphi(s), a^* - s^*\rangle \leqslant \lim_{i\to\infty} i^{-1}(1 + 2\|a^{**}\|) = 0.$$

$\square$

As a simple consequence of the above theorem and Property 2.5 we get that for any maximal monotone subset with relatively compact range, the closure of the range is convex (in the norm topology).

The Debrunner-Flor theorem, see Property 2.6, is a convenient tool for constructing maximal monotone subsets with their range in a given compact set (to construct NI operators), we refer to [2], [4] for more on the method. Below such a construction is presented.

Property 3.2. *Let X be a real Banach space and $C^* \subset X^*$ a compact (nonempty) convex subset. If $M \subset X \times C^*$ is monotone, then there is a maximally monotone subset $T \subset X \times X^*$ such that $M \subset T$ and $T \subset X \times C^*$.*

Proof. Let us define the family of subsets

$$\mathcal{F} := \{S \subset X \times C^*\colon\ S \text{ is monotone and } M \subset S\}.$$

Let us observe that if $\mathcal{F}_1 \subset \mathcal{F}$ is such that

$$\forall\, A, B \in \mathcal{F}_1, \quad A \subset B \ \text{ or } \ B \subset A,$$

then for every $C \in \mathcal{F}_1$ we get

$$C \subset \{(x, x^*) \in X \times C^* \mid \exists\, A \in \mathcal{F}_1\colon\ (x, x^*) \in A\} \in \mathcal{F}.$$

By the Kuratowski-Zorn lemma there is $T \in \mathcal{F}$ which is maximal, i.e.

$$\forall\, C \subset \mathcal{F}, \quad C \subset T \ \text{ or } \ (C \setminus T \neq \emptyset \Rightarrow T \setminus C \neq \emptyset).$$

We shall show that $D(T) = X$. Let $x \in X$. It follows from Property 2.6 that there is $c^* \in C^*$ such that $(x, c^*) \cup T$ is monotone, so $\{(x, c^*) \cup T\} \in \mathcal{F}$. The choice of T implies that $(x, c^*) \in T$, so $D(T) = X$. In order to complete the proof we need to show that T is maximally monotone. To this end let us fix $(a, a^*) \in X \times X^*$ such that

$$\forall\, (t, t^*) \in T, \quad \langle a^* - t^*, a - t\rangle \geqslant 0. \tag{5}$$

Since C^* is compact (in the strong topology of X^*), so it is compact in the $w(X^*, X)$ topology (in the weak* topology). Let us assume that $a^* \notin C^*$. By the Hahn-Banach theorem (see [7], [The Hahn-Banach Theorems]) there are $x_0 \in X$ and $\varepsilon > 0$ such that

$$\forall\, c^* \in C^*, \quad \langle a^*, x_0\rangle + \varepsilon \leqslant \langle c^*, x_0\rangle. \tag{6}$$

Let us put $t_0 := a - x_0$ and take $c_0^* \in C^*$ such that $(t_0, c_0^*) \in T$. By (5) and (6) we get

$$-\varepsilon \geqslant \langle a^* - c_0^*, x_0\rangle = \langle a^* - c_0^*, a - t_0\rangle \geqslant 0,$$

a contradiction, so $a^* \in C^*$ and $\{(a, a^*)\} \cup T \in \mathcal{F}$. By the choice of T we have $(a, a^*) \in T$, thus T is maximally monotone with the desired properties. $\square$

Let us observe that a direct consequence of Property 3.2 is that if $S \subset X \times X^*$ is maximally monotone and $D(S)$ is included in some compact convex subset of X, say C, then $R(S) = X^*$, so S is of type NI. In order to get it let us observe that

$$M^* := \{(s^*, \varphi(s))\colon\ (s, s^*) \in S\} \subset X^* \times \varphi(C) \subset X^* \times X^{**}$$

is monotone, so there exists $T^* \subset X^* \times \varphi(C)$ which is maximally monotone and $M^* \subset T^*$, thus by Property 2.3, $D(T^*) = X^*$. It is easy to observe that $D(T^*) = R(S)$. Let us also notice, that if C is convex, weakly compact then we still have $R(S) = X^*$ (it is enough to repeat the above reasoning using Whitley's construction, see [3] and [4, Lemma 1.7].

Property 3.3. *Let X be a real Banach space and let a sequence $\{(t_i, t_i^*)\}_{i=1}^{\infty}$ be a monotone set, i.e.*

$$\forall\, i, j \in \mathbb{N}, \quad \langle t_j^* - t_i^*, t_j - t_i \rangle \geqslant 0.$$

Then there exists a sequence of maximal monotone sets $\{T_i\}_{i=1}^{\infty}$ such that

(i) $\forall\, i \in \mathbb{N}$, $\{(t_1, t_1^*), \ldots, (t_i, t_i^*)\} \subset T_i \subset X \times X^*$;

(ii) $\forall\, i \in \mathbb{N}$, $\{z^* \in X^* \mid \exists\, z \in \operatorname{conv}\{t_1, \ldots, t_i\} \colon (z, z^*) \in T_i\} = \operatorname{conv}\{t_1^*, \ldots, t_i^*\}$, *where* conv *stands for the convex hull;*

(iii) $\forall\, i \in \mathbb{N}$, $R(T_i) = \operatorname{conv}\{t_1^*, \ldots, t_i^*\}$.

Proof. Let us put $C_i^* := \operatorname{conv}\{t_1^*, \ldots, t_i^*\}$ and $Q_i^{**} := \operatorname{conv}\{\varphi(t_1), \ldots, \varphi(t_i)\}$, where φ is given in (1). Let us put

$$\begin{aligned}\mathcal{F}_i := \{M \subset X^* \times X^{**} \colon\ & M \subset C_i^* \times Q_i^{**} \text{ and} \\ & M \text{ is a monotone set, and } \{(t_1^*, \varphi(t_1)), \ldots, (t_i^*, \varphi(t_i))\} \subset M\}.\end{aligned}$$

It is easy to observe that for every $i \in \mathbb{N}$ there is a maximal with respect to the inclusion set in $\mathcal{F}_i$, say S_i, i.e.

$$\forall\, C \in \mathcal{F}_i,\ C \subset S_i \quad \text{or} \quad (C \setminus S_i \neq \emptyset \Rightarrow S_i \setminus C \neq \emptyset).$$

We shall show that

$$C_i^* \subset D(S_i) := \{x^* \in C_i^* \mid \exists\, x^{**} \in Q_i^{**} \colon (x^*, x^{**}) \in S_i\}.$$

Let $a^* \in C_i^*$. By Property 2.6 we are able to choose $a^{**} \in Q^{**}$ such that the set $\{(a^*, a^{**})\} \cup S_i$ is monotone. Thus, by the choice of S_i, $(a^*, a^{**}) \in S_i$ and $a^* \in D(S_i)$. Let us put

$$M_i := \{(x, x^*) \in X \times X^* \colon (x^*, \varphi(x)) \in S_i\}$$

and observe that $R(M_i) = D(S_i) = C_i^*$. Of course the set M_i is monotone. By Property 3.2 we are able to construct a maximal monotone subset $T_i \subset X \times X^*$ such that $R(T_i) \subset C_i^*$, $M_i \subset T_i$. Hence, (i)–(iii) are satisied for T_i and we are done. $\square$

Below an approximation technique for maximally monotone operators on separable Banach space is presented. It corresponds to the result by Fitzpatrick and Phelps, see [2, Theorem 2.3], which was obtained in the real Banach space set-up for bounded approximants.

Property 3.4. *Let X be a real Banach space such that its dual space X^* is separable, let $T \subset X \times X^*$ be a maximal monotone subset, and let $\{(s_1, s_1^*), \ldots, (s_k, s_k^*)\} \subset T$ be given. Then for every countable and dense subset $\{(t_1, t_1^*), (t_2, t_2^*), \ldots\} \subset T$ there exists a sequence of maximal monotone subsets of $X \times X^*$, say $\{T_i\}_{i=1}^{\infty}$, such that*

(i) $\forall\, i \in \mathbb{N}, \quad \{(s_1, s_1^*), \ldots (s_k, s_k^*)\} \subset T_i$;

(ii) $\forall\, i \in \mathbb{N}, \quad \{(t_1, t_1^*), \ldots, (t_i, t_i^*)\} \subset T_i$ *and* $R(T_i) = \operatorname{conv}\{s_1^*, \ldots, s_k^*, t_1^*, \ldots, t_i^*\}$;

(iii) $\forall\, (t, t^*) \in T, \ \exists \{(x_i, x_i^*)\}_{i=1}^{\infty} \colon \forall\, i \in \mathbb{N}, \ \{(x_1, x_1^*), \ldots, (x_i, x_i^*)\} \subset T_i \cap T$ *and* $\lim\limits_{i\to\infty} (x_i, x_i^*) = (t, t^*)$.

Proof. Let us choose a countable and dense subset of T, say $\{(t_1, t_1^*), (t_2, t_2^*), \ldots\} \subset T$. For every $(t, t^*) \in T$ we are able to find a subsequence $\{(t_{i_m}, t_{i_m}^*)\}_{m=1}^{\infty} \subset \{(t_1, t_1^*), (t_2, t_2^*), \ldots\}$ such that $(t_{i_m}, t_{i_m}^*) \to (t, t^*)$ whenever $m \to \infty$. It follows from Property 3.3 that for every $i \in \mathbb{N}$ there is a maximal monotone subset $T_i \subset X \times X^*$ for which $\{(s_1, s_1^*), \ldots, (s_k, s_k^*)\} \cup \{(t_1, t_1^*), \ldots, (t_i, t_i^*)\} \subset T_i$ and $R(T_i) = \operatorname{conv}\{s_1^*, \ldots, s_k^*, t_1^*, \ldots, t_i^*\}$, which implies (i)–(ii). In order to get (iii), let us fix $(t, t^*) \in T$ and take a subsequence $\{(t_{i_m}, t_{i_m}^*)\}_{m=1}^{\infty} \subset \{(t_1, t_1^*), (t_2, t_2^*), \ldots\}$ such that $(t_{i_m}, t_{i_m}^*) \to (t, t^*)$ with $i_1 = 1$. For every $m, i \in \mathbb{N}$, $i_m \leqslant i < i_{m+1}$ let us put

$$(x_i, x_i^*) := (t_{i_m}, t_{i_m}^*).$$

Of course $\{(x_1, x_1^*), \ldots, (x_i, x_i^*)\} \subset T_i$ and $\lim\limits_{i\to\infty} (x_i, x_i^*) = (t, t^*)$. □

Let us notice that if $S \subset X \times X^*$ is a maximally monotone set with its range included in a compact subset of X^*, then Theorem 3.1 together with Property 2.4 ensures the existence of pairs from S, $(s, s^*) \in S$, which are not far from the graph of the subdifferential of $2^{-1}\|\cdot\|$; we refer to [5], [9], [8] for several facts on subdifferentials of convex functions. In view of that we can use Property 3.3 to get some additional information on S, namely, we have

Corollary 3.5. *Let X be a real Banach space such that its dual space X^* is separable, let $T \subset X \times X^*$ be a maximal monotone subset, and let $\{(s_1, s_1^*), \ldots, (s_k, s_k^*)\} \subset T$, $(w, w^*) \in X \times X^*$ be given. For every countable and dense subset $\{(t_1, t_1^*), (t_2, t_2^*), \ldots\} \subset T$ and every sequence $\{\lambda\}_{i=1}^{\infty} \subset (0, \infty)$ there exists a sequence of maximal monotone subsets of $X \times X^*$, say $\{T_i\}_{i=1}^{\infty}$, such that*

(i) $\forall\, i \in \mathbb{N}, \ \{(s_1, s_1^*), \ldots, (s_k, s_k^*)\} \subset T_i$;

(ii) $\forall\, i \in \mathbb{N}, \ \{(t_1, t_1^*), \ldots, (t_i, t_i^*)\} \subset T_i$ *and* $R(T_i) = \operatorname{conv}\{s_1^*, \ldots, s_k^*, t_1^*, \ldots, t_i^*\}$;

(iii) $\forall\, (t, t^*) \in T, \ \exists \{(x_i, x_i^*)\}_{i=1}^{\infty} \colon \forall\, i \in \mathbb{N}, \ \{(x_1, x_1^*), \ldots, (x_i, x_i^*)\} \subset T_i \cap T$ *and* $\lim\limits_{i\to\infty} (x_i, x_i^*) = (t, t^*)$;

(iv) $\forall\, i \in \mathbb{N}, \ \exists\, (z_i, z_i^*) \in T_i \colon \ \lambda_i^2 \|w - z_i\|^2 + \|w^* - z_i^*\|^2 + 2\lambda_i \langle w^* - z_i^*, w - z_i \rangle \leqslant \min\{i^{-2}, \lambda_i^{-2}, \lambda_i^{-1}\}$.

We present three examples which shed some new lights on the domain of a maximal monotone set.

Example 3.6. Let X be a real Banach space such that its dual space X^* is separable and let $T \subset X \times X^*$ be a maximally monotone subset. Assume that $a \in \operatorname{conv} D(T) \setminus D(T)$ and $\{r_i\}_{i=1}^{\infty} \subset (0,\infty)$ is such that $r_i \to \infty$ whenever $i \to \infty$. For any approximation $\{T_i\}_{i=1}^{\infty}$ the existence of which is ensured in Property 3.4, we are able to choose a sequence $\{(z_i, z_i^*)\}_{i=1}^{\infty} \subset X \times X^*$ such that

$$\forall\, i \in \mathbb{N},\ \ (z_i, z_i^*) \in T_i$$

and

$$z_i \to a,\ \ r_i\|a - z_i\| \to \infty,\ \ \|z_i^*\| \to \infty.$$

In fact, since $a \in \operatorname{conv} D(T)$, so there are $(s_1, s_1^*) \in T, \dots, (s_k, s_k^*) \in T$ such that for some $\lambda_1 > 0, \dots, \lambda_k > 0$ with $\sum\limits_{i=1}^{k} \lambda_i = 1$ we get $a = \sum\limits_{i=1}^{k} \lambda_i s_i$. It follows from Corollary 3.5 that there exists a sequence of maximal monotone subsets of $X \times X^*$, say $\{T_i\}_{i=1}^{\infty}$, such that (i)–(iii) of the corollary are satisfied and for every $i \in \mathbb{N}$ we can find $(z_i, z_i^*) \in T_i$ such that

$$r_i^2\|a - z_i\|^2 + \|-z_i^*\|^2 + 2r_i\langle -z_i^*, a - z_i\rangle \leqslant i^{-2}. \tag{7}$$

Because of $(z_i, z_i^*) \in T_i$ and $\{(s_1, s_1^*), \dots, (s_k, s_k^*)\} \subset T_i$ we have

$$\forall\, i \in \mathbb{N},\ \forall\, j \in \{1, \dots, k\},\quad \langle z_i^* - s_j^*, z_i - s_j\rangle \geqslant 0,$$

hence

$$\langle z_i^*, z_i - a\rangle \geqslant -\sum_{j=1}^{k} \lambda_j \langle -s_j^*, a - s_j\rangle + \Big\langle \sum_{j=1}^{k} \lambda_j s_j^*, z_i - a \Big\rangle.$$

Hence, by (7) we obtain

$$\begin{aligned}\forall\, i \in \mathbb{N},\quad & r_i^2\|a - z_i\|^2 + \|-z_i^*\|^2 \\ & + 2r_i\Big(-\sum_{j=1}^{k} \lambda_j\langle -s_j^*, a - s_j\rangle - \Big\|\sum_{j=1}^{k} \lambda_j s_j^*\Big\| \|z_i - a\|\Big) \leqslant i^{-2},\end{aligned}$$

which implies $\|z_i - a\| \to 0$ whenever $i \to \infty$. If there is a bounded subsequence of the sequence $\{z_i^*\}_{i=1}^{\infty}$, say $\{z_{i_m}^*\}_{m=1}^{\infty}$, then by Property 2.1 we are able to choose a subsequence which is weak* convergent to some $z^* \in X^*$ (we assume that $z_{i_m}^* \xrightarrow{\text{weak}^*} z^*$ in order to avoid too many indices). Let us take any $(t, t^*) \in T$ and a sequence

$\{(x_i, x_i^*)\}_{i=1}^{\infty} \subset X \times X^*$ such that for all $i \in \mathbb{N}$, $\{(x_1, x_1^*), \ldots, (x_i, x_i^*)\} \subset T_i \cap T$, and $(x_i, x_i^*) \to (t, t^*)$ whenever $i \to \infty$. We have

$$\forall\, m \in \mathbb{N}, \quad \langle z_{i_m}^* - x_{i_m}^*, z_{i_m} - x_{i_m} \rangle \geqslant 0,$$

so letting $m \to \infty$ we get $\langle z^* - t^*, a - t \rangle \geqslant 0$, thus $(a, z^*) \in T$ and $a \in D(T)$, which is impossible. Hence $\lim\limits_{i\to\infty} \|z_i^*\| = +\infty$ and by (7) we get

$$\lim_{i\to\infty} \frac{\|z_i^*\|}{r_i\|a - z_i\|} = 1,$$

so $\lim\limits_{i\to\infty} r_i\|a - z_i\| = +\infty$.

Example 3.7. Let X be a real Banach space such that its dual space X^* is separable and let $T \subset X \times X^*$ be a maximally monotone subset. Assume that $a \in \operatorname{conv} D(T)$ and there is a sequence $\{(w_i, w_i^*)\}_{i=1}^{\infty} \subset T$ such that $\|w_i^*\| \to \infty$ whenever $i \to \infty$, and

$$\limsup_{i\to\infty} \|w_i^*\|^{-1}\|w_i - a\|^{-1}\langle w_i^*, w_i - a\rangle < 0. \tag{8}$$

Then $a \in \operatorname{cl} D(T)$. Moreover, if $a \notin D(T)$, then $a = \lim\limits_{i\to\infty} w_i$.

In fact, since $a \in \operatorname{conv} D(T)$, so there are $(s_1, s_1^*) \in T, \ldots, (s_k, s_k^*) \in T$ such that for some $\lambda_1 > 0, \ldots \lambda_k > 0$ with $\sum\limits_{i=1}^{k} \lambda_i = 1$ we get $a = \sum\limits_{i=1}^{k} \lambda_i s_i$. Let us put $a^* := \sum\limits_{i=1}^{k} \lambda_i s_i^*$ and choose a dense countable subset of T, say $Z \subset T$, such that $(s_1, s_1^*) \in Z, \ldots, (s_k, s_k^*) \in Z$, $\{(w_i, w_i^*)\}_{i=1}^{\infty} \subset Z$. By Property 3.3 we are able to construct a sequence of maximally monotone subsets of $X \times X^*$, say $\{T_i\}_{i=1}^{\infty}$, such that

(a) $Z \subset \bigcup\limits_{i=1}^{\infty} T_i$;
(b) $\forall\, i \in \mathbb{N}$, $\{(s_1, s_1^*), \ldots, (s_k, s_k^*)\} \subset T_i$;
(c) $\forall\, i \in \mathbb{N}$, $T_i \cap Z \subset T_{i+1}$;
(d) the range of T_i is a compact subset of X^*.

For every $i \in \mathbb{N}$ let us put $r_i := \|w_i^*\|$ and choose a subsequence $\{j_i\}$ for which

$$\forall\, i \in \mathbb{N}, \quad (w_i, w_i^*) \in T_{j_i}. \tag{9}$$

Since T_{j_i} are maximally monotone of type NI, so the sets

$$S_i := \{(r_i t, t^*) \colon\ (t, t^*) \in T_{j_i}\}$$

are also maximally monotone of type NI. By Property 2.4 we are able to find $(z_i, z_i^*) \in T_{j_i}$ such that

$$\forall\, i \in \mathbb{N}, \quad r_i^2\|a - z_i\|^2 + \|a^* - z_i^*\|^2 + 2r_i\langle a^* - z_i^*, a - z_i\rangle \leqslant i^{-1}, \tag{10}$$

hence, similarly to Example 3.6, $z_i \to a$ whenever $i \to \infty$. If $\liminf_{i\to\infty}(\|a - z_i\|^2 + \|a^* - z_i^*\|^2) > 0$, then by (10)

$$\lim_{i\to\infty} \frac{\|a^* - z_i^*\|}{r_i\|a - z_i\|} = 1. \tag{11}$$

If $\liminf_{i\to\infty}(\|a - z_i\|^2 + \|a^* - z_i^*\|^2) = 0$, then $(a, a^*) \in T$, so $a \in D(T)$.

In order to complete the proof it is enough to consider the case $a \notin D(T)$. Let us observe that in this case the reasoning above yields the implication if $a \notin D(T)$ then (11) holds true. By (9) we have

$$\forall\, i \in \mathbb{N}, \quad 0 \leqslant \langle w_i^* - z_i^*, w_i - z_i\rangle,$$

and by (10) we obtain

$$\begin{aligned}
\forall i \in \mathbb{N}, \quad 0 &\leqslant r_i(\langle w_i^* - a^*, w_i - a\rangle + \langle z_i^* - a^*, a - w_i\rangle \\
&\quad + \langle z_i^* - a^*, z_i - a\rangle + \langle w_i^* - a^*, a - z_i\rangle) \\
&\leqslant r_i\langle w_i^* - a^*, w_i - a\rangle - 2^{-1}(r_i^2\|a - z_i\|^2 + \|a^* - z_i^*\|^2) \\
&\quad + r_i\|a^* - z_i^*\|\|a - w_i\| + r_i\|a^* - w_i^*\|\|a - z_i\| + i^{-1}.
\end{aligned}$$

If $\liminf_{i\to\infty} \|a - w_i\| > 0$, then we have

$$\begin{aligned}
\forall\, i \in \mathbb{N}, \quad 0 &\leqslant (r_i\|a - w_i\|)^{-1}\langle w_i^* - a^*, w_i - a\rangle + r_i^{-1}\|a^* - z_i^*\| \\
&\quad + (r_i\|a - w_i\|)^{-1}\|a^* - w_i^*\|\|a - z_i\| + (r_i\|a - w_i\|i)^{-1}.
\end{aligned}$$

It follows from (11) that $\lim_{i\to\infty} r_i^{-1}\|a^* - z_i^*\| = 0$, thus by (8)

$$\begin{aligned}
0 &\leqslant \liminf_{i\to\infty}(\|w_i^*\|\|a - w_i\|)^{-1}\langle w_i^* - a^*, w_i - a\rangle \\
&= \liminf_{i\to\infty}(\|w_i^*\|\|a - w_i\|)^{-1}\langle w_i^*, w_i - a\rangle < 0,
\end{aligned}$$

a contradiction, so $w_i \to a$ whenever $i \to \infty$.

Example 3.8. Let X be a real Banach space such that its dual space X^* is separable, and let $S \subset X \times X^*$ be a maximally monotone subset with $0 \in \operatorname{conv} D(S)$. Assume that there is a convex function $p\colon [0,\infty) \to \mathbb{R}$ such that $\liminf_{t\to\infty} p(t)/t > 0$ and

$$\exists\, \bar{s} \in D(S)\colon \forall s^* \in R(S), \ \ \langle s^*, \bar{s}\rangle \geqslant p(\|s^*\|). \tag{12}$$

Then $0 \in D(S)$.

In fact, let $(\bar{s}, \bar{s}^*) \in S$. If $0 \notin D(S)$ then a reasoning similar to that in Example 3.6 gives a sequence $\{(z_i, z_i^*)\}_{i=1}^{\infty} \subset X \times X^*$ such that $z_i \to 0$, $\|z_i^*\| \to \infty$ whenever $i \to \infty$, and

$$\forall\, i \in \mathbb{N}, \quad z_i^* \in \operatorname{conv} R(S) \quad \text{and} \quad \langle \bar{s}^* - z_i^*, \bar{s} - z_i \rangle \geqslant 0.$$

It follows from (12) that

$$\forall\, i \in \mathbb{N}, \quad \langle z_i^*, \bar{s} \rangle \geqslant p(\|z_i^*\|)$$

and

$$0 > -\liminf_{i\to\infty} \frac{p(\|z_i^*\|)}{\|z_i^*\|} \geqslant \liminf_{i\to\infty} \frac{\langle \bar{s}^* - z_i^*, \bar{s} - z_i \rangle}{\|z_i^*\|} \geqslant 0,$$

a contradiction, so $0 \in D(S)$.

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Author's address: D. Zagrodny, Faculty of Mathematics and Natural Science, College of Science, Cardinal Stefan Wyszyński University, Dewajtis 5, 01-815 Warsaw, Poland, e-mail: `d.zagrodny@uksw.edu.pl`.