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ON GENERALIZED JORDAN DERIVATIONS OF LIE TRIPLE SYSTEMS

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Abstract. Under some conditions we prove that every generalized Jordan triple derivation on a Lie triple system is a generalized derivation. Specially, we conclude that every Jordan triple θ-derivation on a Lie triple system is a θ-derivation.

Keywords: Lie triple system, (φ,ψ)-derivation, Jordan triple (φ,ψ)-derivation, θ-derivation, Jordan triple θ-derivation

MSC 2010: 16W25, 17A40

1. Introduction

Throughout this paper $\mathcal{R}$ will represent an associative ring. Given an integer $n>1$, a ring $\mathcal{R}$ is said to be *n-torsion free*, if for $x\in\mathcal{R}$, $nx=0$ implies that $x=0$. Recall that a ring $\mathcal{R}$ is *prime* if for $a,b\in\mathcal{R}$, $aRb=(0)$ implies that either $a=0$ or $b=0$, and is *semiprime* in case $a\in\mathcal{R}$, $aRa=(0)$ implies that $a=0$. An additive mapping $D\colon\mathcal{R}\to\mathcal{R}$ is called a *derivation* if $D(ab)=D(a)b+aD(b)$ holds for all pairs $a,b\in\mathcal{R}$. An additive mapping $D\colon\mathcal{R}\to\mathcal{R}$ is called a *Jordan derivation* if $D(a^2)=D(a)a+aD(a)$ holds for all $a\in\mathcal{R}$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true. I. N. Herstein [7] proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. M. Brešar [3] extended this result to 2-torsion free semiprime rings. An additive mapping $D\colon\mathcal{R}\to\mathcal{R}$ is called a *Jordan triple derivation* if $D(aba)=D(a)ba+aD(b)a+abD(a)$ holds for all pairs $a,b\in\mathcal{R}$. M. Brešar [4] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Let θ,φ be automorphisms of $\mathcal{R}$. An additive mapping $\delta\colon\mathcal{R}\to\mathcal{R}$ is called a *(θ,φ)-derivation* if $\delta(xy)=\delta(x)\theta(y)+\varphi(x)\delta(y)$ holds for all $x,y\in\mathcal{R}$. An additive mapping $\delta\colon\mathcal{R}\to\mathcal{R}$ is called a *Jordan (θ,φ)-derivation* if $\delta(x^2)=\delta(x)\theta(x)+\varphi(x)\delta(x)$ holds for all $x\in\mathcal{R}$. An additive mapping $\delta\colon\mathcal{R}\to\mathcal{R}$

is called a *Jordan triple* (θ, φ)*-derivation* if $\delta(xyx) = \delta(x)\theta(y)\theta(x) + \varphi(x)\delta(y)\theta(x) + \varphi(x)\varphi(y)\delta(x)$ holds for all $x, y \in \mathcal{R}$. It is easy to see that every (θ, φ)-derivation is a Jordan (triple) (θ, φ)-derivation. A result of [5] states that every Jordan (θ, φ)-derivation is a Jordan triple (θ, φ)-derivation. An additive mapping $\xi\colon \mathcal{R} \to \mathcal{R}$ is called a *generalized* (θ, φ)*-derivation* if there exists a (θ, φ)-derivation $\delta\colon \mathcal{R} \to \mathcal{R}$ such that $\xi(xy) = \xi(x)\theta(y) + \varphi(x)\delta(y)$ holds for all $x, y \in \mathcal{R}$ (see [6], [11]). In [13], Liu and Shiue proved that every Jordan (triple) (θ, φ)-derivation on a 2-torsion free semiprime ring is a (θ, φ)-derivation. Also, they introduced a concept of generalized Jordan (θ, φ)-derivation and generalized Jordan triple (θ, φ)-derivation (see also [10]). An additive mapping $\xi\colon \mathcal{R} \to \mathcal{R}$ is called a *generalized Jordan* (θ, φ)*-derivation* if $\xi(x^2) = \xi(x)\theta(x) + \varphi(x)\delta(x)$ holds for all $x \in \mathcal{R}$ where $\delta\colon \mathcal{R} \to \mathcal{R}$ is a Jordan (θ, φ)-derivation. An additive mapping $\xi\colon \mathcal{R} \to \mathcal{R}$ is called a *generalized Jordan triple* (θ, φ)*-derivation* if $\xi(xyx) = \xi(x)\theta(y)\theta(x) + \varphi(x)\delta(y)\theta(x) + \varphi(x)\varphi(y)\delta(x)$ holds for all pairs $x, y \in \mathcal{R}$ where $\delta\colon \mathcal{R} \to \mathcal{R}$ is a Jordan triple (θ, φ)-derivation. A result of [1] states that every generalized Jordan (θ, φ)-derivation is a generalized Jordan triple (θ, φ)-derivation. Liu and Shiue [13] proved that every generalized Jordan (triple) (θ, φ)-derivation on a 2-torsion free semiprime ring is a generalized (θ, φ)-derivation (see also [14]).

The concept of Lie triple system was first introduced by N. Jacobson [8], [9] (see also [12]). A Lie triple system is a $\mathcal{R}$-module $\mathcal{L}$ with a $\mathcal{R}$-trilinear mapping $\mathcal{L}\times\mathcal{L}\times\mathcal{L} \ni (x, y, z) \longmapsto [x, y, z] \in \mathcal{L}$ satisfying the following axioms

(a) $[x, x, y] = 0$,

(b) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

(c) $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all $u, v, x, y, z \in \mathcal{L}$. It follows from (a) that $[x, y, z] = -[y, x, z]$ for all $x, y, z \in \mathcal{L}$.

It is clear that every Lie algebra with product $[., .]$ is a Lie triple system with respect to $[x, y, z] := [[x, y], z]$. Conversely, any Lie triple system $\mathcal{L}$ can be considered as a subspace of a Lie algebra (Bertram [2], Jacobson [9]).

In the rest of this paper, we let $\mathcal{L}$ be an $\mathcal{R}$-module which is a Lie triple system and $\theta, \varphi\colon \mathcal{L} \to \mathcal{L}$ be $\mathcal{R}$-linear mappings.

Definition 1.1. An $\mathcal{R}$-linear mapping $D\colon \mathcal{L} \to \mathcal{L}$ is called a (θ, φ)*-derivation* if

$$D([x, y, z]) = [D(x), \theta(y), \theta(z)] + [\varphi(x), D(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$. If $\varphi = \theta$, a (θ, φ)-derivation is called a θ*-derivation.* If $\varphi = \theta = I_{\mathcal{L}}$, where $I_{\mathcal{L}}$ is the identity map on $\mathcal{L}$, a (θ, φ)-derivation is called a derivation (see [12]).

Let $u, v \in \mathcal{L}$ and $D_{u,v}\colon \mathcal{L} \to \mathcal{L}$ be the mapping defined by

$$D_{u,v}(x) := [u, v, x]$$

for all $x \in \mathcal{L}$. It is clear that $D_{u,v}$ is $\mathcal{R}$-linear and we get from (c) that the mapping $D_{u,v}$ is a derivation on $\mathcal{L}$.

Following Jing and Lu [10], we introduce a concept of Jordan triple (θ, φ)-derivation.

Definition 1.2. An $\mathcal{R}$-linear mapping $D\colon \mathcal{L} \to \mathcal{L}$ is called a *Jordan triple* (θ, φ)*-derivation* if

$$D([x, y, x]) = [D(x), \theta(y), \theta(x)] + [\varphi(x), D(y), \theta(x)] + [\varphi(x), \varphi(y), D(x)]$$

for all $x, y \in \mathcal{L}$. If $\varphi = \theta$, a Jordan triple (θ, φ)-derivation is called a *Jordan triple* θ*-derivation.* If $\varphi = \theta = I_{\mathcal{L}}$, a Jordan triple (θ, φ)-derivation is called a *Jordan triple derivation.*

Following Liu and Shiue [13], we introduce a concept of generalized (θ, φ)-derivations and generalized Jordan triple (θ, φ)-derivations on Lie triple systems.

Definition 1.3. Let $\delta\colon \mathcal{L} \to \mathcal{L}$ be a (θ, φ)-derivation. An $\mathcal{R}$-linear mapping $D\colon \mathcal{L} \to \mathcal{L}$ is called a *generalized* (θ, φ)*-derivation with respect to* δ if

$$D([x, y, z]) = [\delta(x), \theta(y), \theta(z)] + [\varphi(x), \delta(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$.

Definition 1.4. Let $\delta\colon \mathcal{L} \to \mathcal{L}$ be a Jordan triple (θ, φ)-derivation. An $\mathcal{R}$-linear mapping $D\colon \mathcal{L} \to \mathcal{L}$ is called a *generalized Jordan triple* (θ, φ)*-derivation with respect to* δ if

$$D([x, y, x]) = [\delta(x), \theta(y), \theta(x)] + [\varphi(x), \delta(y), \theta(x)] + [\varphi(x), \varphi(y), D(x)]$$

for all $x, y \in \mathcal{L}$.

2. Main results

It is clear that every (θ, φ)-derivation on a Lie triple system is a Jordan triple (θ, φ)-derivation. In this section under some conditions we prove that every generalized Jordan triple (θ, φ)-derivation on a Lie triple system $\mathcal{L}$ is a generalized (θ, φ)-derivation. So we conclude that every Jordan triple θ-derivation on $\mathcal{L}$ is a θ-derivation.

Throughout this section $\theta, \varphi, D, \delta\colon \mathcal{L} \to \mathcal{L}$ are $\mathcal{R}$-linear mappings and $A^{\delta,D}_{\theta,\varphi}\colon \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a mapping defined by

$$A^{\delta,D}_{\theta,\varphi}(x, y, z) := [\delta(x), \theta(y), \theta(z)] + [\varphi(x), \delta(y), \theta(z)] + [\varphi(x), \varphi(y), D(z)]$$

for all $x, y, z \in \mathcal{L}$. It is clear that the mapping $A^{\delta,D}_{\theta,\varphi}$ is $\mathcal{R}$-trilinear and $A^{\delta,D}_{\theta,\varphi}(x,x,y) = [\delta(x), (\theta-\varphi)(x), \theta(y)]$ for all $x, y \in \mathcal{L}$. We denote $A^{\delta,\delta}_{\theta,\varphi}$ by $A^{\delta}_{\theta,\varphi}$.

Lemma 2.1. *Let $\mathcal{R}$ be a 3-torsion free ring and let $D\colon \mathcal{L} \to \mathcal{L}$ be a generalized Jordan triple (θ,φ)-derivation with respect to the Jordan triple (θ,φ)-derivation δ. If*

$$(2.1)\quad [\varphi(x), \varphi(y), (D-\delta)(z)] + [\varphi(y), \varphi(z), (D-\delta)(x)] + [\varphi(z), \varphi(x), (D-\delta)(y)] = 0$$

for all $x, y, z \in \mathcal{L}$, then

$$(2.2)\qquad (D-\delta)([x,y,z]) = [\varphi(x), \varphi(y), (D-\delta)(z)],$$

$$(2.3)\qquad B(x,y,z) + B(y,z,x) + B(z,x,y) = 0$$

for all $x, y, z \in \mathcal{L}$, where $B := A^{\delta,D}_{\theta,\varphi} - A^{\delta}_{\theta,\varphi}$.

Proof. It is clear that $(D-\delta)([x,y,x]) = [\varphi(x), \varphi(y), (D-\delta)(x)]$ for all $x, y, z \in \mathcal{L}$. We must show that the $\mathcal{R}$-trilinear mapping

$$F\colon \mathcal{L}\times\mathcal{L}\times\mathcal{L} \to \mathcal{L}, \quad (x,y,z) \longmapsto (D-\delta)([x,y,z]) - [\varphi(x), \varphi(y), (D-\delta)(z)]$$

vanishes identically. We have $F(x,y,x) = 0$ and by (a), $F(x,x,y) = 0$ for all $x, y \in \mathcal{L}$. Since F is $\mathcal{R}$-trilinear and $F(x+y, x, x+y) = 0$, we get that $F(y,x,x) = 0$ for all $x, y \in \mathcal{L}$. Therefore $F(x,y,z) = F(y,z,x) = F(z,x,y)$ and it follows from (b) and (2.1) that

$$3F(x,y,z) = F(x,y,z) + F(y,z,x) + F(z,x,y) = 0,$$

for all $x, y, z \in \mathcal{L}$. So F is identically 0 since there is no 3-torsion. To prove (2.3), we have from (2.2) that

$$B(x,y,z) = (D-\delta)[x,y,z]$$

for all $x, y, z \in \mathcal{L}$. Since $D-\delta$ is additive, we get (2.3) from (b). $\square$

Remark 2.2. If $D\colon \mathcal{L} \to \mathcal{L}$ is a generalized Jordan triple (θ,φ)-derivation with respect to the Jordan triple (θ,φ)-derivation δ satisfying (2.2), then one checks that (2.1) holds.

Lemma 2.3. *Let $\mathcal{R}$ be a 3-torsion free ring and let $F\colon \mathcal{L}\times\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ be an $\mathcal{R}$-trilinear mapping satisfying $F(x,y,x)=F(x,x,y)=0$ and*

$$F(x,y,z)+F(y,z,x)+F(z,x,y)=0 \tag{2.4}$$

for all $x,y,z\in\mathcal{L}$. Then F is identically 0.

Proof. Since F is $\mathcal{R}$-trilinear and $F(x+y,x,x+y)=0$, we get that $F(y,x,x)=0$ for all $x,y\in\mathcal{L}$. Therefore $F(x,y,z)=F(y,z,x)=F(z,x,y)$ for all $x,y,z\in\mathcal{L}$. Hence it follows from (2.4) that $3F(x,y,z)=0$ for all $x,y,z\in\mathcal{L}$. So F is identically 0 since there is no 3-torsion. □

Theorem 2.4. *Let $\mathcal{R}$ be a 3-torsion free ring and let $D\colon \mathcal{L}\to\mathcal{L}$ be a Jordan triple (θ,φ)-derivation. Then D is a (θ,φ)-derivation if and only if*

(i) $[D(x),(\theta-\varphi)(x),\theta(y)]=0$;

(ii) $A^D_{\theta,\varphi}(x,y,z)+A^D_{\theta,\varphi}(y,z,x)+A^D_{\theta,\varphi}(z,x,y)=0$

for all $x,y,z\in\mathcal{L}$.

Proof. Let D be a (θ,φ)-derivation, then $A^D_{\theta,\varphi}(x,y,z)=D([x,y,z])$ for all $x,y,z\in\mathcal{L}$. So $A^D_{\theta,\varphi}(x,x,y)=0$ for all $x,y\in\mathcal{L}$. This proves (i). Since $[x,y,z]+[y,z,x]+[z,x,y]=0$, (ii) is valid for all $x,y,z\in\mathcal{L}$.

Conversely, we prove that D is a (θ,φ)-derivation if (i) and (ii) hold. We show that the $\mathcal{R}$-trilinear mapping

$$F\colon \mathcal{L}\times\mathcal{L}\times\mathcal{L}\to\mathcal{L},\quad (x,y,z)\longmapsto D([x,y,z])-A^D_{\theta,\varphi}(x,y,z)$$

vanishes identically. It follows from (a) and (i) that $F(x,x,y)=0$ and since $D\colon \mathcal{L}\to\mathcal{L}$ is a Jordan triple (θ,φ)-derivation, we have $F(x,y,x)=0$ for all $x,y\in\mathcal{L}$. By (b) and (ii), F satisfies (2.4). Hence by Lemma 2.3, F vanishes identically. □

Theorem 2.5. *Let $\mathcal{R}$ be a 3-torsion free ring and let $D\colon \mathcal{L}\to\mathcal{L}$ be a generalized Jordan triple (θ,φ)-derivation with respect to the Jordan triple (θ,φ)-derivation δ satisfying* (2.1)*. If*

(i) $[\delta(x),(\theta-\varphi)(x),\theta(y)]=0$;

(ii) $A^\delta_{\theta,\varphi}(x,y,z)+A^\delta_{\theta,\varphi}(y,z,x)+A^\delta_{\theta,\varphi}(z,x,y)=0$

for all $x,y,z\in\mathcal{L}$, then δ is a (θ,φ)-derivation and D is a generalized (θ,φ)-derivation with respect to the (θ,φ)-derivation δ.

Proof. It follows from Theorem 2.4 that δ is a (θ,φ)-derivation. Applying Lemma 2.1, we get from (2.3) and (ii) that

$$A^{\delta,D}_{\theta,\varphi}(x,y,z)+A^{\delta,D}_{\theta,\varphi}(y,z,x)+A^{\delta,D}_{\theta,\varphi}(z,x,y)=0 \tag{2.5}$$

for all $x,y,z\in\mathcal{L}$. The rest of the proof is similar to the proof of Theorem 2.4. □

Crollary 2.6. *Let $\mathcal{R}$ be a 3-torsion free ring and let $D\colon \mathcal{L} \to \mathcal{L}$ be a generalized Jordan triple θ-derivation with respect to the Jordan triple θ-derivation δ satisfying*

$$[\theta(x), \theta(y), (D-\delta)(z)] + [\theta(y), \theta(z), (D-\delta)(x)] + [\theta(z), \theta(x), (D-\delta)(y)] = 0$$

for all $x, y, z \in \mathcal{L}$. Then δ is a θ-derivation and D is a generalized θ-derivation with respect to the θ-derivation δ.

P r o o f. It is clear that condition (i) of Theorem 2.5 is valid when $\theta = \varphi$. For condition (ii) of Theorem 2.5, we have from (b) that

$$\begin{aligned} &A^{\delta}_{\theta,\theta}(x,y,z) + A^{\delta}_{\theta,\theta}(y,z,x) + A^{\delta}_{\theta,\theta}(z,x,y) \\ &\quad = \big([\delta(x), \theta(y), \theta(z)] + [\theta(y), \theta(z), \delta(x)] + [\theta(z), \delta(x), \theta(y)]\big) \\ &\qquad + \big([\theta(x), \delta(y), \theta(z)] + [\delta(y), \theta(z), \theta(x)] + [\theta(z), \theta(x), \delta(y)]\big) \\ &\qquad + \big([\theta(x), \theta(y), \delta(z)] + [\theta(y), \delta(z), \theta(x)] + [\delta(z), \theta(x), \theta(y)]\big) = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{L}$. So condition (ii) of Theorem 2.4 is valid if $\varphi = \theta$. Hence δ is a θ-derivation and D is a generalized θ-derivation with respect to δ. $\square$

Corollary 2.7. *Let $\mathcal{R}$ be a 3-torsion free ring. Then $D\colon \mathcal{L} \to \mathcal{L}$ is a Jordan triple θ-derivation if and only if D is a θ-derivation.*

Corollary 2.8. *Let $\mathcal{R}$ be a 3-torsion free ring. Then $D\colon \mathcal{L} \to \mathcal{L}$ is a Jordan triple derivation if and only if D is a derivation.*

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