Timothy Myers

An elementary proof of the theorem that absolute gauge integrability implies Lebesgue integrability

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 3, 621–633

Persistent URL: http://dml.cz/dmlcz/140594

Czechoslovak Mathematical Journal, 60 (135) (2010), 621–633

AN ELEMENTARY PROOF OF THE THEOREM THAT ABSOLUTE GAUGE INTEGRABILITY IMPLIES LEBESGUE INTEGRABILITY

Timothy Myers, Washington

(Received November 4, 2008)

Abstract. It is commonly known that absolute gauge integrability, or Henstock-Kurzweil (H-K) integrability implies Lebesgue integrability. In this article, we are going to present another proof of that fact which utilizes the *basic* definitions and properties of the Lebesgue and H-K integrals.

Keywords: absolute integrability, gauge Integral, H-K integral, Lebesgue integral

MSC 2010: 26A39, 26A42

1. Introduction

Since its introduction in the late 1950's [6], [7], the Henstock-Kurzweil integral has found many applications; especially in the theory of ordinary differential equations as well as the theory of integral equations because in particular it can integrate any real-valued function that is Lebesgue integrable on a closed rectangle I^n in $\mathbb{R}^n$ [2], [3], [4], [5], [8], [15]. In fact, the gauge integral is necessary to yield a solution to the classical initial Cauchy problem that is not obtainable with the classical existence results [Peano existence Theorem, Carathéodory existence Theorem] which depend on Lebesgue integration. The primary purpose of this work is to prove a restricted but potent converse to this relationship between the gauge and Lebesgue integrals which can be used to easily extend important properties of Lebesgue integration to the gauge integral. In particular, the fact that a real valued function is absolutely gauge integrable on $I^n \subset \mathbb{R}^n$ if and only if it is Lebesgue integrable on I^n yields a direct, simple proof of the dominated and monotone convergence theorems for the gauge integral [11], [12]. This converse is the following:

Theorem 1.1. *If a real valued function is absolutely gauge integrable on a cell $I^n \subset \mathbb{R}^n$, then it is Lebesgue integrable on I^n.*

The basic approach to proving Theorem 1.1 presented in this work augments the purposes served by previous proofs of this result. In [9], R. McLeod develops Theorem 1.1 independent of the Lebesgue integral by first proving the dominated convergence theorem for the gauge integral. In [11], [12] W. Pfeffer defines a gauge type of integral in a general setting, and uses the concept of an indefinite gauge integral defined on admissible sets to establish Theorem 1.1. The approach presented here is shorter and more elementary than [9], [11], [12]. This author's proof of Theorem 1.1 depends only on seminal concepts such as the fact that Lebesgue integrability on I^n implies gauge integrability, the linearity of the gauge and Lebesgue integrals, Lebesgue's monotone convergence theorem, and the basic properties for the gauge and Lebesgue integrals listed in the preliminaries.

2. Preliminaries

In this section, we are going to outline some basic definitions and properties which we will use to prove Theorem 1.1.

Definitions 2.1 [1], [13], [16]**.**

(a) Let $a_k, b_k \in \mathbb{R}\colon\ a_k < b_k$ for $k = 1, \ldots, n$. The set $I = \{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon\ a_k \leqslant x_k \leqslant b_k,\ k = 1, \ldots, n\}$ is called a *closed interval*, *n-cell* or a *cell* in $\mathbb{R}^n$, and will often be denoted by the symbol I^n to emphasize dimension. If $a_k < x_k < b_k$ then I is called an *open interval*. A k-th face of I, s_k^p, will denote the set of all points in I where the k-th coordinate is exactly a_k if $p = 0$, or b_k if $p = 1$. Note that the boundary of I equals the union of all faces of I [10].

(b) A *partition* of the n-cell I^n is a finite collection of q sub-cells C_i in $\mathbb{R}^n$ where $I^n = \bigcup_{i=1}^{q} C_i$, and the sub-cells C_i are pairwise disjoint except for common boundary point(s).

(c) A *tagged partition* of I^n, $P = \{(C_i, \mathbf{t}_i)\}_{i=1}^{q}$, is a collection of ordered pairs where the cells C_i form a partition of I^n and the points $\mathbf{t}_i \in C_i$ are defined to be the *tags*.

(d) The *distance* of the cell, C_i, denoted by $\operatorname{diam}(C_i)$, is defined as $\operatorname{diam}(C_i) = \sup\{|\mathbf{x} - \mathbf{y}|\colon\ \mathbf{x}, \mathbf{y} \in C_i\}$, where the diameter is computed using the Euclidean metric on $\mathbb{R}^n$.

(e) If the n-cell $C = \{(x_1, \ldots, x_n)\colon\ a_k \leqslant x_k \leqslant b_k, k = 1, \ldots, n\}$ then the *volume* of C, denoted by $|C|$, is defined as $|C| = \prod_{k=1}^{n} (b_k - a_k)$.

(f) Let f be the real valued function $f\colon I^n \to \mathbb{R}$. The *Riemann sum* of f corresponding to the tagged partition $P = \{(C_i, \mathbf{t}_i)\}_{i=1}^{q}$ of I^n is the real number $\sum_{i=1}^{q} f(\mathbf{t}_i)|C_i|$.

(g) A strictly positive function δ on I^n is called a *gauge* on I^n.

(h) If δ is a gauge on I^n and $P = \{(C_i, \mathbf{t}_i)\}_{i=1}^{q}$ is a tagged partition of I^n, then P is said to be a δ-*fine* partition if $\operatorname{diam}(C_i) < \delta(\mathbf{t}_i)$ for $i = 1, \ldots, q$.

The following lemma helps to make the definition of the gauge integral so versatile.

Cousin's Lemma 2.2 [1]. *Let I^n be an n-cell in $\mathbb{R}^n$. Let δ be a gauge on I^n. Then there exists a δ-fine partition of I^n.*

Definition 2.3 [1]. Let f be the real valued function $f\colon I^n \to \mathbb{R}$. The real number $A \in \mathbb{R}$ is the *generalized Riemann integral* or the *gauge integral* of f on I^n if for every $\varepsilon > 0$ there exists a gauge δ_ε on I^n such that if $P = \{(C_i, \mathbf{t}_i\}_{i=1}^{q}$ is any δ-fine tagged partition of I^n, then $\left|\sum_{i=1}^{q} f(\mathbf{t}_i)|C_i| - A\right| < \varepsilon$.

If such a real number A exists, then f is said to be *gauge integrable* or *generalized Riemann integrable* on I^n, which is indicated by the notation $f \in \mathfrak{R}^*(I^n)$, and the real number A, the gauge integral of f, is denoted as $(G)\int_{I^n} f(\mathbf{x})\,\mathrm{d}V$.

2.4 Basic properties of the gauge integral [9].

(a) Let $m \in \mathbb{N}$, and let $a_1, \ldots, a_m$ be real valued constants. Suppose that the real valued functions $f_1, \ldots, f_m$ are gauge integrable on I^n. Then $a_1 f_1 + \ldots + a_m f_m$ is gauge integrable on I^n, and

$$(G)\int_{I^n} \sum_{i=1}^{m} a_i f_i \,\mathrm{d}V = \sum_{i=1}^{m} a_i (G)\int_{I^n} f_i \,\mathrm{d}V$$

(b) Suppose that f and g are gauge integrable on I^n and that $f \leqslant g$. Then

$$(G)\int_{I^n} f \,\mathrm{d}V \leqslant (G)\int_{I^n} g \,\mathrm{d}V$$

The next lemma is fundamental for the theory of Henstock-Kurzweil integration.

Henstock's Lemma 2.5 [9]. *Let the real valued function f be gauge integrable on I^n. Let $\varepsilon > 0$, and let δ_ε be a gauge on I^n:*

$$\left|\sum_{i=1}^{q} f(\mathbf{t}_i)|C_i| - (G)\int_{I^n} f\,\mathrm{d}V\right| < \varepsilon$$

where $P = \{(C_i, \mathbf{t}_i)\}_{i=1}^{q}$ is any δ_ε fine partition of I^n. Let $\mathcal{P} = \{C_i\}_{i=1}^{q}$, and let $\mathcal{E}$ be any nonempty subset of $\mathcal{P}$. Then the following inequality holds:

$$\left|\sum_{C_i\in\mathcal{E}} \left[f(\mathbf{t}_i)|C_i| - (G)\int_{C_i} f\,\mathrm{d}V\right]\right| \leqslant \varepsilon.$$

The following result, presented by R. McLeod, is reformulated to fit the definition of the gauge integral presented here.

The Covering Lemma 2.6 [9]. *Let $\emptyset \neq E \subset I^n$, and let δ be a gauge defined on I^n. Then there exists a collection of cells $\mathcal{S} = \{C_i\}_{i=1}^{p}$, where $p \in \mathbb{N} \cup \{\infty\}$ and where the following are true:*

(a) *each pair of the cells in $\mathcal{S}$ intersect at most along a common boundary,*

(b) *there exists a point $\mathbf{t}_i \in C_i \cap E$ and $\operatorname{diam}(C_i) < \delta(\mathbf{t}_i)$ for $i = 1, \ldots, p$,*

(c) $E \subset \bigcup_{i=1}^{p} C_j \subset I^n$.

Proposition 2.7 [9]. *Suppose that the real valued function f is gauge integrable on I^n and that $S = \{f_m\}_{m=1}^{\infty}$ is a sequence of real valued functions defined on I^n for which $\lim_{m\to\infty} f_m(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in I^n$. Then there exists $T \in \mathbb{N}$ such that f_m is gauge integrable if $m \geqslant T$.*

Proposition 2.8 [9]. *Let the real valued functions f, g, and h be gauge integrable on I^n where $f(\mathbf{x}) \leqslant h(\mathbf{x})$ and $g(\mathbf{x}) \leqslant h(\mathbf{x})$ for all $\mathbf{x} \in I^n$. Then $f \wedge g$, the pointwise minimum of f and g, is also gauge integrable on I^n.*

The symbols μ^*, $\mathfrak{M}$, and $\mathcal{L}(I^n)$ will respectively denote the Lebesgue outer measure, the collection of all Lebesgue measurable sets in $\mathbb{R}^n$, and the class of all real valued, Lebesgue integrable functions which are defined everywhere on I^n.

Proposition 2.9 [13], [14]. *Let $f\colon I^n \to \mathbb{R}$ be non-negative. Then there exists a monotonically increasing sequence $\{s_m\}_{m=1}^{\infty}$ of simple functions $s_m = \sum_{k=1}^{r_m} c_{m_k}\chi_{E_{m_k}}$, $r_m \in \mathbb{N}$ with non-negative coefficients c_{m_k}, where $E_{m_u} \cap E_{m_v} = \emptyset$ if $u, v \in \{1, \dots, r_m\}\colon u \neq v$, $\bigcup_{k=1}^{r_m} E_{m_k} = I^n$, such that $0 \leqslant s_m(\mathbf{x}) \leqslant f(\mathbf{x})$ and $\lim_{m\to\infty} s_m(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in I^n$. If f is also Lebesgue measurable, this sequence $\{s_m\}_{m=1}^{\infty}$ can be constructed so that each of its simple functions is also Lebesgue measurable.*

Proposition 2.10 [17]. *Let $A \in \mathfrak{M}$, $\mu^*(A) < +\infty$, and $E \subset A$. If $\mu^*(A) = \mu^*(E) + \mu^*(A \setminus E)$, then $E \in \mathfrak{M}$.*

Proposition 2.11 [11], [12]. *If $f\colon I^n \to \mathbb{R}$ is Lebesgue integrable on I^n, then f is gauge integrable on I^n to the same value.*

3. Main result

Theorems 3.1 through 3.6, the most significant part of this proof, will establish the result that a non-negative function $f \in \mathcal{R}^*(I^n)$ is Lebesgue measurable.

The proof of the following theorem is modified to fit the version of the gauge integral presented here.

Theorem 3.1 [9]. *Let $\emptyset \neq E \subset I^n$, and suppose that $(G)\int_{I^n} \chi_E \,\mathrm{d}V$ exists. Then for all $\varepsilon > 0$ there exists a sequence $\{C_j\}_{j=1}^{p}$ of cells where $p \in \mathbb{N} \cup \{\infty\}$ for which $E \subset \bigcup_{j=1}^{p} C_j \subset I^n$ and*

$$\sum_{j=1}^{p} |C_j| < (G)\int_{I^n} \chi_E \,\mathrm{d}V + \varepsilon. \tag{1}$$

P r o o f. Since $(G)\int_{I^n} \chi_E \,\mathrm{d}V$ exists, there exists a gauge, $\delta_{\varepsilon/2}$, such that for any $\delta_{\varepsilon/2}$ fine partition $\{(D_i, \mathbf{t}_i)\}_{i=1}^{q}$ of I^n,

$$\left| \sum_{i=1}^{q} \chi_E(\mathbf{t}_i)|D_i| - (G)\int_{I^n} \chi_E \,\mathrm{d}V \right| < \frac{\varepsilon}{2}. \tag{2}$$

By the covering lemma, there exists a collection $\{C_j\}_{j=1}^{p}$ of cells, such that $p \in \mathbb{N} \cup \{\infty\}$, where each pair of cells intersect at most along a common boundary, and where there exists $\mathbf{t}_j \in E \cap C_j$ such that

$$\text{(3)} \qquad \operatorname{diam}(C_j) < \delta_{\varepsilon/2}(\mathbf{t}_j) \text{ for } j = 1, \ldots, p, \quad \text{and} \quad E \subset \bigcup_{j=1}^{p} C_j \subset I^n.$$

Let $k \in \mathbb{N}\colon\ 1 \leqslant k \leqslant p$ if $p < \infty$, or $1 \leqslant k < p$ if $p = \infty$, and let $S_k = \{(C_j, \mathbf{t}_j)\}_{j=1}^{k}$.

Note that since the boundary of $\bigcup_{j=1}^{k} C_k$ consists of faces of cells, with each such face parallel to $n-1$ coordinate axes; then $I^n \setminus \bigcup_{j=1}^{k} C_k$ can be filled with cells, and each such cell contains a $\delta_{\varepsilon/2}$-fine partition by Cousin's Lemma. These facts and (3) insure that all k of the cells in S_k can be contained in a $\delta_{\varepsilon/2}$ fine partition Q_k of I^n. Thus (2) holds for Q_k, and so Henstock's lemma applied to the subset S_k of Q_k implies that

$$\sum_{j=1}^{k} \chi_E(\mathbf{t}_j)|C_j| \leqslant (G)\int_{I^n} \chi_E \,\mathrm{d}V + \frac{\varepsilon}{2}.$$

Since $\chi_E(\mathbf{t}_j) = 1$ by (3), then

$$\text{(4)} \qquad \sum_{j=1}^{k} |C_j| \leqslant (G)\int_{I^n} \chi_E \,\mathrm{d}V + \frac{\varepsilon}{2}.$$

If $p < \infty$, replace k with p in (4). If $p = \infty$, proceed as follows. Let $r_k = \sum_{j=1}^{k} |C_j|$, and note that $\lim_{k\to\infty} r_k$ exists since the sequence of partial sums $\{r_k\}_{k=1}^{\infty}$ is an increasing sequence that is bounded above by $(G)\int_{I^n} \chi_E \,\mathrm{d}V + \frac{\varepsilon}{2}$. Thus, (4) holds if the series on the left side of the inequality is infinite by taking the limit of both sides as $k \to \infty$. In either case, the inequality in (4) becomes strict by replacing $\frac{\varepsilon}{2}$ with ε, thus establishing (1). □

Theorem 3.2. *Let I^n be any cell, and E any subset of $\mathbb{R}^n$ for which $\emptyset \neq E \subset I^n$, and suppose that χ_E is gauge integrable on I^n. Then*

$$\text{(5)} \qquad (G)\int_{I^n} \chi_E \,\mathrm{d}V = \mu^*(E).$$

Proof. First, we will show that $(G)\int_{I^n} \chi_E \,\mathrm{d}V$ is a lower bound for the set

$$\text{(6)} \qquad \Big\{ \sum_{j=1}^{q} |I_j|\colon\ E \subset \bigcup_{j=1}^{q} I_j\colon\ q \in \mathbb{N} \cup \{\infty\} \text{ and each interval } I_j \text{ is open} \Big\}.$$

Note that since χ_E is gauge integrable on I^n, then $(G)\int_{I^n}\chi_E\,\mathrm{d}V < \sum_{j=1}^{q}|I_j|$ if $\sum_{j=1}^{q}|I_j| = +\infty$. Assume therefore that

$$\sum_{j=1}^{q}|I_j| < +\infty. \tag{7}$$

Since

$$\int_{I^n}\chi_{I_j}\,\mathrm{d}\mu = \mu^*(I_j\cap I^n) \leqslant |I_j| < +\infty, \tag{8}$$

then (7) and (8) imply that

$$\sum_{j=1}^{q}\int_{I^n}\chi_{I_j}\,\mathrm{d}\mu \leqslant \sum_{j=1}^{q}|I_j| < +\infty. \tag{9}$$

The following argument will show that the sum on the left side of (9) is linear even if $q=\infty$. Let $k\in\mathbb{N}$, $f_k = \sum_{j=1}^{k}\chi_{I_j}$, and let $f = \sum_{j=1}^{\infty}\chi_{I_j}$. Since $\{f_k\}_{k=1}^{\infty}$ is a sequence of nonnegative, Lebesgue measurable functions such that for all $\mathbf{x}\in I^n$:

$$\lim_{k\to\infty} f_k(\mathbf{x}) = \lim_{k\to\infty}\sum_{j=1}^{k}\chi_{I_j}(\mathbf{x}) = f(\mathbf{x}),$$

and $0 \leqslant f_1(\mathbf{x}) \leqslant f_2(\mathbf{x}) \leqslant \ldots$, Lebesgue's monotone convergence theorem implies that

$$\lim_{k\to\infty}\int_{I^n} f_k\,\mathrm{d}\mu = \int_{I^n} f\,\mathrm{d}\mu. \tag{10}$$

By linearity of the Lebesgue integral (and substitution on both sides of (10) if $q=\infty$),

$$\int_{I^n}\sum_{j=1}^{q}\chi_{I_j}\,\mathrm{d}\mu = \sum_{j=1}^{q}\int_{I^n}\chi_{I_j}\,\mathrm{d}\mu < +\infty. \tag{11}$$

Since $\chi_{[\bigcup_{j=1}^{q} I_j]} \leqslant \sum_{j=1}^{q}\chi_{I_j}$ and $\bigcup_{j=1}^{q} I_j \in \mathfrak{M}$, then $\chi_{[\bigcup_{j=1}^{q} I_j]}$ is Lebesgue measurable. Therefore

$$\int_{I^n}\chi_{[\bigcup_{j=1}^{q} I_j]}\,\mathrm{d}\mu \leqslant \int_{I^n}\sum_{j=1}^{q}\chi_{I_j}\,\mathrm{d}\mu \tag{12}$$

and so $\chi_{[\bigcup_{j=1}^q I_j]}$ is Lebesgue integrable by transitivity and the final inequality in (11). Since Lebesgue integrability of a real valued function defined on a cell implies gauge integrability,

$$(G)\int_{I^n} \chi_{[\bigcup_{j=1}^q I_j]}\,\mathrm{d}V = \int_{I^n} \chi_{[\bigcup_{j=1}^q I_j]}\,\mathrm{d}\mu. \tag{13}$$

Since χ_E is gauge integrable and $\chi_E \leqslant \chi_{[\bigcup_{j=1}^q I_j]}$, then in turn (13), (12), (11), and (9) imply that

$$(G)\int_{I^n} \chi_E\,\mathrm{d}V \leqslant (G)\int_{I^n} \chi_{[\bigcup_{j=1}^q I_j]}\,\mathrm{d}V \leqslant \sum_{j=1}^{q}\int_{I^n} \chi_{I_j}\,\mathrm{d}\mu \leqslant \sum_{j=1}^{q}|I_j|.$$

Therefore, $(G)\int_{I^n} \chi_E\,\mathrm{d}V$ is a lower bound for the set (6).

Let $\varepsilon > 0$. The following argument will show that $(G)\int_{I^n} \chi_E\,\mathrm{d}V + \varepsilon$ is not a lower bound for (6). By Theorem 3.1, there exists a sequence $\{C_j\}_{j=1}^p$ of cells where $p \in \mathbb{N}\cup\{\infty\}$ for which $E \subset \bigcup\limits_{j=1}^{p} C_j \subset I^n$ and

$$\sum_{j=1}^{p}|C_j| < (G)\int_{I^n} \chi_E\,\mathrm{d}V + \frac{\varepsilon}{2}. \tag{14}$$

Use the definition of Lebesgue measure to choose an open interval $J_j\colon\ C_j \subset J_j$ and $|J_j| < |C_j| + \frac{1}{2}\varepsilon/p$ for each $j = 1,\dots,p$ if $p < \infty$, and $|J_j| < |C_j| + \varepsilon/3^{j+1}$ for each $j \in \mathbb{N}$ if $p = \infty$. In either case,

$$\sum_{j=1}^{p}|J_j| < \sum_{j=1}^{p}|C_j| + \frac{\varepsilon}{2} \quad\Rightarrow\quad \sum_{j=1}^{p}|J_j| - \frac{\varepsilon}{2} < \sum_{j=1}^{p}|C_j|. \tag{15}$$

Inequality (14) and the final inequality in (15) yield the result

$$\sum_{j=1}^{p}|J_j| < (G)\int_{I^n} \chi_E\,\mathrm{d}V + \varepsilon, \tag{16}$$

which means that

$$(G)\int_{I^n} \chi_E\,\mathrm{d}V = \inf\left\{\sum_{j=1}^{q}|I_j|\colon\ E \subset \bigcup_{j=1}^{q} I_j\colon\ \begin{array}{l} q \in \mathbb{N}\cup\{\infty\} \\ \text{and each interval } I_j \text{ is open}\end{array}\right\},$$

thereby establishing (5). □

Theorem 3.3. *Let E be a nonempty subset of I^n, and suppose that χ_E is gauge integrable on I^n. Then $E \in \mathfrak{M}$.*

Proof. Since $(G)\int_{I^n} \chi_{I^n} \,\mathrm{d}V$ and $(G)\int_{I^n} \chi_E \,\mathrm{d}V$ exist and $\chi_{I^n\setminus E} = \chi_{I^n} - \chi_E$, then $(G)\int_{I^n} \chi_{I^n\setminus E} \,\mathrm{d}V$ exists by linearity.

Since $I^n \setminus E$, $E \subset I^n$, Theorem 3.2 implies that $(G)\int_{I^n} \chi_E \,\mathrm{d}V = \mu^*(E)$ and $(G)\int_{I^n} \chi_{I^n\setminus E} \,\mathrm{d}V = \mu^*(I^n \setminus E)$.

Also $(G)\int_{I^n} \chi_{I^n} \,\mathrm{d}V = \mu^*(I^n)$. Therefore,

$$\begin{aligned}\mu^*(E) + \mu^*(I^n \setminus E) &= (G)\int_{I^n} \chi_E \,\mathrm{d}V + (G)\int_{I^n} \chi_{I^n\setminus E} \,\mathrm{d}V \\ &= (G)\int_{I^n} \chi_{I^n} \,\mathrm{d}V = \mu^*(I^n).\end{aligned} \tag{17}$$

Since $I^n \in \mathfrak{M}$, (17) implies that $E \in \mathfrak{M}$ by Proposition 2.10. □

The Lebesgue measurability of a gauge integrable characteristic function will now be extended to a gauge integrable simple function in Theorems 3.4 and 3.5, and then to a non-negative real valued gauge integrable function in Theorem 3.6.

Theorem 3.4. *Let $s = \sum_{k=1}^{m} d_k \chi_{E_k}$, $m \in \mathbb{N}$ be a simple function where $d_k > 0$ for $k = 1, \ldots, m$, $\bigcup_{k=1}^{m} E_k \subset I^n$ and $E_u \cap E_v = \emptyset$ for $u, v \in \{1, \ldots, m\}\colon u \neq v$. If s is gauge integrable on I^n, then the function $\chi_{E_1} + \ldots + \chi_{E_m}$ is also gauge integrable on I^n.*

Proof. Note that since $E_u \cap E_v = \emptyset$ for $u \neq v$, it follows that $\chi_{E_1} + \ldots + \chi_{E_m} = \chi_{\bigcup_{k=1}^m E_k}$. Since each $d_k > 0$ for $k = 1, \ldots, m$, choose $c \in \mathbb{R}$ so that $cd_k > 1$, which implies that $cs(\mathbf{x}) > 1$ for all $\mathbf{x} \in \bigcup_{k=1}^{m} E_k$. Let $d = \max\{cd_k\}_{k=1}^m$. Since $\mathrm{d}\chi_{I^n}$ is gauge integrable on I^n, $\mathrm{d}\chi_{I^n}(\mathbf{x}) > \chi_{I^n}(\mathbf{x})$ and $\mathrm{d}\chi_{I^n}(\mathbf{x}) \geqslant cs(\mathbf{x})$ for all $\mathbf{x} \in I^n$, then $cs \wedge \chi_{I^n}$, the pointwise minimum of the gauge integrable functions cs and χ_{I^n} on I^n, is also gauge integrable on I^n by Proposition 2.8. Thus, the relationship

$$(cs \wedge \chi_{I^n})(\mathbf{x}) = (\chi_{\bigcup_{k=1}^m E_k})(\mathbf{x})$$

for all $\mathbf{x} \in I^n$ implies that the function $\chi_{\bigcup_{k=1}^m E_k} = \chi_{E_1} + \ldots + \chi_{E_m}$ is gauge integrable on I^n.

Theorem 3.5. *Let $s = \sum_{k=1}^{m} d_k \chi_{E_k}$, $m \in \mathbb{N}$ be a simple function with non-negative coefficients d_k, where $\bigcup_{k=1}^{m} E_k \subset I^n$ and $E_u \cap E_v = \emptyset$ for $u \neq v$. If s is gauge integrable on I^n, then s is measurable in the Lebesgue sense.*

Proof. This proof will proceed by induction on m, the number of terms in the simple function s.

Let $d_1 \chi_{F_1}$ be a simple function that is gauge integrable on I^n, where $F_1 \subset I^n$ and $d_1 \geqslant 0$.

If $d_1 = 0$, then $d_1 \chi_{F_1}(\mathbf{x}) = 0$ for each $\mathbf{x} \in I^n$, and so in this case, $d_1 \chi_{F_1}$ is a constant function, which is measurable in the Lebesgue sense.

If $d_1 > 0$, then χ_{F_1} is gauge integrable on I^n by linearity, and hence Theorem 3.3 implies that $F_1 \in \mathfrak{M}$. Therefore, $d_1 \chi_{F_1}$ is Lebesgue measurable.

Let $k \in \mathbb{N}$, and let $s_k = d_1 \chi_{F_1} + \ldots + d_k \chi_{F_k}$ be any simple function where $\bigcup_{j=1}^{k} F_j \subset I^n$, $d_j \geqslant 0$, and $F_u \cap F_v = \emptyset$ if $u, v \in \{1, \ldots, k\}\colon\ u \neq v$. Assume that if any such simple function s_k is gauge integrable on I^n, then s_k is Lebesgue measurable.

Now let $s_{k+1} = b_1 \chi_{E_1} + \ldots + b_{k+1} \chi_{E_{k+1}}$ be any simple function subject to the conditions $\bigcup_{i=1}^{k+1} E_i \subset I^n$, $b_i \geqslant 0$, $E_u \cap E_v = \emptyset$ if $u, v \in \{1, \ldots, k+1\}\colon\ u \neq v$, and s_{k+1} is gauge integrable on I^n. Since s_{k+1} is a simple function, then assume, without loss of generality, that $b_1 > b_2 > \ldots > b_k > b_{k+1} \geqslant 0$. Then the function $r_{k+1} = \chi_{E_1} + \ldots + \chi_{E_{k+1}}$ is gauge integrable on I^n by Theorem 3.4 if $b_{k+1} > 0$, and by the inductive hypothesis if $b_{k+1} = 0$. By linearity, the function $r_{k+1} - b_1^{-1} s_{k+1}$ is also gauge integrable on I^n. Thus, $r_{k+1} - b_1^{-1} s_{k+1} = \bigl(1 - b_2/b_1\bigr)\chi_{E_2} + \ldots + \bigl(1 - b_{k+1}/b_1\bigr)\chi_{E_{k+1}}$.

Now since $0 < 1 - b_2/b_1 < \ldots < 1 - b_{k+1}/b_1 \leqslant 1$, the coefficients in $r_{k+1} - b_1^{-1} s_{k+1}$ are distinct and hence $r_{k+1} - b_1^{-1} s_{k+1}$ is a simple function with non-negative coefficients that is gauge integrable. Thus, the inductive hypothesis implies that $r_{k+1} - {b_1}^{-1} s_{k+1}$ is Lebesgue measurable and so $E_2, \ldots, E_{k+1} \in \mathfrak{M}$. Now since $r_{k+1} = \chi_{\bigcup_{i=1}^{k+1} E_i}$ is gauge integrable on I^n and $\bigcup_{i=1}^{k+1} E_i \subset I^n$, Theorem 3.3 implies that $\bigcup_{i=1}^{k+1} E_i \in \mathfrak{M}$. Since $\mathfrak{M}$ is a sigma ring, $\bigcup_{j=2}^{k+1} E_j \in \mathfrak{M}$, and hence $E_1 = \Bigl(\bigcup_{i=1}^{k+1} E_i\Bigr) \setminus \Bigl(\bigcup_{j=2}^{k+1} E_j\Bigr) \in \mathfrak{M}$ [13], [14]. Thus since $E_1, E_2, \ldots, E_{k+1} \in \mathfrak{M}$, the simple function s_{k+1} is Lebesgue measurable. Therefore by induction, this theorem is true. $\square$

Theorem 3.6. *Let $f\colon I^n \to \mathbb{R}$ be non-negative. If f is gauge integrable on I^n, then f is measurable in the Lebesgue sense.*

P r o o f. Since f is non-negative on I^n, Proposition 2.9 implies that there exists an increasing sequence $\{s_m\}_{m=1}^{\infty}$ of simple functions $s_m = \sum\limits_{k=1}^{r_m} c_{m_k}\chi_{E_{m_k}}$, $r_m \in \mathbb{N}$ with non-negative coefficients c_{m_k}, where $E_{m_u} \cap E_{m_v} = \emptyset$ if $u, v \in \{1, \ldots, r_m\}\colon\ u \neq v$, $\bigcup\limits_{k=1}^{r_m} E_{m_k} = I^n$ and $\lim\limits_{m\to\infty} s_m(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in I^n$. Hence by Proposition 2.7, there exists $T \in \mathbb{N}$: each simple function s_m is gauge integrable on I^n if $m \geqslant T$. Theorem 3.5 then implies that each such s_m is measurable in the Lebesgue sense, and consequently, h is Lebesgue measurable, where $h(\mathbf{x}) = \sup\bigl\{s_m(\mathbf{x})\bigr\}$ for all $\mathbf{x} \in I^n$, $m \geqslant T$ [13], [14]. But since $\{s_m\}_{m=1}^{\infty}$ is monotonically increasing, $\sup\bigl\{s_m(\mathbf{x})\colon\ m \geqslant T\bigr\} = \lim\limits_{m\to\infty} s_m(\mathbf{x})$. Thus $h(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in I^n$, and so f is measurable in the Lebesgue sense. $\square$

Theorem 3.7. *Suppose that $f\colon\ I^n \to \mathbb{R}$ is non-negative and gauge integrable on I^n. Then f is Lebesgue integrable on I^n and*

$$\int_{I^n} f\,\mathrm{d}\mu = (G)\int_{I^n} f\,\mathrm{d}V. \tag{18}$$

P r o o f. Since f is gauge integrable on I^n, then f is Lebesgue measurable on I^n by Theorem 3.6. Since f is also nonnegative, there exists, by Proposition 2.9 a sequence $\{s_m\}_{m=1}^{\infty}$ of simple, Lebesgue measurable functions such that for all $\mathbf{x} \in I^n$,

$$0 \leqslant s_m(\mathbf{x}) \leqslant f(\mathbf{x}) \quad \text{and} \quad \lim_{m\to\infty} s_m(\mathbf{x}) = f(\mathbf{x}) \tag{19}$$

Now since each of these simple functions s_m is Lebesgue measurable, then each s_m is Lebesgue and therefore gauge integrable on I^n to the same value. By this fact and (19),

$$\int_{I^n} s_m\,\mathrm{d}\mu = (G)\int_{I^n} s_m\,\mathrm{d}V \leqslant (G)\int_{I^n} f\,\mathrm{d}V$$

which yields

$$\lim_{m\to\infty}\int_{I^n} s_m\,\mathrm{d}\mu \leqslant (G)\int_{I^n} f\,\mathrm{d}V < +\infty.$$

Also, by Lebesgue's monotone convergence theorem,

$$\lim_{m\to\infty}\int_{I^n} s_m\,\mathrm{d}\mu = \int_{I^n} f\,\mathrm{d}\mu$$

so that $\int_{I^n} f\,\mathrm{d}\mu < +\infty$ by substitution, and (18) follows immediately by Proposition 2.11. $\square$

The principal theorem in this article, Theorem 1.1, now follows trivially from the previous results because all of the intricate work for this proof was completed there. For the sake of generality, Theorem 1.1 is included as part (b) of the following theorem contained in W. Pfeffer's works, which states the complete relationship between the gauge and Lebesgue integrals.

Theorem 3.8 (The relationship between the gauge and Lebesgue integrals) [11], [12]**.** *Let $f\colon I^n \to \mathbb{R}$.*

(a) *If $|f| \in \mathcal{L}(I^n)$, then $f \in \mathcal{L}(I^n)$ and f, $|f| \in \mathcal{R}^*(I^n)$.*

(b) *If $|f| \in \mathcal{R}^*(I^n)$, then $f \in \mathcal{R}^*(I^n)$ and f, $|f| \in \mathcal{L}(I^n)$.*

If either (a) *or* (b) *is true, both the Lebesgue and gauge intergals integrate f and $|f|$ over I^n to the same value.*

Proof. Part (a): Since the hypothesis implies that $f \in \mathcal{L}(I^n)$ [13], [14], then $f, |f| \in \mathcal{R}^*(I^n)$ by Proposistion 2.11.

Part (b): By Theorem 3.7, since $|f|$ is non-negative, then $|f| \in \mathcal{L}(I^n)$ and consequently $f \in \mathcal{L}(I^n)$ which in turn implies that $f \in \mathcal{R}^*(I^n)$ by Proposition 2.11.

Note that if either (a) or (b) is true, it follows that $\int_{I^n} |f| \,\mathrm{d}\mu = (G) \int_{I^n} |f| \,\mathrm{d}V$ and $\int_{I^n} f \,\mathrm{d}\mu = (G) \int_{I^n} f \,\mathrm{d}V$ by Proposition 2.11 since $f, |f| \in \mathcal{L}(I^n)$. □

Acknowledgments. I would like to thank Dr. D. Bugajewski and Dr. Xiao-Xiong Gan for their suggestions and for giving me the opportunity to present these results in the context of a seminar which stimulated discussions about this topic.

References

[1] *R. Bartle*: Return to the Riemann integral. Am. Math. Mon. *103* (1996), 625–632.

[2] *D. Bugajewska*: On the equation of nth order and the Denjoy integral. Nonlinear Anal. *34* (1998), 1111–1115.

[3] *D. Bugajewska and D. Bugajewski*: On nonlinear integral equations and nonabsolute convergent integrals. Dyn. Syst. Appl. *14* (2005), 135–148.

[4] *D. Bugajewski and S. Szufla*: On the Aronszajn property for differential equations and the Denjoy integral. Ann. Soc. Math. *35* (1995), 61–69.

[5] *T. Chew and F. Flordeliza*: On $x' = f(t, x)$ and Henstock-Kurzweil integrals. Differ. Integral Equ. *4* (1991), 861–868.

[6] *R. Henstock*: Definitions of Riemann type of the variational integral. Proc. Lond. Math. Soc. *11* (1961), 404–418.

[7] *R. Henstock*: The General Theory of Integration. Oxford Math. Monogr., Clarendon Press, Oxford, 1991.

[8] *J. Kurzweil*: Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter. Czech. Math. J. *7* (1957), 418–449.

[9] *R. McLeod*: The Generalized Riemann Integral. Carus Math. Monogr., no. 20, Mathematical Association of America, Washington, 1980.

[10] *J. Munkres*: Analysis on Manifolds. Addison-Wesley Publishing Company, Redwood City, CA, 1991.

[11] *W. Pfeffer*: The divergence theorem. Trans. Am. Math. Soc. *295* (1986), 665–685.
[12] *W. Pfeffer*: The multidimensional fundamental theorem of calculus. J. Austral. Math. Soc. (Ser. A) *43* (1987), 143–170.
[13] *W. Rudin*: Principles of Mathematical Analysis. Third Ed., McGraw-Hill, New York, 1976.
[14] *W. Rudin*: Real and Complex Analysis. McGraw-Hill, New York, 1987.
[15] *Š. Schwabik*: The Perron integral in ordinary differential equations. Differ. Integral Equ. *6* (1993), 863–882.
[16] *M. Spivak*: Calculus on Manifolds. W. A. Benjamin, Menlo Park, CA, 1965.
[17] *K. Stromberg*: An Introduction to Classical Real Analysis. Waldworth, Inc, 1981.

Author's address: Timothy Myers, Department of Mathematics, Howard University, Washington DC 20059, e-mail: `timyers@howard.edu`.