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GLOBAL AND NON-GLOBAL EXISTENCE OF SOLUTIONS TO A NONLOCAL AND DEGENERATE QUASILINEAR PARABOLIC SYSTEM

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Abstract. The paper deals with positive solutions of a nonlocal and degenerate quasilinear parabolic system not in divergence form

$$u_t = v^p\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big), \quad v_t = u^q\Big(\Delta v + b\int_\Omega v\,\mathrm{d}x\Big)$$

with null Dirichlet boundary conditions. By using the standard approximation method, we first give a series of fine a priori estimates for the solution of the corresponding approximate problem. Then using the diagonal method, we get the local existence and the bounds of the solution (u,v) to this problem. Moreover, a necessary and sufficient condition for the non-global existence of the solution is obtained. Under some further conditions on the initial data, we get criteria for the finite time blow-up of the solution.

Keywords: strongly coupled, degenerate parabolic system, nonlocal source, global existence, blow-up

MSC 2010: 35K05, 35D55, 45K05

1. Introduction

In this paper we consider the positive solution of the following non-local and degenerate problem not in divergence form:

$$(1.1)\qquad \begin{cases} u_t = v^p\big(\Delta u + a\int_\Omega u\,\mathrm{d}x\big), & x\in\Omega,\ \ t>0,\\ v_t = u^q\big(\Delta v + b\int_\Omega v\,\mathrm{d}x\big), & x\in\Omega,\ \ t>0,\\ u(x,t)=v(x,t)=0, & x\in\partial\Omega,\ \ t>0,\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, the parameters a, b, p, q are positive constants and $p, q \geqslant 1$. The initial data $u_0(x)$ and $v_0(x)$ satisfy

(H1) $u_0(x), v_0(x) \in C^1(\overline{\Omega})$, $u_0(x), v_0(x) > 0$ in Ω;

(H2) $u_0(x) = v_0(x) = 0$, $\partial u_0/\partial\nu < 0$, $\partial v_0/\partial\nu < 0$ on $\partial\Omega$, where ν is the outward normal vector on $\partial\Omega$.

Such a problem can describe a variety of physical phenomena which arise, for example, in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium or in the studies of population dynamics (see [1], [3], [5], [6], [7] and references therein).

As for local problems, a lot of effort has been devoted in the past few years to the study of the type

$$u_t = f(u)(\Delta u + au)$$

(see [2], [8], [9], [10], [13], [16], [17], [18]). The corresponding system

$$\begin{cases} u_t = f_1(u)(\Delta u + av), & x \in \Omega,\ t > 0, \\ v_t = f_2(v)(\Delta v + bu), & x \in \Omega,\ t > 0, \end{cases}$$

with the same initial and boundary conditions as the problem (1.1), has also been investigated ([5], [14]). It is worth pointing out that in [15], Wang et al. investigated the system

$$\begin{cases} u_t = v^p(\Delta u + au), & x \in \Omega,\ t > 0, \\ v_t = u^q(\Delta v + bv), & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

with null Dirichlet boundary conditions and positive initial conditions, where $p, q \geqslant 1$. It was proved that when $\min\{a, b\} \leqslant \lambda_1$ then there exists a global positive classical solution and no positive classical solution can blow up in finite time, whereas when $\min\{a, b\} > \lambda_1$, there is no global positive classical solution if in addition the initial data satisfy some further conditions, then the positive classical solution is unique and blows up in finite time, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition.

Furthermore, for nonlocal problems, the global existence and the blowing-up behavior of the solution have been investigated by many researchers too, but the situation becomes more complicated. In [6], Duan et al. established the local existence of a solution and the finite-time blowup result for the system

$$\begin{cases} u_t = u^p\left(\Delta u + a\displaystyle\int_\Omega v^r\,\mathrm{d}x\right), & x \in \Omega,\ t > 0, \\ v_t = v^q\left(\Delta v + b\displaystyle\int_\Omega u^s\,\mathrm{d}x\right), & x \in \Omega,\ t > 0, \end{cases}$$

where $r, s > 1$ and $0 < p, q < 1$. The scalar case of (1.1), i.e.

$$u_t = u^p\biggl(\Delta u + a\int_\Omega u\,\mathrm{d}x\biggr),$$

with the homogeneous Dirichlet boundary condition and positive initial data, has been studied in [3], [4], where $p > 0$. It was proved that the solution blows up in finite time if and only if $\mu > 1/a$, where μ is defined in (1.4) below.

Motivated by the above papers, in this paper we consider the global existence and finite blow-up of solutions to the problem (1.1). Necessary and sufficient conditions will be given, see Theorem 1. Note that the problem (1.1) is not only degenerate and strongly coupled but also not in the divergence form, hence the standard comparison principle and upper and lower solutions method do not hold. In general, we do not know how to extend the local classical solution in time t so that it becomes a maximum defined classical solution since the uniqueness does not hold. We can only prove the uniqueness of a positive classical solution to (1.1) by adding some further conditions on the initial values.

Throughout this paper we say that (u, v) is a classical solution of the problem (1.1) if $(u, v) \in [C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T))]^2$ for some $T\colon\ 0 < T < \infty$, and (u, v) satisfies the differential equations in $\Omega \times (0, T)$ and the initial and boundary conditions continuously. We say that a positive classical solution (u, v) of (1.1) blows up in finite time if there exists $T\colon\ 0 < T < \infty$ such that (u, v) exists in $(0, T)$ and $\overline{\lim\limits_{t\nearrow T}} \max\limits_{\overline{\Omega}}\{u(\cdot, t) + v(\cdot, t)\} = \infty$. We say that the problem (1.1) has no global positive classical solution if there exist $0 < T_2 \leqslant T_1 < \infty$ and two pairs of functions

$$(u_i, v_i) \in [C^{2+\beta,1+\beta/2}_{\mathrm{loc}}(\Omega \times (0, T_i)) \cap C(\overline{\Omega} \times (0, T_i))]^2$$

satisfying $\overline{\lim\limits_{t\nearrow T_i}} \max\limits_{\overline{\Omega}}\{u_i(\cdot, t) + v_i(\cdot, t)\} = \infty$, such that any positive classical solution (u, v) of (1.1) fulfils

$$u_1(x, t) \leqslant u(x, t) \leqslant u_2(x, t), \quad v_1(x, t) \leqslant v(x, t) \leqslant v_2(x, t) \quad \text{in } \Omega \times (0, T_2)$$

provided that (u, v) exists in $(0, T_2)$. When the problem (1.1) has no global positive classical solution, we cannot say that the positive classical solution must blow up in finite time since, in general, it cannot be extend in time t.

The main purpose of this paper is to get criteria for the finite time blow-up of a solution to the problem (1.1). Set $Q_T = \Omega\times(0, T)$, $S_T = \partial\Omega\times(0, T)$ with $0 < T < \infty$. We denote by $\varphi(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi(x) = 1,\ \ x \in \Omega; \qquad \varphi(x) = 0,\ \ x \in \partial\Omega. \tag{1.3}$$

Let

$$\mu = \int_\Omega \varphi(x)\,\mathrm{d}x. \tag{1.4}$$

The following theorem is the main result of this paper.

Theorem 1. *The solution* (u,v) *of* (1.1) *does not exist globally if and only if*

$$\min\{a,b\} > \frac{1}{\mu}. \tag{1.5}$$

The paper is organized as follows. In Section 2 we study the local existence of a positive classical solution. Section 3 is devoted to the global existence result. In Section 4 we study the finite time blow-up problem.

2. Comparison principle and local existence of solution

First, we state a comparison principle, of which the proof is standard.

Lemma 1. *Suppose that* $w, z \in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ *and satisfy*

$$\begin{cases} w_t - d_1\Delta w \geqslant c_{11}w + c_{12}z + \displaystyle\int_\Omega c_{13}(x,t)w(x,t)\,\mathrm{d}x, & (x,t)\in Q_T,\\ z_t - d_2\Delta z \geqslant c_{21}w + c_{22}z + \displaystyle\int_\Omega c_{23}(x,t)z(x,t)\,\mathrm{d}x, & (x,t)\in Q_T,\\ w(x,t)\geqslant 0,\ \ z(x,t)\geqslant 0, & (x,t)\in S_T,\\ w(x,0)\geqslant 0,\ \ z(x,0)\geqslant 0, & x\in\Omega, \end{cases}$$

where $c_{ij}=c_{ij}(x,t)$ *and* $d_i = d_i(x,t)$ *are continuous functions in* Q_T *and* $c_{ij}(x,t)\geqslant 0$ $(i\neq j)$, $d_i(x,t)>0$ *in* Q_T, $i=1,2$, $j=1,2,3$. *Then* $w(x,t)\geqslant 0$, $z(x,t)\geqslant 0$ *on* $\overline{Q}_T$.

Since $u=v=0$ on the boundary $\partial\Omega$, the equation in (1.1) is not of strictly parabolic type. The standard parabolic theory ([11], [12]), cannot be used directly to prove the local existence of a solution to the problem (1.1). To overcome this difficulty we will use the standard approximate method (see [15]). For $\varepsilon>0$, consider the approximate problem

$$\begin{cases} u_{\varepsilon t} = f_\varepsilon^{(1)}(v_\varepsilon)\Big(\Delta u_\varepsilon + a\displaystyle\int_\Omega u_\varepsilon\,\mathrm{d}x - a\varepsilon|\Omega|\Big), & x\in\Omega,\ \ t>0,\\ v_{\varepsilon t} = g_\varepsilon^{(1)}(u_\varepsilon)\Big(\Delta v_\varepsilon + b\displaystyle\int_\Omega v_\varepsilon\,\mathrm{d}x - b\varepsilon|\Omega|\Big), & x\in\Omega,\ t>0,\\ u_\varepsilon(x,t) = v_\varepsilon(x,t)=\varepsilon, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x)+\varepsilon,\quad v_\varepsilon(x,0)=v_0(x)+\varepsilon, & x\in\Omega, \end{cases} \tag{2.1}$$

where $|\Omega|$ is the volume of Ω, $f_\varepsilon^{(1)}$ and $g_\varepsilon^{(1)}$ are smooth functions, $f_\varepsilon^{(1)} \geqslant (\varepsilon/2)^p$ and $g_\varepsilon^{(1)} \geqslant (\varepsilon/2)^q$ and they satisfy

$$f_\varepsilon^{(1)}(v) = \begin{cases} v^p, & v > \varepsilon, \\ (\varepsilon/2)^p, & v < \varepsilon/2, \end{cases} \qquad g_\varepsilon^{(1)}(u) = \begin{cases} u^q, & u > \varepsilon, \\ (\varepsilon/2)^q, & u < \varepsilon/2. \end{cases}$$

The standard parabolic theory shows that (2.1) admits a unique maximal defined classical solution $(u_\varepsilon, v_\varepsilon)$ which is defined on $[0, T(\varepsilon))$, where $0 < T(\varepsilon) \leqslant \infty$. Applying Lemma 1 it is easy to prove that $u_\varepsilon > \varepsilon$, $v_\varepsilon > \varepsilon$, which gives $f_\varepsilon^{(1)}(v_\varepsilon) = v_\varepsilon^p$ and $g_\varepsilon^{(1)}(u_\varepsilon) = u_\varepsilon^q$. Hence $(u_\varepsilon, v_\varepsilon)$ solves the problem

$$\text{(2.2)} \qquad \begin{cases} u_{\varepsilon t} = v_\varepsilon^p \Big(\Delta u_\varepsilon + a \displaystyle\int_\Omega u_\varepsilon \, \mathrm{d}x - a\varepsilon|\Omega| \Big), & x \in \Omega, \ 0 < t < T(\varepsilon), \\ v_{\varepsilon t} = u_\varepsilon^q \Big(\Delta v_\varepsilon + b \displaystyle\int_\Omega v_\varepsilon \, \mathrm{d}x - b\varepsilon|\Omega| \Big), & x \in \Omega, \ 0 < t < T(\varepsilon), \\ u_\varepsilon(x,t) = v_\varepsilon(x,t) = \varepsilon, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad v_\varepsilon(x,0) = v_0(x) + \varepsilon, & x \in \Omega. \end{cases}$$

To discuss the convergence of $(u_\varepsilon, v_\varepsilon)$, we should give some estimates of their lower and upper bounds. Set $M = \max\{\max_{\overline{\Omega}} u_0(x), \max_{\overline{\Omega}} v_0(x)\}$ and let $(f(t), g(t))$ be the unique solution of the ODE

$$\text{(2.3)} \qquad \begin{cases} f' = a|\Omega| f g^p, \quad g' = b|\Omega| g f^q, \quad t > 0, \\ f(0) = g(0) = M + 1. \end{cases}$$

Then $f(t), g(t) \geqslant M + 1$. Denote by T^*, $0 < T^* < \infty$, its maximal existence time (note that $T^* < \infty$ must hold because $f(t)$, $g(t)$ blow up in finite time).

Applying Lemma 1, it is easy to prove the following lemma (cf. Lemma 1 in [15]).

Lemma 2. *Let $\varepsilon < 1$, and let $(u_\varepsilon, v_\varepsilon)$ be the solution of* (2.2). *Then for any fixed T: $0 < T < \min\{T(\varepsilon), T^*\}$ we have*

$$u_\varepsilon(x,t) \leqslant f(t), \quad v_\varepsilon(x,t) \leqslant g(t), \ \ \forall\, (x,t) \in \overline{\Omega} \times [0, T],$$

which implies that $T(\varepsilon) \geqslant T^*$ for all $\varepsilon < 1$.

In the following we denote $T_* = T^*/2$. Now we give lower bounds for $(u_\varepsilon, v_\varepsilon)$. Let λ_1 and $\psi(x) > 0$ $(x \in \Omega)$ be the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem

$$-\Delta\psi = \lambda\psi, \ \ x \in \Omega; \quad \psi = 0, \ \ x \in \partial\Omega,$$

and assume that $\max\limits_{\overline{\Omega}} \psi(x) = 1$. Then $\lambda_1 > 0$ and $\partial\psi/\partial\nu < 0$ on $\partial\Omega$. By the assumptions on (u_0, v_0) we see that there exists a positive constant k such that

$$u_0(x) \geqslant k\psi(x), \quad v_0(x) \geqslant k\psi(x), \quad x \in \overline{\Omega}. \tag{2.4}$$

Lemma 3. *Let $\varepsilon < 1$ and let $(u_\varepsilon, v_\varepsilon)$ be the solution of* (2.2)*. Then we have*

$$u_\varepsilon(x,t), v_\varepsilon(x,t) \geqslant k\mathrm{e}^{-\varrho t}\psi(x) + \varepsilon, \quad \forall\,(x,t) \in \Omega \times [0, T_*],$$

where $\varrho = \max\{\lambda_1 f^q(T_*), \lambda_1 g^p(T_*)\}$.

Proof. Set $w(x,t) = u_\varepsilon(x,t) - (k\mathrm{e}^{-\varrho t}\psi(x) + \varepsilon)$, $z(x,t) = v_\varepsilon(x,t) - (k\mathrm{e}^{-\varrho t}\psi(x) + \varepsilon)$. Since $u_\varepsilon(x,t) \leqslant f(t) \leqslant f(T_*)$, $v_\varepsilon(x,t) \leqslant g(t) \leqslant g(T_*)$ for all $(x,t) \in \overline{\Omega} \times [0, T_*]$, it follows that

$$\begin{aligned} w_t &= v_\varepsilon^p\Big(\Delta u_\varepsilon + a\int_\Omega u_\varepsilon\,\mathrm{d}x - a\varepsilon|\Omega|\Big) + k\varrho\mathrm{e}^{-\varrho t}\psi \\ &= v_\varepsilon^p\Big(\Delta w + a\int_\Omega w\,\mathrm{d}x\Big) + v_\varepsilon^p\Big(ak\mathrm{e}^{-\varrho t}\int_\Omega \psi\,\mathrm{d}x - k\lambda_1\mathrm{e}^{-\varrho t}\psi\Big) + k\varrho\mathrm{e}^{-\varrho t}\psi \\ &\geqslant v_\varepsilon^p\Big(\Delta w + a\int_\Omega w\,\mathrm{d}x\Big) + k\mathrm{e}^{-\varrho t}\psi(\varrho - \lambda_1 v_\varepsilon^p) \\ &\geqslant v_\varepsilon^p\Big(\Delta w + a\int_\Omega w\,\mathrm{d}x\Big), \quad (x,t) \in \Omega \times (0, T_*], \end{aligned}$$

$$\begin{gathered} z_t \geqslant u_\varepsilon^q\Big(\Delta z + b\int_\Omega z\,\mathrm{d}x\Big), \quad (x,t) \in \Omega \times (0, T_*], \\ w(x,0) \geqslant 0, \quad z(x,0) \geqslant 0, \quad x \in \Omega, \\ w(x,t) = 0, \quad z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, T_*]. \end{gathered}$$

In view of Lemma 1 we have that $w, z \geqslant 0$. The proof is completed. □

In view of Lemma 2 and Lemma 3, applying the standard local Schauder estimates (Theorem 7.1 of Chap. 7 of [11]) and the diagonal method we obtain that there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ and $u, v \in C_{\mathrm{loc}}^{2+\alpha, 1+\alpha/2}(\Omega \times (0, T_*])$ such that

$$(u_{\varepsilon'}, v_{\varepsilon'}) \longrightarrow (u, v) \quad \text{in} \quad [C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_* \times [\tau, T_*])]^2 \ \text{as}\ \varepsilon' \to 0^+$$

for any $\Omega_* \subset\subset \Omega$ and $0 < \tau < T_*$. Hence (u, v) satisfies the differential equations of the problem (1.1).

Fix ε_0: $0 < \varepsilon_0 \ll 1$. For any $\Omega_0 \subset\subset \Omega$ and $0 < \varepsilon' < \varepsilon_0$, thanks to Lemma 2 and

$$u_{\varepsilon'}(x,t) \leqslant f(t), \quad v_{\varepsilon'}(x,t) \leqslant g(t) \ \text{on}\ \overline{\Omega}_0 \times [0, T_*],$$

the L^p theory and the imbedding theorem show that the $C^{\alpha,\alpha/2}(\overline{\Omega}_0 \times [0,T_*])$ norms of $u_{\varepsilon'}$ and $v_{\varepsilon'}$ are uniformly bounded for all $\varepsilon' < \varepsilon_0$. And hence

$$(u_{\varepsilon'}, v_{\varepsilon'}) \longrightarrow (u,v) \quad \text{in } [C^{\beta,\beta/2}(\overline{\Omega}_0 \times [0,T_*])]^2 \quad (0<\beta<\alpha) \text{ as } \varepsilon' \to 0^+,$$

which implies that $u, v \in C(\Omega \times [0,T_*])$. Similarly to the arguments of [9], [15] we can prove that (u,v) is continuous on $\partial\Omega \times (0,T_*]$. Using the initial and boundary conditions of (2.2) we see that (u,v) satisfies the initial and boundary conditions of (1.1), i.e.

$$(u,v) \in [C^{2+\beta,1+\beta/2}_{\mathrm{loc}}(\Omega \times (0,T_*]) \cap C(\overline{\Omega} \times [0,T_*])]^2$$

is a classical solution of (1.1). Thus we have

Theorem 2. *The problem* (1.1) *has a positive classical solution* $(u,v) \in [C^{2+\beta,1+\beta/2}_{\mathrm{loc}}(\Omega \times (0,T_*]) \cap C(\overline{\Omega} \times [0,T_*])]^2$ *for some* $\beta\colon\ 0<\beta<1$.

However, we cannot get the uniqueness result in the general case.

3. Global existence results

In the next two sections we always assume that $a \geqslant b$. In this section we will prove the global existence of a positive classical solution. Applying (H1) and (H2) we see that there exists a large positive constant K such that

$$u_0(x) \leqslant K\varphi(x), \quad v_0(x) \leqslant K\varphi(x), \qquad x \in \overline{\Omega}, \tag{3.1}$$

where $\varphi(x)$ is defined in (1.3).

Theorem 3. *If* $b \leqslant 1/\mu$, *then the problem* (1.1) *has at least one global positive classical solution* (u,v). *Moreover, all positive classical solutions must satisfy the following estimates.*

(i) *If* $a \leqslant 1/\mu$ *then*

$$u(x,t), v(x,t) \leqslant K\varphi(x) \quad \text{on } \overline{\Omega} \times [0,\infty);$$

(ii) *if* $b \leqslant 1/\mu < a$ *then*

$$u(x,t) \leqslant K\varphi(x), \quad v(x,\,t) \leqslant KM\mathrm{e}^{\varrho t} \quad \text{on } \overline{\Omega} \times [0,\infty),$$

where the positive constant K *satisfies* (3.1), *and* $\varrho = b|\Omega|(KM)^q$, $M = \max\limits_{\overline{\Omega}} \varphi(x)$.

P r o o f. For any given ε: $0<\varepsilon<1$, let $(u_\varepsilon, v_\varepsilon)$ be the unique positive classical solution of (2.2) which is defined on $\overline{\Omega}\times[0,T(\varepsilon))$ with $T(\varepsilon)\leqslant\infty$, and let the positive constant K satisfy (3.1).

Step 1: Upper bounds of $(u_\varepsilon, v_\varepsilon)$.

(i) If $a\leqslant 1/\mu$, let $w(x,t)=K\varphi(x)+\varepsilon-u_\varepsilon(x,t)$, $z(x,t)=K\varphi(x)+\varepsilon-v_\varepsilon(x,t)$. Similarly to the proof of Lemma 3 we can prove that $w(x,t), z(x,t)\geqslant 0$ on $\overline{\Omega}\times[0,T(\varepsilon))$.

(ii) If $b\leqslant 1/\mu<a$, similarly to the proof of Lemma 3 we obtain that $u_\varepsilon(x,t)\leqslant K\varphi(x)+\varepsilon$ on $\overline{\Omega}\times[0,T(\varepsilon))$. Let $z=KM\mathrm{e}^{\overline{\varrho}t}+\varepsilon-v_\varepsilon$, $\overline{\varrho}=b|\Omega|(KM+\varepsilon)^q$. In view of $u_\varepsilon(x,t)\leqslant K\varphi(x)+\varepsilon\leqslant KM+\varepsilon$ we have that

$$
\begin{aligned}
z_t=KM\overline{\varrho}\mathrm{e}^{\overline{\varrho}t}-v_{\varepsilon t}&=u_\varepsilon^q\left(\Delta z+b\int_\Omega z\,\mathrm{d}x\right)+KM\mathrm{e}^{\overline{\varrho}t}(\overline{\varrho}-b|\Omega|u_\varepsilon^q)\\
&\geqslant u_\varepsilon^q\left(\Delta z+b\int_\Omega z\,\mathrm{d}x\right),\quad (x,t)\in\Omega\times(0,T(\varepsilon)),\\
z(x,t)&=KM\mathrm{e}^{\overline{\varrho}t}>0,\quad (x,t)\in\partial\Omega\times(0,T(\varepsilon)),\\
z(x,0)&=KM-v_0(x)\geqslant 0,\quad x\in\Omega.
\end{aligned}
$$

By Lemma 1 we get that $v_\varepsilon(x,t)\leqslant KM\mathrm{e}^{\overline{\varrho}t}+\varepsilon$ on $\overline{\Omega}\times[0,T(\varepsilon))$.

Step 2: Lower bounds of $(u_\varepsilon, v_\varepsilon)$.

We can prove

$$
u_\varepsilon(x,t), v_\varepsilon(x,t)\geqslant k\psi(x)\mathrm{e}^{-rt}+\varepsilon,\quad \forall\,(x,t)\in\overline{\Omega}\times[0,T(\varepsilon)),
$$

where $r=\max\{\lambda_1(KM+\varepsilon)^q,\ \lambda_1(KM+\varepsilon)^p\}$ if $a\leqslant 1/\mu$, and $r=\max\{\lambda_1(KM+\varepsilon)^q,\ \lambda_1(KM\mathrm{e}^{\overline{\varrho}t}+\varepsilon)^p\}$ if $b\leqslant 1/\mu<a$.

In fact, if $a\leqslant 1/\mu$, by Step 1 (i) we have $u_\varepsilon, v_\varepsilon\leqslant K\varphi(x)+\varepsilon$ on $\overline{\Omega}\times[0,T(\varepsilon))$. Set $w(x,t)=u_\varepsilon(x,t)-(k\mathrm{e}^{-rt}\psi(x)+\varepsilon)$, $z(x,t)=v_\varepsilon(x,t)-(k\mathrm{e}^{-rt}\psi(x)+\varepsilon)$; it follows that

$$
\begin{aligned}
w_t&=v_\varepsilon^p\left(\Delta u_\varepsilon+a\int_\Omega u_\varepsilon\,\mathrm{d}x-a\varepsilon|\Omega|\right)+kr\mathrm{e}^{-rt}\psi\\
&=v_\varepsilon^p\left(\Delta w+a\int_\Omega w\,\mathrm{d}x\right)+v_\varepsilon^p\left(ak\mathrm{e}^{-rt}\int_\Omega\psi\,\mathrm{d}x-k\lambda_1\mathrm{e}^{-rt}\psi\right)+kr\mathrm{e}^{-rt}\psi\\
&\geqslant v_\varepsilon^p\left(\Delta w+a\int_\Omega w\,\mathrm{d}x\right),\quad (x,t)\in\Omega\times(0,T(\varepsilon)),\\
z_t&\geqslant u_\varepsilon^q\left(\Delta z+b\int_\Omega z\,\mathrm{d}x\right),\quad (x,t)\in\Omega\times(0,T(\varepsilon)),\\
&w(x,0)\geqslant 0,\quad z(x,0)\geqslant 0,\quad x\in\Omega,\\
&w(x,t)=0,\quad z(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T(\varepsilon)).
\end{aligned}
$$

In view of Lemma 1 we have that $w, z \geqslant 0$. The proof of the case $b \leqslant 1/\mu < a$ is similar.

Step 3: The upper bounds of $(u_\varepsilon, v_\varepsilon)$ obtained by Step 1 show that $(u_\varepsilon, v_\varepsilon)$ exists globally, i.e. $T(\varepsilon) = \infty$ for all $0 < \varepsilon < 1$. For any $\Omega_n \subset\subset \Omega$ and $0 < \tau < T_n < \infty$, it follows from the results of Steps 1 and 2 that there exist positive constants $\sigma(n, \tau)$ and $M(n, \tau)$ such that

$$\sigma(n,\tau) \leqslant u_\varepsilon, v_\varepsilon \leqslant M(n,\tau) \quad \text{on } \overline{\Omega}_n \times [\tau, T_n]$$

for all $0 < \varepsilon < 1$. Applying the standard local Schauder estimates ([11]) and the diagonal method we conclude that there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ and $u, v \in C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\Omega \times (0,\infty))$ such that

$$(u_{\varepsilon'}, v_{\varepsilon'}) \longrightarrow (u, v) \quad \text{in } [C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_* \times [\tau, T_0])]^2 \ \text{ as } \varepsilon' \to 0^+$$

for any $\Omega_* \subset\subset \Omega$ and $0 < \tau < T_0 < \infty$. And hence (u, v) satisfies the differential equations of (1.1) in $\Omega \times (0, \infty)$.

Similarly to the arguments of Section 2 we see that

$$(u, v) \in [C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))]^2$$

is a classical solution of (1.1).

Estimates (i) and (ii) can be proved in a way similar to that of Step 1. The proof is completed. □

4. Blow-up results

Theorem 4. *Assume that $b > 1/\mu$. Then the problem* (1.1) *has no global positive classical solution. If in addition the initial data satisfy $(u_0, v_0) \in [C^2(\overline{\Omega})]^2$ and $\Delta u_0 + a\int_\Omega u_0 \mathrm{d}x \geqslant 0$, $\Delta v_0 + b\int_\Omega v_0 \,\mathrm{d}x \geqslant 0$ in Ω, then the positive classical solution of* (1.1) *is unique and blows up in finite time.*

We divide the proof of Theorem 4 into two lemmas: Lemma 4 and Lemma 6.

Lemma 4. *Assume that $b > 1/\mu$ and the initial data satisfy $(u_0, v_0) \in [C^2(\overline{\Omega})]^2$. If $\Delta u_0 + a\int_\Omega u_0 \,\mathrm{d}x \geqslant 0$, $\Delta v_0 + b\int_\Omega v_0 \,\mathrm{d}x \geqslant 0$ in Ω, then the positive classical solution of* (1.1) *is unique and blows up in finite time.*

To prove Lemma 4, we need the following lemma. A similar proof can be found in [3].

Lemma 5. *Let $p > 0$, and let u be the unique positive (local) solution of the problem*

$$(4.1)\qquad \begin{cases} u_t = Au^p\Big(\Delta u + B\displaystyle\int_\Omega u\,\mathrm{d}x\Big), & x \in \Omega,\ \ t > 0,\\ u(x,t) = C, & x \in \partial\Omega,\ \ t > 0,\\ u(x,0) = C, & x \in \Omega, \end{cases}$$

where A and C are positive constants. Then u blows up in finite time if $B > 1/\mu$.

P r o o f of Lemma 4. *Step 1*: Monotonicity of (u,v) in t.

Let $\varepsilon \in (0,1)$ and let $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.1). Then

$$(u_\varepsilon, v_\varepsilon) \in [C(\overline{\Omega} \times [0,T(\varepsilon))) \cap C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\Omega \times (0,T(\varepsilon)))]^2$$

(see Theorem 7.1 of Chap. 7 of [11]). Therefore, $(u_\varepsilon, v_\varepsilon)$ satisfies (2.2). Let $w = u_{\varepsilon t}$, $z = v_{\varepsilon t}$. Then we have, in the "weak" sense,

$$(4.2)\qquad \begin{cases} w_t = pv_\varepsilon^{p-1}z\Big(\Delta u_\varepsilon + a\displaystyle\int_\Omega u_\varepsilon\,\mathrm{d}x - a|\Omega|\varepsilon\Big) + v_\varepsilon^p\Big(\Delta w + a\displaystyle\int_\Omega w\,\mathrm{d}x\Big)\\ \quad = v_\varepsilon^p\Big(\Delta w + a\displaystyle\int_\Omega w\,\mathrm{d}x\Big) + (p/v_\varepsilon)zv_\varepsilon^p\Big(\Delta u_\varepsilon + a\displaystyle\int_\Omega u_\varepsilon\,\mathrm{d}x - a|\Omega|\varepsilon\Big)\\ \quad = v_\varepsilon^p\Big(\Delta w + a\displaystyle\int_\Omega w\,\mathrm{d}x\Big) + (pz/v_\varepsilon)w, \quad x \in \Omega,\ \ 0 < t < T(\varepsilon),\\ z_t = u_\varepsilon^q\Big(\Delta z + b\displaystyle\int_\Omega z\,\mathrm{d}x\Big) + (qw/u_\varepsilon)z, \quad x \in \Omega,\ \ 0 < t < T(\varepsilon),\\ w(x,0) = (v_0(x)+\varepsilon)^p\Big(\Delta u_0(x) + a\displaystyle\int_\Omega u_0(x)\,\mathrm{d}x\Big) \geqslant 0, \quad x \in \Omega,\\ z(x,0) = (u_0(x)+\varepsilon)^q\Big(\Delta v_0(x) + b\displaystyle\int_\Omega v_0(x)\,\mathrm{d}x\Big) \geqslant 0, \quad x \in \Omega,\\ w(x,t) = z(x,t) = 0, \quad x \in \partial\Omega, \quad 0 < t < T(\varepsilon). \end{cases}$$

In view of $u_\varepsilon \geqslant \varepsilon$, $v_\varepsilon \geqslant \varepsilon$ and $w, z \in C(\overline{\Omega} \times [0,T(\varepsilon)))$, the L^p-theory and the Schauder Theory imply that (w,z) is a classical solution of (4.2), i.e. $w, z \in C(\overline{\Omega} \times [0,T(\varepsilon))) \cap C^{2,1}(\Omega \times (0,T(\varepsilon)))$. Lemma 1 shows that $w \geqslant 0$, $z \geqslant 0$, i.e. $u_{\varepsilon t} \geqslant 0$, $v_{\varepsilon t} \geqslant 0$. Since

$$(u_\varepsilon, v_\varepsilon) \longrightarrow (u,v) \quad \text{in } [C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\Omega \times (0,T))]^2 \quad \text{as } \varepsilon \longrightarrow 0^+,$$

we know that $u_t \geqslant 0$, $v_t \geqslant 0$ and hence $u \geqslant u_0(x)$, $v \geqslant v_0(x)$ in $\Omega \times (0,T)$.

Step 2: The uniqueness.

Let (u,v) be the solution of (1.1) obtained by Theorem 2, then $u_t \geqslant 0$, $v_t \geqslant 0$ by Step 1, which implies $\Delta u + a\int_\Omega u\,\mathrm{d}x \geqslant 0$, $\Delta v + b\int_\Omega v\,\mathrm{d}x \geqslant 0$. Let $(\tilde{u},\tilde{v})$, which is defined on $\overline{\Omega} \times [0, \tilde{T})$, be another positive classical solution of (1.1) with the same initial data (u_0, v_0). Set $\xi = \tilde{u} - u$, $\eta = \tilde{v} - v$, then for any $0 < T_0 < \min\{T, \tilde{T}\}$ we have

$$\begin{aligned}
\xi_t = \tilde{u}_t - u_t &= \tilde{v}^p\Big(\Delta\tilde{u} + a\int_\Omega \tilde{u}\,\mathrm{d}x\Big) - v^p\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) \\
&= \tilde{v}^p\Big(\Delta\xi + a\int_\Omega \xi\,\mathrm{d}x\Big) + (\tilde{v}^p - v^p)\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) \\
&= \tilde{v}^p\Big(\Delta\xi + a\int_\Omega \xi\,\mathrm{d}x\Big) \\
&\quad + \Big(p\int_0^1 [v + s(\tilde{v} - v)]^{p-1}\,\mathrm{d}s\Big[\Delta u + a\int_\Omega u\,\mathrm{d}x\Big]\Big)\eta, \quad (x,t) \in \Omega \times (0, T_0), \\
\eta_t = \tilde{v}_t - v_t &= \tilde{u}^q\Big(\Delta\eta + b\int_\Omega \eta\,\mathrm{d}x\Big) \\
&\quad + \Big(q\int_0^1 [u + s(\tilde{u} - u)]^{q-1}\,\mathrm{d}s\Big[\Delta v + b\int_\Omega v\,\mathrm{d}x\Big]\Big)\xi, \quad (x,t) \in \Omega \times (0, T_0), \\
&\xi = \eta = 0, \quad (x,t) \in \partial\Omega \times (0, T_0) \cup \Omega \times \{0\}.
\end{aligned}$$

Since $\Delta u + a\int_\Omega u\,\mathrm{d}x \geqslant 0$, $\Delta v + b\int_\Omega v\,\mathrm{d}x \geqslant 0$, Lemma 1 implies that $\xi \geqslant 0$, $\eta \geqslant 0$, i.e. $\tilde{u} \geqslant u$, $\tilde{v} \geqslant v$. Similarly, we have $\tilde{u} \leqslant u$, $\tilde{v} \leqslant v$. Hence, $\tilde{u} = u$, $\tilde{v} = v$. The uniqueness is proved.

Step 3: (u, v) blows up in finite time.

For any smooth subdomain $\Omega_* \subset\subset \Omega$, denote by $\varphi_1(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_1(x) = 1, \ \ x \in \Omega_*; \quad \varphi_1(x) = 0, \ \ x \in \partial\Omega_*. \tag{4.3}$$

Since $b > 1/\mu$, by the continuity of the solution of (4.3) on the domain Ω_* we can choose Ω_* such that $b > 1/\mu_1$, where $\mu_1 = \int_{\Omega_*} \varphi_1(x)\,\mathrm{d}x$. Applying $u \geqslant u_0 > 0$, $v \geqslant v_0 > 0$ in Ω we know that

$$u(x,t), v(x,\,t) \geqslant \sigma \quad \text{on } \overline{\Omega}_* \times (0, T)$$

for some positive constant σ. It follows that

$$\begin{cases}
u_t = v^p\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) \geqslant \sigma^{p-\alpha}v^\alpha\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) & \text{on } \overline{\Omega}_* \times (0, T), \\
v_t = u^q\Big(\Delta v + b\int_\Omega v\,\mathrm{d}x\Big) \geqslant \sigma^{q-\alpha}u^\alpha\Big(\Delta v + b\int_\Omega v\,\mathrm{d}x\Big) & \text{on } \overline{\Omega}_* \times (0, T), \\
u(x,0) = u_0(x) \geqslant \sigma, \ \ v(x,0) = v_0(x) \geqslant \sigma \quad \text{in } \Omega_*, \\
u(x,t) \geqslant \sigma, \ \ v(x,t) \geqslant \sigma \quad \text{on } \partial\Omega_* \times (0, T).
\end{cases} \tag{4.4}$$

Choose

$$\alpha\colon\ 0<\alpha<1,\quad c\colon\ b>c>1/\mu_1,\quad l=\min\{\sigma^{p-\alpha},\sigma^{q-\alpha}\},$$

and let w be the unique positive solution of the problem

$$(4.5)\qquad \begin{cases} w_t = lw^\alpha\Big(\Delta w + c\displaystyle\int_\Omega w\,\mathrm{d}x\Big), & x\in\Omega_*,\ t>0,\\ w(x,0)=w_0(x)=\sigma, & x\in\Omega_*,\\ w(x,t)=\sigma, & x\in\partial\Omega_*,\ t>0.\end{cases}$$

Applying $c>1/\mu_1$, we know that w blows up in finite time T^* (see Lemma 5). Since the initial data w_0 satisfies $\Delta w_0 + c\int_\Omega w_0\,\mathrm{d}x = c\sigma|\Omega|>0$, it follows that $w_t\geqslant 0$, i.e. $\Delta w + c\int_\Omega w\,\mathrm{d}x\geqslant 0$.

On the contrary, let us assume that (u,v) exists globally, i.e. $T=\infty$. Let

$$\hat u = u-w,\quad \hat v = v-w.$$

Then we have, by (4.4), (4.5), and $c<b\leqslant a$,

$$\begin{aligned}\hat u_t &= u_t - w_t \geqslant \sigma^{p-\alpha}v^\alpha\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) - lw^\alpha\Big(\Delta w + c\int_\Omega w\,\mathrm{d}x\Big)\\ &\geqslant lv^\alpha\Big(\Delta u + a\int_\Omega u\,\mathrm{d}x\Big) - lw^\alpha\Big(\Delta w + a\int_\Omega w\,\mathrm{d}x\Big)\\ &= lv^\alpha\Big(\Delta\hat u + a\int_\Omega \hat u\,\mathrm{d}x\Big) + l(v^\alpha - w^\alpha)\Big(\Delta w + a\int_\Omega w\,\mathrm{d}x\Big)\\ &= lv^\alpha\Big(\Delta\hat u + a\int_\Omega \hat u\,\mathrm{d}x\Big) + \Big(l\alpha\int_0^1[w+s(v-w)]^{\alpha-1}\,\mathrm{d}s\Big[\Delta w + a\int_\Omega w\,\mathrm{d}x\Big]\Big)\hat v,\\ &\quad x\in\Omega_*,\ 0<t<T^*,\\ \hat v_t &\geqslant lu^\alpha\Big(\Delta\hat v + b\int_\Omega \hat v\,\mathrm{d}x\Big) + \Big(l\alpha\int_0^1[w+s(u-w)]^{\alpha-1}\,\mathrm{d}s\Big[\Delta w + a\int_\Omega w\,\mathrm{d}x\Big]\Big)\hat u,\\ &\quad x\in\Omega_*,\ 0<t<T^*,\\ &\qquad \hat u(x,t)\geqslant 0,\quad \hat v(x,t)\geqslant 0,\quad (x,t)\in\partial\Omega_*\times(0,T^*)\cup\Omega_*\times\{0\}.\end{aligned}$$

Lemma 1 shows that $\hat u,\hat v\geqslant 0$, i.e. $u\geqslant w$, $v\geqslant w$ in $\overline{\Omega}_*\times(0,T^*)$. It is a contradiction. Therefore, (u,v) blows up in finite time since the uniqueness holds. The proof is completed. □

Lemma 6. *Assume that $b > 1/\mu$. Then the problem* (1.1) *does not have any global positive classical solution.*

Proof. Applying (H1) and (H2) we see that there exists a small constant κ such that

$$u_0(x) \geqslant \kappa\varphi(x), \quad v_0(x) \geqslant \kappa\varphi(x), \quad x \in \overline{\Omega},$$

where $\varphi(x)$ is defined in (1.3). Now let the positive constant K satisfy (3.1), and let $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ be the positive classical solutions of (1.1) with initial data $(\overline{u}_0, \overline{v}_0) = (K\varphi, K\varphi)$ and $(\underline{u}_0, \underline{v}_0) = (\kappa\varphi, \kappa\varphi)$, respectively. Since

$$\begin{aligned}
\Delta\overline{u}_0 + a\int_\Omega \overline{u}_0 \,\mathrm{d}x &= K(a\mu - 1) > 0, \quad \Delta\overline{v}_0 + b\int_\Omega \overline{v}_0 \,\mathrm{d}x = K(b\mu - 1) > 0 \quad \text{in } \Omega,\\
\Delta\underline{u}_0 + a\int_\Omega \underline{u}_0 \,\mathrm{d}x &= k(a\mu - 1) > 0, \quad \Delta\underline{v}_0 + b\int_\Omega \underline{v}_0 \,\mathrm{d}x = k(b\mu - 1) > 0 \quad \text{in } \Omega,
\end{aligned}$$

in view of Lemma 4 we see that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ blow up in finite times $\overline{T}$ and $\underline{T}$, respectively. Moreover,

$$\begin{aligned}
\Delta\overline{u} + a\int_\Omega \overline{u} \,\mathrm{d}x > 0, \quad \Delta\overline{v} + b\int_\Omega \overline{v} \,\mathrm{d}x > 0 \quad \text{in } \Omega \times (0, \overline{T}),\\
\Delta\underline{u} + a\int_\Omega \underline{u} \,\mathrm{d}x > 0, \quad \Delta\underline{v} + b\int_\Omega \underline{v} \,\mathrm{d}x > 0 \quad \text{in } \Omega \times (0, \underline{T}).
\end{aligned}$$

Since

$$\begin{aligned}
\underline{u}_0(x) = \kappa\varphi(x) \leqslant u_0(x) \leqslant K\varphi(x) = \overline{u}_0(x),\\
\underline{v}_0(x) = \kappa\varphi(x) \leqslant v_0(x) \leqslant K\varphi(x) = \overline{v}_0(x),
\end{aligned}$$

by the same arguments as those at the end of Step 3 of Lemma 4 we can prove that $\overline{T} \leqslant \underline{T}$ and

$$\underline{u}(x,t) \leqslant u(x,\, t) \leqslant \overline{u}(x,\, t), \quad \underline{v}(x,t) \leqslant v(x,t) \leqslant \overline{v}(x,t) \quad \text{in } \Omega \times (0, \overline{T}).$$

This shows that (u, v) cannot exist globally. Lemma 6 is proved. □

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