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A CHARACTERIZATION OF COMMUTATIVE BASIC ALGEBRAS

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Abstract. A basic algebra is an algebra of the same type as an MV-algebra and it is in a one-to-one correspondence to a bounded lattice having antitone involutions on its principal filters. We present a simple criterion for checking whether a basic algebra is commutative or even an MV-algebra.

Keywords: lattice with section antitone involution, basic algebra, commutative basic algebra, MV-algebra

MSC 2010: 06D35, 06F35, 03G10

1. Introduction

The concept of a basic algebra was introduced in [3] and used in [4] and [5] for a lattice theoretical approach to MV-algebras (see e.g. [6] for this concept). For reader's convenience, we repeat basic definitions.

By a *lattice with section antitone involutions* we mean a system $\mathscr{L} = (L;\vee,\wedge,({}^a)_{a\in L},0,1)$ where $(L;\vee,\wedge,0,1)$ is a bounded lattice such that for each $a\in L$ there is an antitone involution $x\mapsto x^a$ in the principal filter $[a,1]$ (the so-called *section*), i.e. $x^{aa}=x$ and $x\leqslant y$ implies $y^a\leqslant x^a$ for $x,y\in[a,1]$.

The family $({}^a)_{a\in L}$ of section antitone involutions that are partial unary operations on L can be equivalently replaced by a single binary operation $\rightarrow$ defined by

$$x\rightarrow y := (x\vee y)^y.$$

Hence, a lattice with section antitone involution can be considered an algebra $(L;\vee,\wedge,\rightarrow,0,1)$ of type $(2,2,2,0,0)$, see [3] and [5] for detailes.

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Let us recall that an MV-algebra is an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(MV2) $x \oplus y = y \oplus x$;

(MV3) $x \oplus 0 = x$;

(MV4) $\neg\neg x = x$;

(MV5) $x \oplus \neg 0 = \neg 0$;

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

This algebra forms an algebraic counterpart of Łukasiewicz many-valued logic, see e.g. [6] as the source. This concept was generalized as follows (see e.g. [5]).

By a *basic algebra* we mean an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(A1) $x \oplus 0 = x$;

(A2) $\neg\neg x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$ where $1 = \neg 0$;

(A4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$;

(A5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

A basic algebra $\mathscr{A}$ is *commutative* if it satisfies the commutativity identity

$$x \oplus y = y \oplus x.$$

It was shown in [5] that a basic algebra is an MV-algebra if it is commutative and associative, i.e. if, moreover, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

The following essential result was proved in [5], see also [2], [3] or [1]:

Proposition. (a) *Let $\mathscr{L} = (L; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$ be a lattice with section antitone involutions. Then the assigned algebra $\mathscr{A}(L) = (L; \oplus, \neg, 0)$, where*

$$x \oplus y = (x^0 \vee y)^y \qquad \text{and} \qquad \neg x = x^0,$$

is a basic algebra.

(b) *Conversely, given a basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, we can assign a bounded lattice with section antitone involutions $\mathscr{L}(A) = (A; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$, where $1 = \neg 0$,*

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

and for each $a \in A$, the mapping $x \mapsto x^a = \neg x \oplus a$ is an antitone involution on the principal filter $[a, 1]$, where the order is given by

$$x \leqslant y \qquad \text{if and only if} \qquad \neg x \oplus y = 1.$$

(c) *The assignments are in a one-to-one correspondence, i.e.* $\mathscr{A}(\mathscr{L}(A)) = \mathscr{A}$ *and* $\mathscr{L}(\mathscr{A}(L)) = \mathscr{L}$.

We can notice that, given a basic algebra $\mathscr{A}$, then $x \to y = \neg x \oplus y$, i.e. $\neg x = x \to 0$ and $x \oplus y = (x \to 0) \to y$.

Hence, when investigating basic algebras, we can switch to lattices with section antitone involution whenever it is useful.

For example, it was shown in [5] that a basic algebra $\mathscr{A}$ is an MV-algebra if and only if it satisfies the so-called *Exchange Identity*

$$\text{(EI)} \qquad x \to (y \to z) = y \to (x \to z).$$

It was proved in [4] that a basic algebra is an MV-algebra if and only if it is a BCC-algebra with respect to the term operation $\to$.

2. Commutative basic algebras

According to [5] (see also [1]), if a basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is commutative then the assigned lattice $\mathscr{L}(A)$ is distributive. The converse is not true in general.

Example 1. Consider the lattice $\mathscr{H}$ as shown in Fig. 1. The section antitone

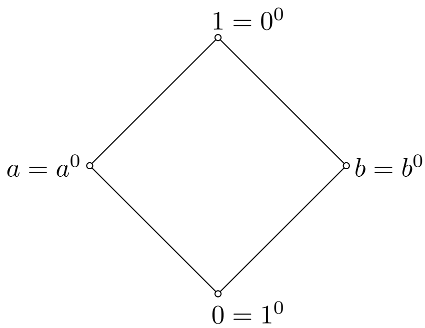

Fig. 1

involutions on two-element sections $[a, 1]$ and $[b, 1]$ are determined uniquely. The lattice $\mathscr{H}$ is distributive but the assigned basic algebra $\mathscr{A}(H)$ is not commutative since

$$a \oplus b = (a^0 \vee b)^b = 1^b = b \qquad \text{and}$$
$$b \oplus a = (b^0 \vee a)^a = 1^a = a.$$

We are going to characterize commutative basic algebras. First we state

Lemma 1. *Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra, $\mathscr{L}(A)$ the assigned lattice. Then*
(a) *$x \oplus y = (x \wedge y^0) \oplus y$ for every $x, y \in A$;*
(b) *if x^0, y are comparable then $x \oplus y = x \oplus (y \wedge x^0)$.*

Proof. (a): By (b) of Proposition, we can apply the de Morgan law to compute

$$(x \wedge y^0) \oplus y = ((x \wedge y^0)^0 \vee y)^y = ((x^0 \vee y) \vee y)^y = (x^0 \vee y)^y = x \oplus y.$$

(b) if $y \leqslant x^0$ then clearly $x \oplus (y \wedge x^0) = x \oplus y$. Assume $x^0 \leqslant y$. Then

$$x \oplus y = (x^0 \vee y)^y = y^y = 1 = (x^0)^{(x^0)} = (x^0 \vee x^0)^{(x^0)} = x \oplus x^0 = x \oplus (y \wedge x^0).$$

□

Using the formulas for $\rightarrow$ (after Proposition), one can easily check that a basic algebra $\mathscr{A}$ is commutative if and only if it satisfies the so-called "law of contraposition", i.e.

$$a \rightarrow b = \neg b \rightarrow \neg a.$$

Namely,

$$x \oplus y = \neg x \rightarrow y = \neg y \rightarrow x = y \oplus x.$$

A simple conclusion is that if a corresponding logic satisfies the law of contraposition and the double negation law then its lattice is distributive.

We are going to show that the afore mentioned condition can be weakened.

Theorem 1. *A basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is commutative if and only if it satisfies the following two conditions:*
(i) *$(x^0)^{(y^0)} = y^x$ for $x \leqslant y$;*
(ii) *$x \oplus (y \wedge x^0) = x \oplus y$.*

Proof. Assume that $\mathscr{A}$ is commutative and $x \leqslant y$. Then $y^0 \leqslant x^0$ and $(x^0)^{(y^0)} = (x^0 \vee y^0)^{(y^0)} = x \oplus y^0 = y^0 \oplus x = (y \vee x)^x = y^x$ (see also Claim 2 in [1]). Further, using (a) of Lemma 1,

$$x \oplus (y \wedge x^0) = (y \wedge x^0) \oplus x = y \oplus x = x \oplus y.$$

Hence, $\mathscr{A}$ satisfies both (i) and (ii).

Conversely, let $\mathscr{A}$ satisfy (i) and (ii). Denote $c = x \wedge y^0$. Then $c^0 = x^0 \vee y$ and hence $y \leqslant c^0$. Applying (i) we obtain

$$c \oplus y = (c^0 \vee y)^y = (c^0)^y = (y^0)^c = (y^0 \vee c)^c = y \oplus c.$$

Using (a) of Lemma 1 and (ii), we conclude

$$x \oplus y = (x \wedge y^0) \oplus y = c \oplus y = y \oplus c = y \oplus (x \wedge y^0) = y \oplus x,$$

thus $\mathscr{A}$ is commutative. $\square$

Example 2. Denote by $\mathbb{R}_0$ the set of all non-negative real numbers and set $\mathbb{R}_\infty = \mathbb{R}_0 \cup \{\infty\}$. For numbers from $\mathbb{R}_0$ we apply arithmetic operations and we define $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, $\infty \pm b = \infty$, $b < \infty$ for any real number b. Then $\mathbb{R}_\infty$ is a chain which can be viewed as a lattice where

$$x \vee y = \max(x, y), \qquad x \wedge y = \min(x, y).$$

Define the section antitone involutions as

$$a^b = \frac{1}{a-b} + b \quad \text{for} \quad b \leqslant a \quad \text{and} \quad \infty^\infty = \infty.$$

It is apparent that the mapping $x \mapsto x^b$ is antitone for any $b \in \mathbb{R}_\infty$ and for $x \in [b, \infty]$ we have

$$x^{bb} = \frac{1}{(1/(x-b)+b)-b} + b = (x-b) + b = x,$$

thus it is really an involution. Further,

$$\begin{aligned} b^b &= \frac{1}{b-b} + b = \frac{1}{0} + b = \infty + b = \infty \qquad \text{and} \\ \infty^b &= \frac{1}{\infty - b} + b = \frac{1}{\infty} + b = 0 + b = b. \end{aligned}$$

Altogether, $\mathscr{R}_\infty = (\mathbb{R}_\infty; \max, \min, ({}^b)_{b\in\mathbb{R}_\infty}, 0, \infty)$ is a lattice with section antitone involutions.

Consider the assigned basic algebra $\mathscr{A}(\mathbb{R}_\infty)$. One can easily see that $\mathscr{A}(\mathbb{R}_\infty)$ satisfies (ii) of Theorem 1 due to (b) of Lemma 1 (since $\mathbb{R}_\infty$ is a chain). On the contrary, $\mathscr{A}(\mathbb{R}_\infty)$ does not satisfy (i) of Theorem 1, e.g. for $b = 1$ and $a = 2$ we have

$$\begin{aligned} a^b &= \frac{1}{2-1} + 1 = 2, \qquad a^0 = \frac{1}{2}, \qquad b^0 = 1 \qquad \text{but} \\ (b^0)^{(a^0)} &= \frac{1}{1-\frac{1}{2}} + \frac{1}{2} = 2 + \frac{1}{2} \neq 2 = a^b. \end{aligned}$$

Due to Theorem 1, $\mathscr{A}(\mathbb{R}_\infty)$ is not commutative.

Theorem 2. *The conditions* (i) *and* (ii) *of Theorem* 1 *are independent.*

P r o o f. As pointed out in Example 3, $\mathscr{A}(\mathbb{R}_\infty)$ satisfies (ii) but not (i). On the contrary, $\mathscr{A}(H)$ of Example 1 satisfies (i), the verification is almost trivial. However, $\mathscr{A}(H)$ does not satisfy (ii):

$$a \oplus (b \wedge a^0) = a \oplus 0 = a \neq b = a \oplus b.$$

$\square$

R e m a r k. It is easy to check that every section involution $x \mapsto x^b$ of $\mathscr{A}(\mathbb{R}_\infty)$ has just one fix-point which is equal to $b + 1$.

Lemma 2. *Let* $\mathscr{A} = (A; \oplus, \neg, 0)$ *be a basic algebra satisfying* (i), *let* $x, y \leqslant a$ *be elements of* A. *Then*

(a) *if* $a^x = a^y$ *then* $x = y$;

(b) $a^{x\wedge y} = a^x \wedge a^y$ *and* $a^{x\vee y} = a^x \vee a^y$.

P r o o f. (a) By (i) we have $(x^0)^{(a^0)} = a^x = a^y = (y^0)^{(a^0)}$. Using the section involution in $[a^0, 1]$ we obtain $x^0 = y^0$, thus $x = x^{00} = y^{00} = y$.

(b) Since the section involutions are antitone, we apply (i) and the de Morgan laws to compute

$$a^{x\vee y} = ((x \vee y)^0)^{(a^0)} = (x^0 \wedge y^0)^{(a^0)} = (x^0)^{(a^0)} \vee (y^0)^{(a^0)} = a^x \vee a^y,$$
$$a^{x\wedge y} = ((x \wedge y)^0)^{(a^0)} = (x^0 \vee y^0)^{(a^0)} = (x^0)^{(a^0)} \wedge (y^0)^{(a^0)} = a^x \wedge a^y.$$

$\square$

Corollary 1. *Let* $\mathscr{A} = (A; \oplus, \neg, 0)$ *be a basic algebra satisfying* (i), *where the involution* $x \mapsto x^0$ *in* $\mathscr{L}(A)$ *has two distinct fix-points. Then* $\mathscr{A}$ *is not commutative.*

P r o o f. Assume $a \neq b$ are fix-points of $x \mapsto x^0$, i.e. $a^0 = a$, $b^0 = b$. Then $a \oplus b = (a^0 \vee b)^b = (a \vee b)^b$ and $b \oplus a = (b^0 \vee a)^a = (b \vee a)^a = (a \vee b)^a$. Clearly $a, b \leqslant a \vee b$. If $a \oplus b = b \oplus a$ then $(a \vee b)^a = (a \vee b)^b$ and, by Lemma 2, $a = b$, a contradiction. $\square$

If a basic algebra $\mathscr{A}$ is an MV-algebra then there is at most one fix-point for every section antitone involution of $\mathscr{L}(A)$. Hence, we incline to recognize that $\mathscr{A}(H)$ of Example 1 is not commutative since the involution $x \mapsto x^0$ has two distinct fix-points (namely a and b). However, the following example shows that this is not the case.

E x a m p l e 3. Let $\mathscr{L}$ be a lattice with section antitone involutions depicted in Fig. 2, where the section involutions (in more than two-element sections) are as follows:

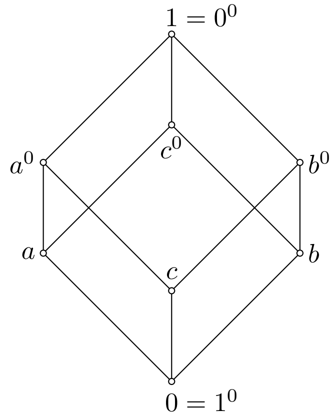

Fig. 2

$[a, 1]\colon\ a \to 1,\ a^0 \to c^0,\ c^0 \to a^0,\ 1 \to a,$
$[b, 1]\colon\ b \to 1,\ b^0 \to c^0,\ c^0 \to b^0,\ 1 \to b,$
$[c, 1]\colon\ c \to 1,\ a^0 \to b^0,\ b^0 \to a^0,\ 1 \to c.$

The only section involution having a fix-point is the trivial one on the trivial section $[1, 1]$. On the other hand, the assigned basic algebra $\mathscr{A}(L)$ is not commutative since $a \oplus b = b$ and $b \oplus a = a$.

A basic algebra $\mathscr{A}$ is called *linearly ordered* if the assigned lattice $\mathscr{L}(A)$ is a chain. By (b) of Lemma 1, every linearly ordered basic algebra satisfies (ii). Hence, we conclude

Corollary 2. *A linearly ordered basic algebra is commutative if and only if it satisfies* (i) *of Theorem* 1.

3. MV-algebras

As mentioned in the introduction, a basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is an MV-algebra if and only if $\mathscr{A}$ is commutative and associative. In what follows, we will characterize whether $\mathscr{A}$ is an MV-algebra in a way similar to that used for commutativity in the previous chapter.

Theorem 3. *A basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is an* MV*-algebra if and only if it satisfies the condition*
(iii) $a^{(b^c)} = b^{(a^c)}$ *for* $c \leqslant b$ *and* $b^c \leqslant a$.

P r o o f. Assume $\mathscr{A}$ satisfies (iii). Let $x, y, z \in A$. An immediate reflexion shows that

$$x \to y = (x \vee y)^y = ((x \vee y) \vee y)^y = (x \vee y) \to y.$$

Hence, $x \to (y \to z)$ can be rewritten as

$$x \to (y \to z) = (x \vee ((y \vee z) \to z)) \to ((y \vee z) \to z) = a \to (b \to z)$$

where $b = y \vee z \geqslant z$ and $a = x \vee b^z \geqslant b^z$, i.e. $x \to (y \to z) = a^{(b^z)}$. By (iii) we have $x \to (y \to z) = b^{(a^z)}$ and, analogously, we can derive $y \to (x \to z) = b^{(a^z)}$. Hence (EI) holds, thus $\mathscr{A}$ is an MV-algebra.

The converse follows directly from the fact that every MV-algebra satisfies (EI) and hence also (iii), see e.g. Theorem 8.5.10 in [5]. □

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