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# ON HÖLDER REGULARITY FOR VECTOR-VALUED MINIMIZERS OF QUASILINEAR FUNCTIONALS

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*Abstract.* We discuss the interior Hölder everywhere regularity for minimizers of quasilinear functionals of the type

$$\mathcal{A}(u;\Omega)=\int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j\,\mathrm{d}x$$

whose gradients belong to the Morrey space $L^{2,n-2}(\Omega,\mathbb{R}^{nN})$.

*Keywords*: quasilinear functional, minimizer, regularity, Campanato-Morrey space

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## 1. Introduction

In this paper we study the interior everywhere regularity of functions minimizing variational integrals

$$\mathcal{A}(u;\Omega)=\int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j\,\mathrm{d}x \tag{1.1}$$

where $u\colon\ \Omega\to\mathbb{R}^N$, $N>1$, $\Omega\subset\mathbb{R}^n$, $n\geqslant 3$ is a bounded open set, $x=(x_1,\dots,x_n)\in\Omega$, $u(x)=(u^1(x),\dots,u^N(x))$, $Du=\{D_\alpha u^i\}$, $D_\alpha=\partial/\partial x_\alpha$, $\alpha=1,\dots,n$, $i=1,\dots,N$.

Throughout the whole text we use the summation convention over repeated indices. We call a function $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ a minimizer of the functional $\mathcal{A}(u;\Omega)$ if

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and only if $\mathcal{A}(u;\Omega) \leqslant \mathcal{A}(v;\Omega)$ for every $v \in W^{1,2}(\Omega,\mathbb{R}^N)$ with $u - v \in W_0^{1,2}(\Omega,\mathbb{R}^N)$. For more information see [6], [9].

On the functional $\mathcal{A}$ we assume the following conditions:

(i) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, $A_{ij}^{\alpha\beta}$ are continuous functions in $u \in \mathbb{R}^N$ for every $x \in \Omega$ and there exists $M > 0$ such that $|A_{ij}^{\alpha\beta}(x,u)| \leqslant M$, $\forall\, x \in \Omega$, $\forall\, u \in \mathbb{R}^N$.

(ii) (ellipticity) There exists $\nu > 0$ such that

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i\xi_\beta^j \geqslant \nu|\xi|^2, \quad \forall\, x \in \Omega, \quad \forall\, u \in \mathbb{R}^N, \quad \forall\, \xi \in \mathbb{R}^{nN}. \tag{1.2}$$

(iii) (oscillation of coefficients) There exists a real function $\omega$ continuous on $[0,\infty)$ which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and $u$, $v \in \mathbb{R}^N$

$$|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)| \leqslant \omega\left(|u-v|\right). \tag{1.3}$$

We set $\omega_\infty = \lim\limits_{t\to\infty} \omega(t) \leqslant 2M$.

(iv) For all $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot,u) \in \mathrm{VMO}(\Omega)$ (uniformly with respect to $u \in \mathbb{R}^N$).

It is well known (see [6], p. 169) that (iii) implies absolute continuity of $\omega$ on $[0,\infty)$. In what follows, by pointwise derivative $\omega'$ of $\omega$ we will understand the right derivative which is finite on $(0,\infty)$. Considering the assumption (iv) it is worth recalling that since $C^0$ is a proper subset of VMO, the continuity of coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$ with respect to $x$ is not supposed.

In this paper we deal with the case $n \geqslant 3$ because for $n = 2$ higher integrability of the gradient of minimizer (see Preliminaries, Lemma 2.4) and the Sobolev imbedding theorem imply that $u$ is locally Hölder continuous in $\Omega$. From many examples (see [4], [6], [9], [10], [12], [14]) for $n \geqslant 3$ it is known that the minimizer $u$ of the functional (1.1) need not be continuous or bounded even in the case of smooth coefficients $A_{ij}^{\alpha\beta}$. For this reason the so called partial regularity for minimizers of the functional (1.1) was studied by many authors ([7], [8], [5]). In our paper (which is motivated by [3]) we concentrate on conditions that imply an everywhere regularity result. More precisely, we state conditions which imply that the minimizer $u$ with gradient $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ belongs to $C^{0,\gamma}(\Omega,\mathbb{R}^N)$. The condition $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ seems to be natural with respect to the paper [2].

Now we can state the following result:

**Theorem 1.1.** *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimizer of the functional* (1.1) *such that $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ and let the hypotheses* (i), (ii), (iii), (iv) *be satisfied. Assume that there exists $p > 1$ such that*

$$Q_p := \min\left\{ \sup_{t\in(0,\infty)} \frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p/(p-1)})(t), \int_0^\infty t^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p/(p-1)})(t)\,\mathrm{d}t \right\} < \infty$$

*and let* $\gamma \in (0,1)$. *Then the inequality*

$$(Q_p \|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})})^{1-1/p} \leqslant \nu C \tag{1.4}$$

*implies that* $u \in C^{0,\gamma}(\Omega, \mathbb{R}^N)$.

*Here*

$$C = \frac{2}{3c(n,N,p,M/\nu)(2^{n+3}L)^{\frac{1}{2}n/(1-\gamma)}},$$

*where* $L$ *is from Lemma* 2.3.

## 2. Preliminaries

If $x \in \mathbb{R}^n$ and $r$ is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n \colon |y-x| < r\}$, $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by

$$u_{x,r} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} u(y)\,\mathrm{d}y = \fint_{\Omega_r(x)} u(y)\,\mathrm{d}y$$

the mean value of the function $u \in L^1(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$, where $|\Omega_r(x)|$ is the $n$-dimensional Lebesgue measure of $\Omega_r(x)$.

Beside the standard space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$ we use Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ (for more detail see e.g. [11]).

For $f \in L^1(\Omega)$, $0 < a < \infty$ we set

$$\mathcal{M}_a(f,\Omega) := \sup_{x\in\Omega, r<a} \fint_{\Omega_r(x)} |f(y) - f_{x,r}|\,\mathrm{d}y.$$

**Definition 2.1** (see [13]). A function $f \in L^1(\Omega)$ is said to belong to $\mathrm{BMO}(\Omega)$ if

$$\mathcal{M}_{\operatorname{diam}\Omega}(f,\Omega) < \infty;$$

a function $f \in L^1(\Omega)$ is said to belong to $\mathrm{VMO}(\Omega)$ if

$$\lim_{a\to 0} \mathcal{M}_a(f,\Omega) = 0.$$

In the proof of the theorem we will use the following results.

**Lemma 2.1** ([15], p. 37). *Let $\psi\colon [0,\infty) \to [0,\infty]$ be a non decreasing function which is absolutely continuous on every closed interval of finite length, $\psi(0)=0$. If $w \geqslant 0$ is measurable and $E(t)=\{y\in\mathbb{R}^n\colon\ w(y)>t\}$ then*

$$\int_{\mathbb{R}^n} \psi\circ w\,\mathrm{d}y = \int_0^\infty \mu\left(E(t)\right)\psi'(t)\,\mathrm{d}t.$$

**Proposition 2.1** (see [1], [6], [11]). *For a bounded domain $\Omega\subset\mathbb{R}^n$ with a Lipschitz boundary, for $q\in[1,\infty)$ and $0<\lambda<\mu\leqslant n$ we have*

(a) $L^{q,\mu}(\Omega,\mathbb{R}^N)\subsetneq L^{q,\lambda}(\Omega,\mathbb{R}^N)$;

(b) $L^{q,n}(\Omega,\mathbb{R}^N)$ *is isomorphic to the* $L^\infty(\Omega,\mathbb{R}^N)$;

(c) *if* $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *and* $Du\in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$, $\lambda\in(n-2,n)$ *then* $u\in C^{0,\alpha}(\Omega,\mathbb{R}^N)$, $\alpha=(\lambda+2-n)/2$.

**Lemma 2.2** (see [1]). *Let $A$, $d$ be positive constants, $\beta\in(0,n)$. Then there exist $\varepsilon_0$, $C$ positive such that for any nonnegative, nondecreasing function $\varphi$ defined on $[0,2d]$ and satisfying the inequality*

$$\varphi(\sigma)\leqslant\Big(A\Big(\frac{\sigma}{R}\Big)^n+K\Big)\varphi(2R)\qquad\forall\,0<\sigma<R\leqslant d \tag{2.1}$$

*with $K\in(0,\varepsilon_0]$ we have*

$$\varphi(\sigma)\leqslant C\sigma^\beta(2d)^{-\beta}\varphi(2d),\qquad\forall\sigma\colon\ 0<\sigma\leqslant d. \tag{2.2}$$

**Lemma 2.3** (see e.g. [1], [6]). *Let $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to the system*

$$-D_\alpha(A^{\alpha\beta}_{ij}D_\beta u^j)=0,\quad i=1,\dots,N$$

*where $A^{\alpha\beta}_{ij}$ are constants satisfying* (i) *and* (ii). *Then there exists a constant $L=L(n,M/\nu)\geqslant 1$ such that for every weak solution $v\in W^{1,2}(\Omega,\mathbb{R}^N)$, for every $x\in\Omega$ and $0<\sigma\leqslant R\leqslant\operatorname{dist}(x,\partial\Omega)$ the estimate*

$$\int_{B_\sigma(x)}|Du(y)|^2\,\mathrm{d}y\leqslant L\Big(\frac{\sigma}{R}\Big)^n\int_{B_R(x)}|Du(y)|^2\,\mathrm{d}y$$

*holds.*

**Lemma 2.4** (see [6], [9]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of the functional* (1.1) *under the assumptions* (i) *and* (ii). *Then $Du \in L^{2p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ for some $p > 1$ and there exists a constant $c = c(n, p, M/\nu)$ such that for all balls $B_{2R}(x) \subset \Omega$*

$$\left( \fint_{B_R(x)} |Du|^{2p} \,\mathrm{d}y \right)^{1/2p} \leqslant c \left( \fint_{B_{2R}(x)} |Du|^2 \,\mathrm{d}y \right)^{1/2}$$

*holds.*

Let $x_0$ be any fixed point of $\Omega$, $0 < R \leqslant \operatorname{dist}(x_0, \partial\Omega)$. We set

$$(A^{\alpha\beta}_{ij}(u_{x_0,R}))_{x_0,R} = \fint_{B_R(x_0)} A^{\alpha\beta}_{ij}(y, u_{x_0,R}) \,\mathrm{d}y.$$

If $v$ is a solution to the system

$$\begin{cases} D_\alpha((A^{\alpha\beta}_{ij}(u_{x_0,R}))_{x_0,R} D_\beta v^j) = 0 \text{ in } B_R(x_0), \\ v - u \in W^{1,2}_0(B_R(x_0), \mathbb{R}^N) \end{cases} \tag{2.3}$$

then the next lemma is true.

**Lemma 2.5** (see [6], [9]). *Let $v \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ be a solution to the problem* (2.3) *with $u \in W^{1,2p}(B_R(x_0), \mathbb{R}^N)$, $p \geqslant 1$. Then*

$$\int_{B_R(x)} |Dv|^{2p} \,\mathrm{d}y \leqslant c(M/\nu) \int_{B_R(x)} |Du|^{2p} \,\mathrm{d}y$$

*holds.*

Remark 2.1. Revising proofs of Lemmas 2.4 and 2.5 one can see that the constants from the above estimates depend increasingly on $M/\nu$.

## 3. Proof of theorem

We set $\varphi(r) = \varphi(x_0, r) = \int_{B_r(x_0)} |Du(y)|^2 \,\mathrm{d}y$ for $B_r(x_0) \subset \Omega$. Now let $x_0$ be any fixed point of $\Omega$, dist $(x_0, \partial\Omega) \geqslant 2d > 0$, $R \leqslant d$ and let $v$ be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x_0)) = \int_{B_R(x_0)} (A^{\alpha\beta}_{ij}(u_R))_R D_\alpha v^i D_\beta v^j \,\mathrm{d}x$$

among all functions in $W^{1,2}(B_R(x_0), \mathbb{R}^N)$ taking the values $u$ on $\partial B_R(x_0)$.

From the Euler equation for $v$ and from Lemma (2.3) we have

$$\int_{B_\sigma(x_0)} |Dv|^2 \,\mathrm{d}x \leqslant L\Big(\frac{\sigma}{R}\Big)^n \int_{B_R(x_0)} |Dv|^2 \,\mathrm{d}x, \quad \forall\, 0<\sigma\leqslant R. \tag{3.1}$$

Put $w = u - v$. It is clear that $w \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N)$. Using (3.1), by standard arguments we obtain

(3.2)

$$\int_{B_\sigma(x_0)} |Du|^2 \,\mathrm{d}x \leqslant 2\Big(1 + 2L\Big(\frac{\sigma}{R}\Big)^n\Big) \int_{B_R(x_0)} |Dw|^2 \,\mathrm{d}x + 4L\Big(\frac{\sigma}{R}\Big)^n \int_{B_R(x_0)} |Du|^2 \,\mathrm{d}x.$$

In the sequel we will estimate the first integral on the right hand side of (3.2). From [8] (see Lemma 2.1) we have

$$\begin{aligned}
\int_{B_R(x_0)} |Dw|^2 \,\mathrm{d}x &\leqslant \frac{2}{\nu}(\mathcal{A}^0\,(u; B_R(x_0)) - \mathcal{A}^0\,(v; B_R(x_0))) \\
&\leqslant \frac{2}{\nu}\Big\{ \int_{B_R(x_0)} ((A_{ij}^{\alpha\beta}(u_R))_R - A_{ij}^{\alpha\beta}(x,u_R)) D_\alpha u^i D_\beta u^j \,\mathrm{d}x \\
&\qquad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x,u_R) - A_{ij}^{\alpha\beta}(x,u)) D_\alpha u^i D_\beta u^j \,\mathrm{d}x \\
&\qquad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x,u_R) - (A_{ij}^{\alpha\beta}(u_R))_R) D_\alpha v^i D_\beta v^j \,\mathrm{d}x \\
&\qquad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x,v) - A_{ij}^{\alpha\beta}(x,u_R)) D_\alpha v^i D_\beta v^j \,\mathrm{d}x \\
&\qquad + \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0))\Big\} \\
&= \frac{2}{\nu}\left\{\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathcal{A}\,(u; B_R(x_0)) - \mathcal{A}\,(v; B_R(x_0))\right\} \\
&\leqslant \frac{2}{\nu}\left(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\right).
\end{aligned} \tag{3.3}$$

Notice that $\mathcal{A}\,(u; B_R(x_0)) - \mathcal{A}\,(v; B_R(x_0)) \leqslant 0$, since $u$ is a minimizer.

Now we will estimate the terms I, II, III and IV from (3.3). We will denote $(A_{ij}^{\alpha\beta}) =: A$. Using the Hölder inequality and higher integrability of the gradient of minima ($p > 1$, $p' = p/(p-1)$) we obtain

$$\begin{aligned}
|I| &\leqslant \int_{B_R(x_0)} |(A(u_R))_R - A(x,u_R)|\,|Du|^2 \,\mathrm{d}x \\
&\leqslant cR^{n/p}\Big(\int_{B_R(x_0)} |(A(u_R))_R - A(x,u_R)|^{p'} \,\mathrm{d}x\Big)^{1/p'} \Big(\fint_{B_R(x_0)} |Du|^{2p} \,\mathrm{d}x\Big)^{1/p} \\
&\leqslant c(n,N,p,M/\nu) R^{n/p}\Big(\int_{B_R(x_0)} |(A(u_R))_R - A(x,u_R)|^{p'} \,\mathrm{d}x\Big)^{1/p'} \fint_{B_{2R}(x_0)} |Du|^2 \,\mathrm{d}x.
\end{aligned}$$

Taking into account the assumptions (i), (iv) and Definition 2.1 we obtain

$$|\mathrm{I}| \leqslant c(n,N,p,M/\nu)(2M)^{1/p}\left(\mathcal{M}_R\left(A(\cdot,u_R)\right)\right)^{1/p'}\varphi(2R). \tag{3.4}$$

A similarity of the terms I and III enables us to write (by means of Lemma 2.5, see [2] for details) the inequality

$$|\mathrm{III}| \leqslant c(n,N,p,M/\nu)(2M)^{1/p}\left(\mathcal{M}_R\left(A(\cdot,u_R)\right)\right)^{1/p'}\varphi(2R). \tag{3.5}$$

Using the Hölder inequality, property (iii) and Lemma 2.4 we get

$$|\mathrm{II}| \leqslant c(n,N,p,M/\nu)\biggl(\frac{1}{R^n}\int_{B_R(x_0)}\omega^{p'}(|u-u_R|)\,\mathrm{d}x\biggr)^{1/p'}\varphi(2R).$$

Taking in Lemma 2.1 $\psi(t)=\omega^{p'}(t)$, $w=|u-u_R|$ on $B_R(x_0)$ and $w=0$ out of $B_R(x_0)$, we have $E_R(t)=\{y\in B_R\colon\ |u-u_R|>t\}$ and so we get

$$\int_{B_R(x_0)}\omega^{p'}\left(|u-u_R|\right)\,\mathrm{d}x=\int_0^\infty\Bigl[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\Bigr]\mu\left(E_R(t)\right)\,\mathrm{d}t.$$

Now under the assumptions of Theorem 1.1 if we suppose

$$Q_p=\int_0^\infty t^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\,\mathrm{d}t<\infty,$$

then (taking into account that $\mu(E_R(t))\leqslant t^{-1}\int_0^t\mu(E_R(s))\,\mathrm{d}s$) we have

$$\begin{aligned}\int_0^\infty\Bigl[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\Bigr]\mu(E_R(t))\,\mathrm{d}t&\leqslant\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\biggl(\frac{1}{t}\int_0^t\mu(E_R(s))\,\mathrm{d}s\biggr)\,\mathrm{d}t\\&\leqslant Q_p\int_{B_R(x_0)}|u-u_R|\,\mathrm{d}x.\end{aligned}$$

On the other hand, if we suppose $Q_p=\sup\limits_{t\in(0,\infty)}(\mathrm{d}/\mathrm{d}t)(\omega^{p'})(t)<\infty$ then

$$\int_0^\infty\Bigl[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\Bigr]\mu(E_R(t))\,\mathrm{d}t\leqslant Q_p\int_{B_R(x_0)}|u-u_R|\,\mathrm{d}x$$

holds as well. So in both the cases we have

$$\int_{B_R(x_0)}\omega^{p'}\left(|u-u_R|\right)\,\mathrm{d}x\leqslant Q_p\int_{B_R(x_0)}|u-u_R|\,\mathrm{d}x.$$

Using the Poincaré inequality and the assumption about $Du$ we finally get

$$|\mathrm{II}| \leqslant c(n,N,p,M/\nu)Q_p^{1/p'}\|Du\|^{1/p'}_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\varphi(2R). \tag{3.6}$$

Combining the last arguments with Lemma 2.4 and Lemma 2.5 we can conclude in a similar way

$$|\mathrm{IV}| \leqslant c(n,N,p,M/\nu)Q_p^{1/p'}\|Du\|^{1/p'}_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\varphi(2R). \tag{3.7}$$

Estimates (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) lead to the following inequality

$$\begin{aligned}\varphi(\sigma) &= \int_{B_\sigma(x_0)} |Du|^2\,\mathrm{d}x \\ &\leqslant \Big\{4L\Big(\frac{\sigma}{R}\Big)^n + \frac{8}{\nu}\Big(1+2L\Big(\frac{\sigma}{R}\Big)^n\Big) \\ &\quad \times c[(2M)^{1/p}\left(\mathcal{M}_R\left(A(\cdot,u_R)\right)\right)^{1/p'} + \big(Q_p\|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\big)^{1/p'}]\Big\}\varphi(2R)\end{aligned}$$

where $c = c(n,N,p,M/\nu)$.

Now we can use Lemma 2.2 in the following manner:

We take $\gamma \in (0,1)$ and set

$$A = 4L, \qquad \varepsilon_0 = \frac{1}{2(2^{n+3}L)^{(n-2+2\gamma)/2(1-\gamma)}}$$

and

$$K = \frac{8}{\nu}\,(1+2L)\,c[(2M)^{1/p}\left(\mathcal{M}_R\left(A(\cdot,u_R)\right)\right)^{1/p'} + \big(Q_p\|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\big)^{1/p'}].$$

Then the assumption (1.4) and a suitable small $d > 0$ (remember the condition (iv) and Definition 2.1) imply that $K < \varepsilon_0$ and hence

$$\varphi(\sigma) \leqslant c\sigma^{n-2+2\gamma}.$$

The result is then a consequence of Proposition 2.1.(c)

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