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ON THE OSCILLATION OF CERTAIN CLASS OF THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider the third-order nonlinear delay differential equation

$$(*)\qquad (a(t)\left(x''(t)\right)^{\gamma})' + q(t)x^{\gamma}(\tau(t)) = 0, \quad t \geqslant t_0,$$

where $a(t)$, $q(t)$ are positive functions, $\gamma > 0$ is a quotient of odd positive integers and the delay function $\tau(t) \leqslant t$ satisfies $\lim\limits_{t\to\infty} \tau(t) = \infty$. We establish some sufficient conditions which ensure that $(*)$ is oscillatory or the solutions converge to zero. Our results in the nondelay case extend and improve some known results and in the delay case the results can be applied to new classes of equations which are not covered by the known criteria. Some examples are considered to illustrate the main results.

Keywords: third-order differential equation, oscillation, nonoscillation, disconjugacy

MSC 2010: 34K11, 34C10

1. Introduction

In the recent years, the qualitative theory of differential equations and their applications have received intensive attention. Although the second-order differential equations have been studied extensively, the study of qualitative behavior of third-order differential equations has received considerably less attention. We mention here papers [3], [4], [5], [8], [10], [12], [13], [15], [16], [17], [18], [19], [20] and the references cited therein.

In this paper we are concerned with oscillation of the third-order delay differential equations of the form

$$(1.1)\qquad (a(t)\left(x''(t)\right)^{\gamma})' + q(t)x^{\gamma}(\tau(t)) = 0, \quad t \geqslant t_0.$$

In the sequel we will assume that the following conditions are satisfied:

(H) $a(t)$, $q(t)$, $\tau(t) \in C([t_0,\infty))$ are positive, γ is a quotient of odd positive integers, $\tau(t) \leqslant t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$, and $\int_{t_0}^{\infty} 1/a^{1/\gamma}(s)\,\mathrm{d}s = \infty$.

Let $T_0 = \min\{\tau(t)\colon\ t \geqslant 0\}$ and $\tau^{-1}(t) = \sup\{s \geqslant 0\colon\ \tau(s) \leqslant t\}$ for $t \geqslant T_0$. Clearly $\tau^{-1}(t) \geqslant t$ for $t \geqslant T_0$, $\tau^{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau(t)$ when the latter exists. By a solution of (1.1) we mean a nontrivial real-valued function $x(t)$ which has the properties $x'(t) \in C^1([\tau^{-1}(t_0),\infty))$ and $a(t)\,(x'(t))^{\gamma} \in C^1([\tau^{-1}(t_0),\infty))$. Our attention is restricted to the solutions of (1.1) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|\colon\ t > t_1\} > 0$ for any $t_1 \geqslant t_x$. We make a standing hypothesis that (1.1) does possess such solutions. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is nonoscillatory. We recall that equation (1.1) is disconjugate on an interval $I = [t_0,\infty)$ provided no nontrivial solution has more than two zeros on $I = [t_0,\infty)$, counting their multiplicity.

In [7] the authors studied oscillation of the third order delay differential equation

$$(a(t)\,(x''(t))^{\gamma})' + q(t)f(x(\tau(t))) = 0, \quad t \geqslant t_0, \tag{1.2}$$

by comparing this equation with the first order delay equation, so that oscillation of the first order equation implied oscillation of the third order equation. But such comparison principle always requires $\tau(t) < t$.

The purpose of this paper is to introduce a different technique to establish some sufficient conditions which guarantee that every solution of (1.1) oscillates or converges to zero. Some illustrative examples are also included.

2. Main results

In this section we establish some new oscillatory criteria for (1.1). First, we state and prove some useful lemmas which we will use later in the proofs of our main results. We note that if $x(t)$ is a solution of (1.1), then $z = -x$ is also a solution of (1.1). Thus, concerning nonoscillatory solutions of (1.1) we can restrict our attention only to the positive ones.

Lemma 1. *Let $x(t)$ be an eventually positive solution of* (1.1). *Then there are only the following two cases for $t \geqslant t_1$, t_1 sufficiently large:*

(i) $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$;
(ii) $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$.

P r o o f. Assume that $x(t)$ is a positive solution of (1.1) on $[t_0, \infty)$. Pick $T \in [t_0, \infty)$ such that $T \geqslant t_0$ and $x(\tau(t)) > 0$ on $[T, \infty)$. From (1.1) and (H) we have

$$(a(t)\,(x''(t))^{\gamma})' = -q(t)x^{\gamma}(\tau(t)) < 0 \quad \text{for } t \geqslant T.$$

Thus $a(t)(x''(t))^{\gamma}$ is nonincreasing and of one sign, which implies that $x''(t)$ is of one sign. If we admit that $x''(t) \leqslant 0$ eventually, then there exists a negative constant d such that

$$a(t)(x''(t))^{\gamma} \leqslant d < 0 \ \text{ for, say, } \ t \geqslant t_1 \geqslant T.$$

Integrating from t_1 to t, we obtain

$$x'(t) \leqslant x'(t_2) + d^{1/\gamma} \int_{t_1}^{t} a^{-1/\gamma}(s)\,\mathrm{d}s. \tag{2.1}$$

Letting $t \to \infty$ and using (H), we get $x'(t) \to -\infty$. Thus, $x'(t) < 0$ eventually. But $x'(t) < 0$ and $x''(t) < 0$ eventually imply $x(t) \to -\infty$ as $t \to \infty$, a contradiction. Hence $x''(t) > 0$. The proof is complete. □

Definition 1. We say that a solution $x(t)$ of (1.1) is from the *class* A if it satisfies (i), and $x(t)$ is from the *class* B if it satisfies (ii)

Lemma 2. *Let $x(t)$ be from the class B. If*

$$\int_{t_0}^{\infty} \int_{z}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} q(s)\,\mathrm{d}s\right]^{1/\gamma} \mathrm{d}u\,\mathrm{d}z = \infty, \tag{2.2}$$

then $\lim\limits_{t\to\infty} x(t) = 0$.

P r o o f. Since $x(t)$ is from the *class* B, so $x(t)$ is positive and decreasing. Therefore there exists a finite

$$\lim_{t\to\infty} x(t) = l.$$

We prove that $l = 0$. Assume this is not the case, i.e., $l > 0$. Hence $x(\tau(t)) \geqslant x(t) > l$ for all $t \geqslant t_2 > t_1$. Integrating (1.1) from t to ∞ and using $x^{\gamma}(\tau(t)) \geqslant l^{\gamma}$, we get

$$x''(t) \geqslant l \left[\frac{1}{a(t)} \int_{t}^{\infty} q(s)\,\mathrm{d}s\right]^{1/\gamma}.$$

Integrating again from t to ∞, we have

$$-x'(t) \geqslant l \int_{t}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} q(s)\,\mathrm{d}s\right]^{1/\gamma} \mathrm{d}u.$$

Integrating from t_2 to ∞, we obtain

$$x(t_2) \geqslant l \int_{t_2}^{\infty} \int_{z}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} q(s)\,\mathrm{d}s\right]^{1/\gamma} \mathrm{d}u\,\mathrm{d}z. \tag{2.3}$$

This is a contradiction with (2.2). Hence $l = 0$ and the proof is complete. □

Lemma 3. *Assume that $z(t) > 0$, $z'(t) \geqslant 0$, $z''(t) \leqslant 0$ on (t_0, ∞). Then for each $l \in (0,1)$ there exists a $T_l \geqslant t_0$ such that*

$$\frac{z(\tau(t))}{\tau(t)} \geqslant l\frac{z(t)}{t} \quad \text{for } t \geqslant T_l.$$

Proof. It follows from the mean value theorem and the monotone properties of $z'(t)$ that

$$z(t) - z(\tau(t)) \leqslant z'(\tau(t))(t - \tau(t))$$

or

$$\frac{z(t)}{z(\tau(t))} \leqslant 1 + \frac{z'(\tau(t))}{z(\tau(t))}(t - \tau(t)). \tag{2.4}$$

Using the mean value theorem once more, we see that

$$z(\tau(t)) \geqslant z(\tau(t)) - z(t_0) \geqslant z'(\tau(t))(\tau(t) - t_0).$$

So for each $l \in (0,1)$ there is a $T_l \geqslant t_0$ such that

$$\frac{z(\tau(t))}{z'(\tau(t))} \geqslant l\tau(t), \quad t \geqslant T_l. \tag{2.5}$$

Combining (2.4) with (2.5), we get

$$\frac{z(t)}{z(\tau(t))} \leqslant 1 + \frac{1}{l\tau(t)}(t - \tau(t)) \leqslant \frac{t}{l\tau(t)}$$

and the proof is complete. □

Lemma 4. *Assume that $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$, $x'''(t) \leqslant 0$ on (T_l, ∞). Then*

$$\frac{x(t)}{x'(t)} \geqslant \frac{t - T_l}{2} \qquad \text{for } t \geqslant T_l.$$

P r o o f. Set

$$X(t) := (t - T_l)x(t) - \frac{(t - T_l)^2}{2} x'(t).$$

Then $X(T_l) = 0$, and

$$X'(t) = x(t) - \frac{(t - T_l)^2}{2} x''(t).$$

We shall prove that $X(t) > 0$. By Taylor's Theorem, since $x''(t)$ is nonincreasing, we have

$$x(t) \geqslant x(T_l) + (t - T_l)x'(T_l) + \frac{(t - T_l)^2}{2} x''(t).$$

This implies

$$X'(t) = x(t) - \frac{(t - T_l)^2}{2} x''(t) \geqslant x(T_l) + (t - T_l)x'(T_l) > 0.$$

Since $X(T_l) = 0$, one gets $X(t) \geqslant 0$ for $t \geqslant T_l$, which implies the desired inequality. $\square$

Lemma 5. *Assume that $x'(t) > 0$, $x''(t) > 0$, $x'''(t) \leqslant 0$ on (T_l, ∞). Then*

$$(t - T_l)\frac{x''(t)}{x'(t)} \leqslant 1 \qquad \text{for } t \geqslant T_l.$$

P r o o f. The result follows from the inequality

$$x'(t) \geqslant \int_{T_l}^{t} x''(s)\,\mathrm{d}s \geqslant x''(t)(t - T_l).$$

$\square$

Now, we present the main results. For simplicity, we introduce the following notation:

$$p_* := \liminf_{t\to\infty} \frac{t^\gamma}{a(t)} \int_t^\infty P_l(s)\,\mathrm{d}s, \quad q_* := \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s, \tag{2.6}$$

where $P_l(s) = l^\gamma q(s)(\tau(s)/s)^\gamma(\frac{1}{2}(\tau(s) - T_l))^\gamma$ with $l \in (0,1)$ arbitrarily chosen and T_l large enough. Moreover, for $x(t)$ from the *class A* we define

$$w(t) := a(t)\Big(\frac{x''(t)}{x'(t)}\Big)^\gamma, \tag{2.7}$$

and

$$r := \liminf_{t\to\infty} \frac{t^{\gamma} w(t)}{a(t)} \quad \text{and} \quad R := \limsup_{t\to\infty} \frac{t^{\gamma} w(t)}{a(t)}. \tag{2.8}$$

Theorem 1. *Assume that $a'(t) \geqslant 0$. Let $x(t)$ be a positive solution of* (1.1).

(I) *Let $p_* < \infty$ and $q_* < \infty$. Suppose that $x(t)$ is from the class A. Then*

$$p_* \leqslant r - r^{1+1/\gamma} \quad \textit{and} \quad p_* + q_* \leqslant 1. \tag{2.9}$$

(II) *If $p_* = \infty$ or $q_* = \infty$, then $x(t)$ is not from the class A.*

Proof. *Part* (I). Assume that $x(t)$ is from the *class A*. First note that $a'(t) \geqslant 0$, which together with

$$0 \geqslant (a(t)(x''(t))^{\gamma})'$$

implies $x'''(t) \leqslant 0$. So there exists a $T \geqslant t_0$ such that $x(t)$ satisfies

$$x(\tau(t)) > 0, \quad x'(t) > 0, \quad x''(t) > 0, \quad x''' \leqslant 0, \quad \text{for } t \in [T, \infty).$$

From the definition of $w(t)$ and (1.1) we see that $w(t)$ is positive and satisfies

$$\begin{aligned} w'(t) &= \frac{(x'(t)))^{\gamma}\left(a(t)\,(x''(t))^{\gamma}\right)' - \left(a(t)\,(x''(t))^{\gamma}\right)\gamma\,(x'(t))^{\gamma-1}\,x''(t)}{(x'(t))^{2\gamma}} \\ &= \frac{\left(a(t)\,(x''(t))^{\gamma}\right)'}{(x(\tau(t)))^{\gamma}}\frac{(x(\tau(t)))^{\gamma}}{(x'(t)))^{\gamma}} - \gamma\frac{\left(a(t)\,(x''(t))^{\gamma}\right)}{(x'(t))^{\gamma}}\frac{x''(t)}{x'(t)} \\ &= -q(t)\frac{(x(\tau(t)))^{\gamma}}{(x'(t))^{\gamma}} - \frac{\gamma}{a^{1/\gamma}(t)} w^{1+1/\gamma}(t). \end{aligned} \tag{2.10}$$

From Lemma 3 with $z(t) = x'(t)$, we have for the same l as in P_l

$$\frac{1}{x'(t)} \geqslant l\frac{\tau(t)}{t}\frac{1}{x'(\tau(t))}, \quad t \geqslant T_l,$$

which together with (2.10) gives

$$w'(t) \leqslant -l^{\gamma} q(t)\Big(\frac{\tau(t)}{t}\Big)^{\gamma}\frac{(x(\tau(t)))^{\gamma}}{(x'(\tau(t))^{\gamma}} - \frac{\gamma}{a^{1/\gamma}(t)} w^{(\gamma+1)/\gamma}(t).$$

Using the fact from Lemma 4 that $x(t) \geqslant \frac{1}{2}(t - T_l)x'(t)$, we have

$$w'(t) + P_l(t) + \frac{\gamma}{a^{1/\gamma}(t)} w^{(\gamma+1)/\gamma}(t) \leqslant 0. \tag{2.11}$$

Since $P_l(t) > 0$ and $w(t) > 0$ for $t \geqslant T_l$, we have from (2.11) that $w'(t) \leqslant 0$, and

$$-(w'(t)/\gamma w^{(\gamma+1)/\gamma}(t)) > \frac{1}{a^{1/\gamma}(t)} \quad \text{for} \quad t \geqslant T_l. \tag{2.12}$$

This implies that

$$(1/w^{1/\gamma}(t))' > \frac{1}{a^{1/\gamma}(t)}. \tag{2.13}$$

Integrating the last inequality from T_l to t, we obtain

$$w(t) < \frac{1}{\left(\int_{T_l}^{t} \mathrm{d}s/a^{1/\gamma}(s)\right)^{\gamma}}, \tag{2.14}$$

which in view of (H) implies that $\lim\limits_{t\to\infty} w(t) = 0$. On the other hand, from the definition of $w(t)$ and Lemma 5 we see that

$$0 \leqslant r \leqslant R < k < \infty. \tag{2.15}$$

Now, we prove that the first inequality in (2.9) holds. Let $\varepsilon > 0$, then by the definitions of p_* and r we can pick $t_2 \in [T_l, \infty)$ sufficiently large so that

$$\frac{t^\gamma}{a(t)} \int_t^\infty P_l(s)\,\mathrm{d}s \geqslant p_* - \varepsilon, \quad \text{and} \quad \frac{t^\gamma w(t)}{a(t)} \geqslant r - \varepsilon \quad \text{for } t \in [t_2, \infty).$$

Integrating (2.11) from t to ∞ and using $\lim\limits_{t\to\infty} w(t) = 0$, we have

$$w(t) \geqslant \int_t^\infty P_l(s)\,\mathrm{d}s + \gamma \int_t^\infty \frac{w^{1+1/\gamma}(s)}{a^{1/\gamma}(s)}\,\mathrm{d}s, \quad \text{for} \quad t \in [t_2, \infty). \tag{2.16}$$

By virtue of the fact that $a'(t) \geqslant 0$, it follows from (2.16) that

$$\begin{aligned}
\frac{t^\gamma}{a(t)} w(t) &\geqslant \frac{t^\gamma}{a(t)} \int_t^\infty P_l(s)\,\mathrm{d}s + \gamma \frac{t^\gamma}{a(t)} \int_t^\infty \frac{a(s)(w(s))^{1/\gamma+1} s^{\gamma+1}}{s^{\gamma+1} a^{1/\gamma+1}(s)}\,\mathrm{d}s \\
&\geqslant (p_* - \varepsilon) + \frac{t^\gamma (r-\varepsilon)^{1+1/\gamma}}{a(t)} \int_t^\infty \frac{\gamma a(s)}{s^{\gamma+1}}\,\mathrm{d}s \\
&\geqslant (p_* - \varepsilon) + (r-\varepsilon)^{1+1/\gamma}\, t^\gamma \int_t^\infty \frac{\gamma}{s^{\gamma+1}}\,\mathrm{d}s,
\end{aligned}$$

so that

$$\frac{t^\gamma}{a(t)} w(t) \geqslant (p_* - \varepsilon) + (r-\varepsilon)^{1+1/\gamma}\, t^\gamma \int_t^\infty \frac{\gamma}{s^{\gamma+1}}\,\mathrm{d}s. \tag{2.17}$$

From (2.17) we have

$$\frac{t^\gamma w(t)}{a(t)} \geqslant (p_* - \varepsilon) + (r - \varepsilon)^{1+1/\gamma}.$$

Taking the lim inf of both sides as $t \to \infty$, we get that

$$r \geqslant p_* - \varepsilon + (r - \varepsilon)^{1+1/\gamma}.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired result

$$p_* \leqslant r - (r)^{1+1/\gamma}.$$

To complete the proof of *Part* (I) it remains to prove the second inequality in (2.9). To do this we will use the inequality (2.11). Multiplying (2.11) by $t^{\gamma+1}/a(t)$ and integrating from t_2 to t $(t \geqslant t_2)$, we get

$$\int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} w'(s)\,\mathrm{d}s \leqslant -\int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s - \gamma \int_{t_2}^t \Big(\frac{s^\gamma w(s)}{a(s)}\Big)^{(\gamma+1)/\gamma} \mathrm{d}s. \tag{2.18}$$

Using integration by parts, we obtain

$$\begin{aligned}\frac{t^{\gamma+1}}{a(t)} w(t) \leqslant \frac{t_2^{\gamma+1} w(t_2)}{a(t_2)} &- \int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s - \gamma \int_{t_2}^t \Big(\frac{s^\gamma w(s)}{a(s)}\Big)^{(\gamma+1)/\gamma} \mathrm{d}s \\ &+ \int_{t_2}^t \Big(\frac{s^{\gamma+1}}{a(s)}\Big)' w(s)\,\mathrm{d}s.\end{aligned}$$

Since $a'(t) \geqslant 0$, we have

$$\Big(\frac{s^{\gamma+1}}{a(s)}\Big)' = \frac{a(s)(\gamma+1)s^\gamma - a'(s)s^{\gamma+1}}{(a(s))^2} \leqslant \frac{(\gamma+1)s^\gamma}{a(s)}.$$

Hence

$$\begin{aligned}\frac{t^{\gamma+1}}{a(t)} w(t) \leqslant \frac{t_2^{\gamma+1} w(t_2)}{a(t_2)} &- \int_{t_2}^t \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s \\ &+ \int_{t_2}^t \Big[(\gamma+1)\frac{s^\gamma w(s)}{a(s)} - \gamma\Big(\frac{s^\gamma w(s)}{a(s)}\Big)^{(\gamma+1)/\gamma}\Big] \mathrm{d}s.\end{aligned}$$

Using the inequality

$$Bu - Au^{(\gamma+1)/\gamma} \leqslant \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$$

with $u(s) := s^\gamma w(s)/a(s) > 0$, and positive constants $A = \gamma$, $B = \gamma + 1$, we get

$$\frac{t^{\gamma+1}}{a(t)} w(t) \leqslant \frac{t_2^{\gamma+1} w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s + (t - t_2).$$

It follows that

$$\frac{t^{\gamma}}{a(t)} w(t) \leqslant \frac{1}{t} \frac{t_2^{\gamma+1} w(t_2)}{a(t_2)} - \frac{1}{t} \int_{t_2}^{t} \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s + \frac{t - t_2}{t}. \tag{2.19}$$

Taking the lim sup of both sides as $t \to \infty$, we obtain

$$R \leqslant -q_* + 1.$$

Combining this with the first inequality in (2.15), we get

$$p_* \leqslant r - r^{1+1/\gamma} \leqslant r \leqslant R \leqslant -q_* + 1,$$

which gives the desired second inequality in (2.9). The proof of *Part* (I) is complete.

Part (II). Assume that $x(t)$ is a positive solution of (1.1). We shall show that $x(t)$ is not from the *class* A. Assume the contrary. First we admit that $p_* = \infty$. Then exactly as in the proof of the first part we get (2.16). Then

$$\frac{t^{\gamma}}{a(t)} w(t) \geqslant \frac{t^{\gamma}}{a(t)} \int_{t}^{\infty} P_l(s)\,\mathrm{d}s.$$

Taking the lim inf of both sides as $t \to \infty$, we get in view of (2.15) that

$$k > r \geqslant \infty,$$

a contradiction. Now we admit that $q_* = \infty$. Then taking lim inf and lim sup on the left and right hand sides of (2.19), respectively, we get

$$0 \leqslant R \leqslant -\infty.$$

This contradiction completes the proof of *Part* (II). $\square$

Now we are ready to present the following oscillation criterion for (1.1).

Theorem 2. *Let $a'(t) \geqslant 0$. Assume that (2.2) holds. Let $x(t)$ be a solution of (1.1). If*

$$p_* = \liminf_{t\to\infty} \frac{t^\gamma}{a(t)} \int_t^\infty P_l(s)\,\mathrm{d}s > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{2.20}$$

then $x(t)$ is oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a positive solution of equation (1.1). If $p_* = \infty$, then Theorem 1 ensures that $x(t)$ is from the *class* B, and from Lemma 2 we see that $\lim\limits_{t\to\infty} x(t) = 0$.

Next, we assume that $p_* < \infty$. We shall discuss two possibilities. If $x(t)$ is from the *class* B, then exactly as above we are led by Lemma 2 to $\lim\limits_{t\to\infty} x(t) = 0$.

Now we assume that $x(t)$ is from the *class* A. Let $w(t)$ and r be defined by (2.7) and (2.8), respectively. Then from Theorem 1 we see that r satisfies the inequality

$$p_* \leqslant r - r^{(\gamma+1)/\gamma}.$$

Using the inequality

$$Bu - Au^{(\gamma+1)/\gamma} \leqslant \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$$

with $A = B = 1$, we get that

$$p_* \leqslant \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

which contradicts (2.20). This completes the proof. $\square$

The proof of the next result is similar to that of Theorem 2, so it can be omitted.

Theorem 3. *Let $a'(t) \geqslant 0$. Assume that (2.2) holds. Let $x(t)$ be a solution of (1.1). If*

$$p_* + q_* > 1,$$

then $x(t)$ is oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

As a consequence of Theorem 3, we have the following result.

Corollary 1. *Let $a'(t) \geqslant 0$. Assume that* (2.2) *holds. Let $x(t)$ be a solution of* (1.1). *If*

$$q_* = \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\gamma+1}}{a(s)} P_l(s)\,\mathrm{d}s > 1,$$

then $x(t)$ is oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Theorems 2 and 3 provide new oscillation criteria also for the following partial case of (1.1):

$$(a(t)x''(t))' + q(t)x(t) = 0.$$

R e m a r k 1. In Lemma 4 we have proved that if $x(t)$ is from the *class A* and $x'''(t) \leqslant 0$, then

$$x(t) \geqslant \frac{(t-T_l)^2}{2} x''(t).$$

So for t sufficiently large, we have

$$x(\tau(t)) \geqslant \frac{(\tau(t)-T_l)^2}{2} x''(\tau(t)),$$

which together with (1.1) provides

$$\big(a(t)\,(x''(t))^\gamma\big)' + q(t)\bigg(\frac{(\tau(t)-T_l)^2}{2}\bigg)^\gamma (x''(\tau(t)))^\gamma \leqslant 0.$$

Setting $y(t) = a(t)\,(x''(t))^\gamma > 0$, we have

$$y'(t) + \frac{q(t)}{a(\tau(t))}\bigg(\frac{(\tau(t)-T_l)^2}{2}\bigg)^\gamma y(\tau(t)) \leqslant 0. \tag{2.21}$$

This means that if $x(t)$ is a positive solution of (1.1), then $y(t)$ is a positive solution of the first order delay differential inequality (2.21), which is the reduction of order. Using well-known oscillation criteria (see e.g. Theorem 2.1.1 and 2.1.3 in [14]) for (2.21), we have the following oscillation results for (1.1).

Theorem 4. *Let $a'(t) \geqslant 0$. Assume that* (2.2) *holds. Let $x(t)$ be a solution of* (1.1). *If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{q(s)}{a(\tau(s))}\bigg(\frac{(\tau(s)-T)^2}{2}\bigg)^\gamma \mathrm{d}s > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \frac{q(s)}{a(\tau(s))}\bigg(\frac{(\tau(s)-T)^2}{2}\bigg)^\gamma \mathrm{d}s > 1,$$

then $x(t)$ is either oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

E x a m p l e 1. Consider the third-order linear differential equation

$$x'''(t)+\frac{a}{t^3}x(\lambda t)=0,\quad a>0,\quad 0<\lambda\leqslant 1,\quad t\geqslant 1. \tag{2.22}$$

Since (2.2) holds, it follows from Theorem 2 that any solution of (2.22) oscillates or satisfies $\lim\limits_{t\to\infty}x(t)=0$ provided that

$$a\lambda^2>1.$$

One can easily see that the basis of the solution space of (2.22) with $a=6$, $\lambda=1$ is given by

$$\{t^{-1},t^2\cos\sqrt{2}\log t,t^2\sin\sqrt{2}\log t\}.$$

R e m a r k 2. We mention here that the results given in [4], [5], [10], [15], [16] cannot be applied to the equation (2.22).

E x a m p l e 2. Consider the nonlinear differential equation

$$((x''(t))^3)'+\frac{a}{t^7}x^3(\lambda t)=0,\quad a>0,\ 0<\lambda\leqslant 1,\ t\geqslant 1. \tag{2.23}$$

It is clear that (2.2) holds. Applying Theorem 2, we see that the solutions of (2.23) oscillate or satisfy $\lim\limits_{t\to\infty}x(t)=0$ if

$$a\lambda^6>\frac{3^4}{4^2}.$$

Note that the results by Grace [7] cannot be applied to the equations (2.22) and (2.23).

R e m a r k 3. It remains an open problem how to extend the above results to cover the case $\int_{t_0}^{\infty}1/a^{1/\gamma}(s)\,\mathrm{d}s<\infty$.

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