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LOCAL STABILITY AND DIFFERENTIABILITY OF THE MEAN–CONDITIONAL VALUE AT RISK MODEL DEFINED ON THE MIXED–INTEGER LOSS FUNCTIONS

Martin Branda

In this paper, we study local stability of the mean-risk model with Conditional Value at Risk measure where the mixed-integer value function appears as a loss variable. This model has been recently introduced and studied in Schulz and Tiedemann [16]. First, we generalize the qualitative results for the case with random technology matrix. We employ the contamination techniques to quantify a possible effect of changes in the underlying probability distribution on the optimal value. We use the generalized qualitative results to express the explicit formula for the directional derivative of the local optimal value function with respect to the underlying probability measure. The derivative is used to construct the bounds. Similarly, we can approximate the behavior of the local optimal value function with respect to the changes of the risk-aversion parameter which determines our aversion to risk.

Keywords: mean-CVaR model, mixed-integer value function, stability analysis, contamination techniques, derivatives of optimal value function

Classification: 90C15, 91B28, 90C11

1. INTRODUCTION

Mean-risk investment models can be perceived as a special class of stochastic programming problems. Stochastic programming solves many real-life problems where optimization and randomness appear together. Such problems arise in economy, finance, industry, agriculture and logistics, cf. [18]. Any successful application of stochastic programming problems requires full knowledge of the underlying probability distribution of the random parts. However, the distribution is usually estimated or approximated. Hence, the stability analysis with respect to some changes of the distribution is necessary, cf. [7]. A successful approach is based on the probability metrics [12] which enable us to bound the distance between the optimal values of our optimization problem with different underlying measures using an appropriate probability metric. However, it can be difficult to compute the metric, especially in integer stochastic programming where very complicated metrics usually appear. On the other hand, the contamination techniques are more computationally tractable

than the approach based on the probability metrics. They provide a way how to construct the contamination bounds for the optimal value, which quantify the effect of considered change in the probability distribution, mostly the behavior under some extreme events. We may refer to [5, 6] for the introduction and the main theoretical results, to [4] for the applications in stochastic integer programming and to [8, 9] for the risk modeling with Value at Risk, Conditional Vale at Risk and for the general class of polyhedral risk measures.

Incorporating integer variables into optimization problems leads in many cases to more realistic models, however, the resulting problems are much more theoretically and computationally demanding. Integer variables help us to model indivisible assets (we can buy only integer number of assets), transaction costs, cardinality constraints (restrictions on maximal number of kinds of assets), logical relations (if you buy certain asset, you must not buy other) etc. There was a large development in stochastic integer programming in the fields of theory and algorithms during the last decade, see [13, 15].

We will concentrate on the application of the contamination techniques on the mean-risk model with Conditional Value at Risk (CVaR) risk measure applied to random variables occurring in recourse stochastic integer programming. This problem was introduced in [16]. In particular, the authors studied the continuity properties of the objective function, both with respect to the first-stage decisions and the integrating probability measure. They also introduced a decomposition algorithm for solving such problems. The mixed-integer value function models the final outcome (loss) connected with our first stage decision, which does not depend on the future realization of the random parts and can represent our initial decision on the portfolio composition. On the other hand, the second stage decisions are made after the realizations of the random parts are observed. For example, they can represent sales. We partly generalize the qualitative results for the case with random technology matrix, which can contain random returns in our settings. Then we apply the results to express the explicit formula for the derivative of the optimal value function which is employed to construct the contamination bounds.

The paper is organized as follows. In Section 2, the mean-risk model with Conditional Value at Risk on general loss random variables is introduced. The aggregate function approach for solving multi-objective programming problems is used. Then the basic properties of the mixed-integer value function are summarized. We investigate the continuity properties of the objective function jointly in the decision vector, the risk-aversion parameter and the underlying probability distribution. As a consequence, the qualitative stability of the local optimal value function with respect to the underlying distribution and the risk-aversion parameter is derived. In Section 3, it is shown how the contamination techniques can be applied to our problem. Then the explicit formula for the directional derivative of the optimal value function is expressed. Similar idea is used to approximate the behaviour of the local optimal value function with respect to the changes of the risk-aversion parameter.

2. MEAN–CVAR MODEL

2.1. Mean–CVaR model

We denote $Z(x,\omega)$ a *loss random variable* dependent on the decision vector $x \in X$, where $X \subseteq \mathbb{R}^n$ is a nonempty closed set, and on the random vector ω defined on the probability space $(\Omega, \mathcal{F}, P)$ with a support in $\mathbb{R}^k$. We assume that the expected loss is finite, i.e.

$$\mathcal{E}(x) := \mathrm{E}_P|Z(x,\omega)| = \int_\Omega |Z(x,\omega)|\mathrm{d}P(\omega) < \infty, \ \forall x \in X.$$

Conditional Value at Risk (CVaR) is often proposed as an alternative measure for Value at Risk (VaR), which is widely used in practice, even though it is not an adequate risk measure, cf. [17]. CVaR is roughly defined as the conditional mean of losses on the condition that we are beyond VaR. Below we will provide the formal definition and summarize the basic properties under general loss distributions, cf. [11].

Let F denote the distribution function of the loss random variable, i.e.

$$F(x,\eta) = P(\{\omega : \ Z(x,\omega) \leq \eta\}), \ \eta \in \mathbb{R}.$$

Then Value at Risk (VaR) and upper Value at Risk are defined as

$$\begin{aligned} \mathrm{VaR}_\alpha(x) &= \min\{\eta : F(x,\eta) \geq \alpha\}, \\ \mathrm{VaR}^+_\alpha(x) &= \min\{\eta : F(x,\eta) > \alpha\} \end{aligned}$$

for some probability level $\alpha \in (0,1)$, usually 0.95 or 0.99. CVaR is defined as the mean of losses in the α-tail distribution

$$\begin{aligned} F_\alpha(x,\eta) &= \frac{F(x,\eta)-\alpha}{1-\alpha}, \text{ if } \eta \geq \mathrm{VaR}_\alpha(x), \\ &= 0, \text{ otherwise.} \end{aligned}$$

For application of CVaR in optimization problems, the following *minimization formula* is of crucial importance [11, Theorem 10]:

$$\mathrm{CVaR}_\alpha(x) = \min_{\eta\in\mathbb{R}} \eta + \frac{1}{1-\alpha}\mathrm{E}_P[Z(x,\omega)-\eta]^+ \tag{1}$$

where $[\cdot]^+ = \max\{\cdot, 0\}$ denotes the positive part and η is a real auxiliary variable. The optimal solution belongs to the closed interval $[\mathrm{VaR}_\alpha(x), \mathrm{VaR}^+_\alpha(x)]$. We can use the *optimization shortcut* [11, Theorem 14] to minimize CVaR, i.e.

$$\min_{x\in X} \mathrm{CVaR}_\alpha(x) = \min_{(\eta,x)\in\mathbb{R}\times X} \eta + \frac{1}{1-\alpha}\mathrm{E}_P[Z(x,\omega)-\eta]^+.$$

Minimizing the risk $\mathrm{CVaR}_\alpha(x)$ and the expected loss $\mathcal{E}(x)$ at the same time under some common constraints X on the portfolio composition leads to multi-objective optimization problem. We are looking for the efficient solutions, i.e. the solutions

$\hat{x} \in X$ such that there is no element $x \in X$ with $\mathrm{CVaR}_\alpha(x) \leq \mathrm{CVaR}_\alpha(\hat{x})$ and $\mathcal{E}(x) \leq \mathcal{E}(\hat{x})$ with at least one strict inequality. There are two main approaches for solving the multi-objective problems, both leading to the single objective problem and under mild conditions to the efficient solutions, see [10]: aggregate function (weighted sum) and ε-constraint approach. Using the aggregate function approach and the optimization shortcut, the objective function of our problem can be written as

$$f(\eta, x; \rho, P) = (1-\rho)\mathrm{E}_P Z(x,\omega) + \rho\left(\eta + \frac{1}{1-\alpha}\mathrm{E}_P[Z(x,\omega)-\eta]^+\right), \tag{2}$$

where $\rho \in (0,1)$ can be seen as the risk-aversion parameter. If we set $\rho = 0$, we minimize the expected loss without involving the risk minimization. On the other hand, if we set $\rho = 1$, we are absolutely risk averse, i.e. we minimize the risk only without considering the expected loss. Hence, throughout this paper $\rho \in [0,1]$. We solve the following problem

$$\min_{(\eta,x)\in\mathbb{R}\times X} f(\eta, x; \rho, P), \tag{3}$$

where ρ and P are the above mentioned parameters.

2.2. Mixed–integer value function

We consider the following *mixed-integer value function*

$$Z(x,\omega) = c^T x + \Phi(h(\omega) - T(\omega)x),\ x \in X, \omega \in \Omega \tag{4}$$

with random right-hand side vector $h[s \times 1]$ and random technology matrix $T[s \times n]$ with dimensions stated in the brackets and with the second stage problem which is a parametric mixed-integer linear problem which can be written as follows

$$\Phi(z) = \min\left\{q^T y + q'^T y' : Wy + W'y' = z,\ y \in \mathbb{Z}_+^m,\ y' \in \mathbb{R}_+^{m'}\right\}, \forall z \in \mathbb{R}^s, \tag{5}$$

where $c \in \mathbb{R}^n$, $q \in \mathbb{R}_+^m$, $q' \in \mathbb{R}_+^{m'}$ are vectors, $W[s \times m]$, $W'[s \times m']$ are matrices, and $\mathbb{Z}_+, \mathbb{R}_+$ denote the nonnegative integers and the nonnegative real numbers. We assume that the matrices W, W' have rational entries only and the random parts depend affinely linearly on the random vector ω.

We denote $\mu = P \circ (h,T)^{-1}$ the image measure on $\mathbb{R}^{s(n+1)}$ which belongs to the general class of Borel measures $\mathcal{P}(\mathbb{R}^{s(n+1)})$. The function $\Phi(z)$ is real-valued on $\mathbb{R}^s$ if we further assume, see [14]:
(A1) complete recourse: $W(\mathbb{Z}_+^m) + W'(\mathbb{R}_+^{m'}) = \mathbb{R}^s$, i.e. for any $z \in \mathbb{R}^s$ there exists $y \in \mathbb{Z}_+^m$ and $y' \in \mathbb{R}_+^{m'}$ such that $Wy + W'y' = z$.
(A2) dual feasibility: $\{u \in \mathbb{R}^s : W^T u \leq q,\ W'^T u \leq q'\} \neq \emptyset$.
We also denote
(A3) finite first moment: $\int_{\mathbb{R}^{s(n+1)}} \|h\| + \|T\|\,\mu(\mathrm{d}(h,T)) < \infty$, where $\|h\|$ denotes the Euclidean norm and $\|T\| = \max\{\|Tx\| :\ x \in \mathbb{R}^n, \|x\| \leq 1\}$ is the induced matrix norm.

Previous assumptions ensure that $\mathrm{E}_P\left[|c^T x+\Phi(h(\omega)-T(\omega)x)|\right]<\infty$, $\forall_{x\in X}$, cf. [14]. Hence, the mixed-integer value function can be considered as the loss random variable depending on x and ω. To proceed further analysis we denote

$$M_d(x)=\{(h,T)\in\mathbb{R}^{s(n+1)}:\Phi \text{ is discontinuous at } h-Tx\}$$

the set of the discontinuity points of Φ for a given $x\in X$. To obtain continuity of the objective function jointly in the decision vector and in the underlying measure, the following subclass of Lebesgue measures must be considered. For arbitrary fixed $p>1$ and $C>0$, we denote

$$\Delta_{p,C}(\mathbb{R}^{s(n+1)})=\{\mu\in\mathcal{P}(\mathbb{R}^{s(n+1)}):\int_{\mathbb{R}^{s(n+1)}}\|(h,T)\|^p\,\mu(\mathrm{d}(h,T))\le C\}.$$

Under the assumptions (A1), (A2) the second stage function (5) has the following property, see [3]: there exists $\beta,\gamma>0$ such that for all $z_1,z_2\in\mathbb{R}^s$ it holds

$$|\Phi(z_1)-\Phi(z_2)|\le\beta\,\|z_1-z_2\|+\gamma. \tag{6}$$

The function Z appears in the expectation as the objective function in two-stage stochastic models with mixed-integer linear recourse, cf. [14, 15]:

$$\min_{x\in X}Q_{\mathrm{E}}(x;\mu):=\int_{\mathbb{R}^{s(n+1)}}c^Tx+\Phi(h-Tx)\;\mu(\mathrm{d}(h,T)).$$

2.3. Qualitative stability

We propose the properties of our mean-CVaR problem that we will use to investigate stability of the optimal value function, especially to express the explicit formulas for the derivatives of the optimal value function.

We start the section with the definition of the weak convergence. We say that a sequence of measures $\{\mu_{n'}\}$ from $\mathcal{P}(\mathbb{R}^{s(n+1)})$ converges weakly to a measure $\mu\in\mathcal{P}(\mathbb{R}^{s(n+1)})$ if for any bounded continuous function $g:\mathbb{R}^{s(n+1)}\to\mathbb{R}$ it holds

$$\lim_{n'\to\infty}\int_{\mathbb{R}^{s(n+1)}}g(\xi)\mu_{n'}(\xi)=\int_{\mathbb{R}^{s(n+1)}}g(\xi)\mu(\mathrm{d}\xi).$$

We use the notation $\mu_{n'}\to\mu$ as $n'\to\infty$.

The objective function (2) can be decomposed into two main parts

$$f(\eta,x;\rho,\mu)=(1-\rho)Q_{\mathrm{E}}(x;\mu)+\rho\left(\eta+\frac{1}{1-\alpha}Q_{\mathrm{E}_\eta}(\eta,x;\mu)\right), \tag{7}$$

where $Q_{\mathrm{E}}(x;\mu)$ is well studied in [14] and the η-Expected Excess is defined as

$$Q_{\mathrm{E}_\eta}(\eta,x)=\int_{\mathbb{R}^{s(n+1)}}[c^Tx+\Phi(h-Tx)-\eta]^+\;\mu(\mathrm{d}(h,T)).$$

The following propositions generalize slightly the qualitative results of [16] for the case with random technology matrix.

Proposition 2.1. Let assumptions (A1), (A2), (A3) be fulfilled and $\mu(M_d(x)) = 0$. Then the η-Expected Excess $Q_{\mathrm{E}_\eta} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is continuous at (η, x) for any $\eta \in \mathbb{R}$. Furthermore, if $\mu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$, then $Q_{\mathrm{E}_\eta} : \mathbb{R} \times \mathbb{R}^m \times \Delta_{p,C}(\mathbb{R}^{s(n+1)}) \to \mathbb{R}$ is continuous at (η, x, μ) for any $\eta \in \mathbb{R}$.

Proof. The η-Expected Excess can be written in the following form, see [16, Lemma 4.4]:

$$\tilde{\Phi}(\tilde{z}) = \min\{v : Wy + W'y' = \tilde{z}_1, v \geq q^T y + q'^T y' - \tilde{z}_2, \\ y \in \mathbb{Z}_+^m,\ y' \in \mathbb{R}^{m'},\ v \in \mathbb{R}\},\ \tilde{z} \in \mathbb{R}^{s+1}$$

and

$$\begin{aligned} Q_{\mathrm{E}_\eta}(\eta, x; \mu) &= \int_{\mathbb{R}^{s(n+1)}} \tilde{\Phi}\left(\begin{pmatrix} h \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & T \\ -1 & c^T \end{pmatrix}\begin{pmatrix} \eta \\ x \end{pmatrix}\right)\mu(\mathrm{d}(h,T)) \\ &= \int_{\mathbb{R}^{(s+1)(n+2)}} \tilde{\Phi}\left(\tilde{h} - \tilde{T}\tilde{x}\right)\tilde{\mu}(\mathrm{d}(\tilde{h},\tilde{T})) = \tilde{Q}_{\mathrm{E}}(\tilde{x};\tilde{\mu}), \end{aligned}$$

where we denote

$$\tilde{h} = \begin{pmatrix} \tilde{h}_1[s\times 1] \\ \tilde{h}_2[1\times 1] \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix},\ \tilde{T} = \begin{pmatrix} \tilde{t}_{11}[s\times 1] & \tilde{T}_{12}[s\times n] \\ \tilde{t}_{21}[1\times 1] & \tilde{t}_{22}[1\times n] \end{pmatrix} = \begin{pmatrix} 0 & T \\ -1 & c^T \end{pmatrix},$$

and the measure $\tilde{\mu}(\tilde{h}, \tilde{T}) = \mu(\tilde{h}_1, \tilde{T}_{12}) \times \delta(\tilde{t}_{11}, \tilde{t}_{21}, \tilde{t}_{22}, \tilde{h}_2)$, where δ is the Dirac measure, which is equal to 1 for $(\tilde{h}_2, \tilde{t}_{11}^T, \tilde{t}_{21}, \tilde{t}_{22}) = (0, 0, -1, c^T)$, and 0 otherwise.

The assumptions ensure the complete recourse and the dual feasibility for Q_{E_η}, cf. [16, Lemma 4.4]. We denote $\tilde{M}_d(\tilde{x}) = \{(\tilde{h}, \tilde{T}) \in \mathbb{R}^{(m+2)(s+1)} : \tilde{\Phi}$ is discontinuous at $\tilde{h} - \tilde{T}\tilde{x}\}$. From the definition $\tilde{\Phi}$ is discontinuous at $\tilde{h} - \tilde{T}\tilde{x}$ if and only if the Φ is discontinuous at $h - Tx$. Then, from the definition of the measure $\tilde{\mu}$, we obtain $\tilde{\mu}(\tilde{M}_d(\tilde{x})) \leq \mu(M_d(x)) = 0$.
We have verified all the assumptions of [14, Proposition 3.8], which shows joint continuity of the recourse function in the decision vector and the underlying measure. Hence, the same is valid for the η-Expected Excess. □

As an immediate consequence, joint continuity of the objective function can be obtained.

Corollary 2.2. Assume that (A1), (A2) are fulfilled, and let $\mu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ and $x \in X$ be such that $\mu(M_d(x)) = 0$. Then the objective function $f : \mathbb{R} \times \mathbb{R}^m \times [0,1] \times \Delta_{p,C}(\mathbb{R}^{s(n+1)}) \to \mathbb{R}$ is continuous jointly at (η, x, ρ, μ) for all $(\eta, \rho) \in \mathbb{R} \times [0,1]$.

Proof. Corollary follows from (7) using previous proposition and the qualitative results valid for the objective function of two stage mixed-integer programming problems, cf. [14, Proposition 3.8]. □

We must pay a special attention to the optimal solutions of the minimization formula, i.e. to the closed interval which is bounded by VaR and upper VaR. We show that the interval is uniformly bounded with respect to the image measures which enables us to study continuity of local minimizers using Berge theory.

Proposition 2.3. Assume that (A1), (A2) are fulfilled. Let $(x_{n'}, \mu_{n'}) \to (x_0, \mu_0)$ as $n' \to \infty$, where $(x_{n'}, \mu_{n'}) \in \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^{s(n+1)})$ for all $n' \in \mathbb{N}$. Then there exist compact subsets K and K^+ such that $\mathrm{VaR}_\alpha(x_{n'}, \mu_{n'}) \in K$ and $\mathrm{VaR}^+_\alpha(x_{n'}, \mu_{n'}) \in K^+$ for all $n' \in \mathbb{N}$.

P r o o f . We modify the proof of [16, Lemma 4.9] for our case with random technology matrix. Let $r = \max_{n'} \|x_{n'}\|$. Then, using the property (6) of the second stage function (5), we obtain

$$\begin{aligned} |c^T x + \Phi(h - Tx)| &\le |c^T x| + |\Phi(h - Tx)| \le |c^T x| + \beta \|h - Tx\| + \gamma \\ &\le r\|c\| + \beta\|h\| + \beta r \|T\| + \gamma \le \overline{\beta}(\|h\| + \|T\|) + \overline{\gamma} \end{aligned}$$

$\forall x \in \{x_{n'}\}$ and for $\beta, \gamma > 0$ taken from (6), $\overline{\beta} := \max\{1, \beta\}$ and $\overline{\gamma} := r\|c\| + \gamma$. Using previous estimate, we can find an upper and a lower bound for $\mathrm{VaR}_\alpha(x, \mu)$ for all $\mu \in \{\mu_{n'}\}$ as follows.

$$\begin{aligned} \mathrm{VaR}_\alpha(x,\mu) &= \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : c^T x + \Phi(h - Tx) \le \eta\}) \ge \alpha\}, \\ &\le \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : \overline{\beta}(\|h\| + \|T\|) + \overline{\gamma} \le \eta\}) \ge \alpha\} \\ &= \overline{\beta} \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : (\|h\| + \|T\|) \le \eta\}) \ge \alpha\} + \overline{\gamma}, \end{aligned}$$

and similarly

$$\begin{aligned} \mathrm{VaR}_\alpha(x,\mu) &= \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : c^T x + \Phi(h - Tx) \le \eta\}) \ge \alpha\}, \\ &\ge \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : -\overline{\beta}(\|h\| + \|T\|) - \overline{\gamma} \le \eta\}) \ge \alpha\} \\ &= \overline{\beta} \min\{\eta :\ \mu(\{(h,T) \in \mathbb{R}^{s(n+1)} : (\|h\| + \|T\|) \ge -\eta\}) \ge \alpha\} + \overline{\gamma} \\ &= -\overline{\beta} \max\{\eta :\ \mu(\{(h,T) : (\|h\| + \|T\|) < \eta\}) \le 1 - \alpha\} + \overline{\gamma}. \end{aligned}$$

Since the elements of $\{\mu_{n'}\}$ are weakly convergent, Prohorov's theorem [2, Theorem 6.2] ensures that there exists, for each $\varepsilon > 0$, a compact set $\mathcal{C} \subset \mathbb{R}^{s(n+1)}$ such that $\mu(\mathcal{C}) > 1 - \varepsilon$ for all $\mu \in \{\mu_{n'}\}$. Let $\varepsilon = 1 - \alpha$, then there exist a compact set $\mathcal{C} \subset \mathbb{R}^{s(n+1)}$ such that $\mu(\mathcal{C}) > \alpha$ for all $\mu \in \{\mu_{n'}\}$, i.e. $\{(h,T) \in \mathbb{R}^{s(n+1)} : (\|h\| + \|T\|) \le \eta\}$ has to be compact. Similarly, setting $\varepsilon = \alpha$ ensures existence of a compact set $\mathcal{C}' \subset \mathbb{R}^{s(n+1)}$ such that $\mu(\mathcal{C}') > 1 - \alpha$ for all $\mu \in \{\mu_{n'}\}$, i.e. $\{(h,T) \in \mathbb{R}^{s(n+1)} : (\|h\| + \|T\|) < \eta\}$ has to be compact too. Using both arguments, it yields the finite lower and upper bound for $\mathrm{VaR}_\alpha(x, \mu)$ for all $\mu \in \{\mu_{n'}\}$.

Using similar argument, we can obtain a compact set $K^+ \subset \mathbb{R}$ which bounds upper VaR for all $\mu \in \{\mu_{n'}\}$. □

Since the mixed-integer recourse function is nonconvex in general, our model lacks convexity, i.e. the objective function is nonconvex. Hence, we study local stability with respect to the risk-aversion parameter ρ and the image measure μ. We use the concept of local minimizers, i.e. the extreme value function and the optimal value function with respect to some bounded open set $V \subset \mathbb{R}^n$ which are defined by

$$\begin{aligned} \varphi_V(\rho,\mu) &= \inf\{f(\eta, x; \rho, \mu) :\ (\eta, x) \in \mathbb{R} \times (X \cap \mathrm{cl}\ V)\}, \\ \Psi_V(\rho,\mu) &= \{(\eta, x) \in \mathbb{R} \times (X \cap \mathrm{cl}\ V) :\ f(\eta, x; \rho, \mu) = \varphi_V(\rho, \mu)\} \end{aligned}$$

and depend on the risk-aversion parameter $\rho \in [0,1]$ and the image measure $\mu \in \mathcal{P}(\mathbb{R}^{s(n+1)})$. We say that $S \subset \mathbb{R}^n$ is a *complete local minimizer* set with respect to the set V of the mean-CVaR problem if

$$\emptyset \neq S = \Psi_V(\rho, \mu) \subset V.$$

The set of optimal solutions can be decomposed into two parts

$$\Psi_V(\rho,\mu) = \mathrm{Proj}_\eta(\Psi_V(\rho,\mu)) \times \mathrm{Proj}_x(\Psi_V(\rho,\mu))$$

where

$$\begin{aligned} \mathrm{Proj}_\eta(\Psi_V(\rho,\mu)) &= \{\eta \in \mathbb{R} :\ \exists x \in X \cap \mathrm{cl}\ V : (\eta, x) \in \Psi_V(\rho,\mu)\} \\ &= \{\eta \in [\mathrm{VaR}_\alpha(x,\mu), \mathrm{VaR}^+_\alpha(x,\mu)] : \\ &\quad \exists x \in X \cap \mathrm{cl}\ V : (\eta, x) \in \Psi_V(\rho,\mu)\} \end{aligned}$$

and

$$\mathrm{Proj}_x(\Psi_V(\rho,\mu)) = \{x \in X \cap \mathrm{cl}\ V :\ \exists \eta \in \mathbb{R} : (\eta, x) \in \Psi_V(\rho,\mu)\} =: \psi_V(\rho,\mu).$$

The following stability results hold due to the joint continuity of the function f in (x, η, ρ, μ) and follows from Berge's theory.

Proposition 2.4. Assume that (A1), (A2) are fulfilled, and let $\mu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ with $\mu(M_d(x)) = 0$ for all $x \in X$. Then

(i) the function $\varphi_V : [0,1] \times \Delta_{p,C}(\mathbb{R}^{s(n+1)}) \to \mathbb{R}$ is continuous at (ρ, μ);

(ii) the set-valued mapping $\Psi_V : [0,1] \times \Delta_{p,C}(\mathbb{R}^{s(n+1)}) \to 2^{\mathbb{R}^{n+1}}$ is Berge upper semicontinuous at (ρ, μ), i.e. for each open set $O \subset \mathbb{R}^{n+1}$ containing $\Psi_V(\rho,\mu)$ there exists a neighbourhood $N(\rho,\mu)$ of (ρ,μ) in $[0,1] \times \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ such that

$$\Psi_V(\rho',\mu') \subset O,\ \forall (\rho',\mu') \in N(\rho,\mu);$$

(iii) there exists a neighbourhood $N(\rho,\mu)$ of (ρ,μ) in $[0,1] \times \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ such that for all pairs $(\rho'\mu') \in N(\rho,\mu)$ the set $\Psi_V(\rho'\mu')$ is a complete local minimizer set with respect to V.

P r o o f. We can restrict considered η to the compact interval $\hat{K} = [\min\{K \cup K'\}, \max\{K \cup K'\}]$ where K, K' were obtained in Proposition 2.3. Hence, we know that the set of optimal solutions belongs to the compact set $\hat{K} \times \mathrm{cl}\, V$. The proposition follows then from Berge's theory, see [1, Theorem 4.2.2], using the joint continuity of the objective function f in (x, η, ρ, μ), cf. Corollary 2.2. We can also employ [16, Lemma 4.1]. □

Remark 2.5. A natural question arises: When is the assumption $\mu(M_d(x)) = 0, \forall x \in X$ fulfilled for a measure $\mu \in \mathcal{P}(\mathbb{R}^{s(n+1)})$? In [14], it is shown that the sufficient condition is that the conditional distribution of h given T is absolutely continuous.

3. CONTAMINATION TECHNIQUES AND DIFFERENTIABILITY

3.1. Contamination techniques

In this section, we show how the contamination techniques, cf. [5, 6], can be applied to investigate stability of our mean-CVaR problem.

Let $\mu \in \mathcal{P}(\mathbb{R}^{s(n+1)})$ be our best fixed estimate of the underlying probability distribution and $\nu \in \mathcal{P}(\mathbb{R}^{s(n+1)})$ represent the distribution under some extreme events which is used to stress the best estimate. Then the contaminated distribution μ^t is defined for all $t \in [0,1]$ by

$$\mu^t = (1-t)\mu + t\nu.$$

In our case, the objective function is linear in the underlying probability measure, i.e. it holds

$$f(\eta, x; \rho, \mu^t) = (1-t)f(\eta, x; \rho, \mu) + tf(\eta, x; \rho, \nu).$$

The directional derivative of the extreme value function at μ in the direction $\nu - \mu$ is defined as

$$\varphi_V^{'}(\rho, \mu; \nu - \mu) = \lim_{t \to 0_+} \frac{\varphi_V(\rho, \mu^t) - \varphi_V(\rho, \mu)}{t}.$$

If the optimal value $\varphi_V(\rho, \mu^t)$ is finite for all $t \in [0,1]$, the linearity of the objective function in the underlying distribution ensures concavity of the extreme value function. Hence, we can construct the contamination bounds for the extreme value function of the contaminated problem as follows

$$(1-t)\varphi_V(\rho, \mu) + t\varphi_V(\rho, \nu) \leq \varphi_V(\rho, \mu^t) \leq \varphi_V(\rho, \mu) + t\varphi_V^{'}(\rho, \mu; \nu - \mu), \ t \in [0,1].$$

In order to evaluate these bounds, we need to solve the original problem with the measure μ and the fully contaminated problem with the distribution ν, and to compute the directional derivative of the optimal value function or at least an upper bound for the derivative. We do not need to solve any contaminated problem which is always larger than the above mentioned problems. We propose an explicit formula for the directional derivative of the local optimal value function for our mean-CVaR problem below.

Similar idea can be used to construct the bounds for the optimal value function with respect to the changes of the risk-aversion parameter ρ. If we assume that the optimal value $\varphi_V(\rho, \mu)$ is finite for all $\rho \in [0,1]$, we can get the bounds:

$$(1-\rho)\varphi_V(0, \mu) + \rho\varphi_V(1, \mu) \leq \varphi_V(\rho, \mu) \leq \varphi_V(0, \mu) + \rho\varphi_V^{'}(\rho, \mu; \rho = 0_+), \ \rho \in [0,1],$$

where we used the derivative defined by

$$\varphi_V^{'}(\rho, \mu; \rho = 0_+) = \lim_{\rho \to 0_+} \frac{\varphi_V(\rho, \mu) - \varphi_V(0, \mu)}{\rho}.$$

3.2. Derivatives of optimal value function

Due to the properties of the objective function f, the explicit formula for the directional derivative of the optimal value function of the mixed-integer CVaR optimization problem can be found. The derivative describes the local behaviour of the optimal value function and enables us to construct the contamination bounds. Similar result was obtained by [4] for the case $\rho = 0$, i.e. for the two stage mixed-integer programming problem.

Theorem 3.1. Assume that (A1), (A2) are fulfilled, and let $\mu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ with $\mu(M_d(x)) = 0$ for all $x \in X$ and $\rho \in [0,1]$ be fixed. Suppose further that $\Psi_V(\rho,\mu)$ is a complete local minimizer set with respect to some bounded open set $V \subset \mathbb{R}^n$. Let $\nu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ be a fixed contamination distribution.
Then the directional derivative of the optimal value function is equal to

$$\varphi_V^{'}(\rho,\mu;\nu-\mu) = \min_{(x,\eta)\in\Psi_V(\rho,\mu)} f(\eta,x;\rho,\nu) - \varphi_V(\mu,\rho).$$

P r o o f. The proof is similar to [6, Theorem 8]. However, special attention must be paid to the specific structure of our problem. Define

$$\mu^t = (1-t)\mu + t\nu,\ t \in [0,1].$$

Both measures μ and ν belong to the class $\Delta_{p,C}(\mathbb{R}^{s(n+1)})$, hence $\mu^t \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ for every $t \in (0,1)$.
According to the Proposition 2.4, Ψ_V is Berge upper semicontinuous at μ, which yields the existence of t_0 such that $\Psi_V(\mu^t) \subset V$ for every $t \in (0,t_0)$.
Due to the linearity of $f(\eta,x;\rho,\cdot)$, we have

$$\frac{f(\eta,x;\rho,\mu^t) - f(\eta,x;\rho,\mu)}{t} = f(\eta,x;\rho,\nu) - f(\eta,x;\rho,\mu).$$

For arbitrary $\overline{x} \in \psi_V(\rho,\mu)$, $\overline{\eta}(\overline{x}) \in [\mathrm{VaR}_\alpha(\overline{x},\mu), \mathrm{VaR}^+_\alpha(\overline{x},\mu)]$ and $t \in (0,t_0)$, it holds that

$$\begin{aligned}\varphi_V(\mu^t) \le f(\overline{\eta}(\overline{x}),\overline{x};\rho,\mu^t) &= f(\overline{\eta}(\overline{x}),\overline{x};\rho,\mu) + t\big(f(\overline{\eta}(\overline{x}),\overline{x};\rho,\nu) - f(\overline{\eta}(\overline{x}),\overline{x};\rho,\mu)\big)\\ &= \varphi_U(\mu,\lambda) + t\big(f(\overline{\eta}(\overline{x}),\overline{x};\rho,\nu) - f(\overline{\eta}(\overline{x}),\overline{x};\rho,\mu)\big),\end{aligned}$$

and so

$$\frac{\varphi_V(\mu^t) - \varphi_V(\mu)}{t} \le f(\eta,x;\rho,\nu) - f(\eta,x;\rho,\mu) \tag{8}$$

for all $x \in \psi_V(\mu)$, $\eta \in [\mathrm{VaR}_\alpha(x,\mu), \mathrm{VaR}^+_\alpha(x,\mu)]$ and all $t \in (0,t_0)$.
Analogously, for arbitrary $\hat{x} \in \psi_V(\mu^t)$, $\hat{\eta}(\hat{x}) \in [\mathrm{VaR}_\alpha(\hat{x},\mu^t), \mathrm{VaR}^+_\alpha(\hat{x},\mu^t)]$ and $t \in (0,t_0)$, we obtain

$$\begin{aligned}\varphi_V(\mu^t) = f(\hat{\eta}(\hat{x}),\hat{x};\rho,\mu^t) &= f(\hat{\eta}(\hat{x}),\hat{x};\rho,\mu) + t\big(f(\hat{\eta}(\hat{x}),\hat{x};\rho,\nu) - f(\hat{\eta}(\hat{x}),\hat{x};\rho,\mu)\big)\\ &\ge \varphi_V(\mu) + t\big(f(\hat{\eta}(\hat{x}),\hat{x};\rho,\nu) - f(\hat{\eta}(\hat{x}),\hat{x};\rho,\mu)\big),\end{aligned}$$

thus

$$\frac{\varphi_V(\mu^t) - \varphi_V(\mu)}{t} \geq f(\eta, x; \rho, \nu) - f(\eta, x; \rho, \mu) \tag{9}$$

for all $x \in \psi_V(\mu^t)$, $\eta \in [\mathrm{VaR}_\alpha(x, \mu^t), \mathrm{VaR}^+_\alpha(x, \mu^t)]$ and all $t \in (0, t_0)$.
From (8) and (9) we obtain

$$\begin{aligned} & \min_{(x,\eta)\in\Psi_V(\mu^t)} \big(f(\eta, x; \rho, \nu) - f(\eta, x; \rho, \mu)\big) \leq \\ \leq \quad & \frac{\varphi_V(\mu^t, \lambda) - \varphi_V(\mu, \lambda)}{t} \leq \\ \leq \quad & \min_{(x,\eta)\in\Psi_V(\mu)} \big(f(\eta, x; \rho, \nu) - f(\eta, x; \rho, \mu)\big) \end{aligned}$$

for $t \in (0, t_0)$.
From our assumptions and from Corollary 2.2 and Proposition 2.4, it follows that the function defined as

$$\phi(t) = \min_{(x,\eta)\in\Psi_V(\mu^t)} \big(f(\eta, x; \rho, \nu) - f(\eta, x; \rho, \mu)\big)$$

is lower semicontinuous at $t = 0$, see [1, Theorem 4.2.3]. This implies that

$$\phi(0) = \min_{(x,\eta)\in\Psi_V(\mu)} \big(f(\eta, x; \rho, \nu) - f(\eta, x; \rho, \mu)\big) \leq \liminf_{t\to 0+} \phi(t),$$

thus

$$\lim_{t\to 0+} \frac{\varphi_V(\mu^t) - \varphi_V(\mu)}{t} = \min_{x\in\psi_V(\mu)} \min_{\eta\in[\mathrm{VaR}_\alpha(x,\mu), \mathrm{VaR}^+_\alpha(x,\mu)]} f(\eta, x; \rho, \nu) - \varphi_V(\mu).$$

This finishes the proof. □

Theorem 3.2. Assume that (A1), (A2) are fulfilled, and let $\mu \in \Delta_{p,C}(\mathbb{R}^{s(n+1)})$ be fixed with $\mu(M_d(x)) = 0$ for all $x \in X$. Suppose further that $\Psi_V(0, \mu)$ is a complete local minimizer set with respect to some bounded open set $V \subset \mathbb{R}^n$.
Then the derivative of the optimal value function is equal to

$$\varphi'_V(\rho, \mu; \rho = 0_+) = \min_{(x,\eta)\in\Psi_V(0,\mu)} f(\eta, x; 1, \mu) - \varphi_V(\mu, 0).$$

Proof. The proof is similar to the proof of previous theorem. □

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