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Fuhai Zhu; Zhiqi Chen

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NOVIKOV SUPERALGEBRAS WITH $A_0 = A_1A_1$

Fuhai Zhu, Zhiqi Chen, Tianjin

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Abstract. Novikov superalgebras are related to quadratic conformal superalgebras which correspond to the Hamiltonian pairs and play a fundamental role in completely integrable systems. In this note we show that the Novikov superalgebras with $A_0 = A_1A_1$ and $\dim A_1 = 2$ are of type N and give a class of Novikov superalgebras of type S with $A_0 = A_1A_1$.

Keywords: Novikov algebra, Novikov superalgebra, type N, type S

MSC 2010: 17A70, 17A30

1. Introduction

Novikov superalgebras are a super variant of Novikov algebras. They are closely related to popular algebraic objects such as conformal superalgebras [5], vertex operator superalgebras [8] and super Gel'fand-Dorfman bialgebras [7] which play an important role in the quantum field theory and the theory of completely integrable systems.

A Novikov superalgebra A is a $\mathbb{Z}_2$-graded vector space $A = A_0 + A_1$ with a bilinear product $(u, v) \mapsto uv$ for any $u \in A_i$, $v \in A_j$, $w \in A$ satisfying

$$(uv)w - u(vw) = (-1)^{ij}((vu)w - v(uw)), \tag{1.1}$$

$$(wu)v = (-1)^{ij}(wv)u. \tag{1.2}$$

The even part of a given Novikov superalgebra is what is said to be a Novikov algebra introduced in connection with the Poisson brackets of hydrodynamic type [1] and Hamiltonian operators in the formal variational calculus [2], [3], [4], [9], [10].

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Novikov superalgebras are classified into two types: N and S. Let $A = A_0 + A_1$ be a Novikov superalgebra with multiplication $(u, v) \mapsto uv$. If A is also a Novikov algebra with respect to the same product and with superstructure forgotten, then A is called a Novikov superalgebra of type N, otherwise A is said to be of type S. It is proved in [6] that all Novikov superalgebras of dimensions up to 3 are of type N. Up to now, all examples of Novikov superalgebras have been of type N.

In this paper we show that the Novikov superalgebras with $A_0 = A_1A_1$ and $\dim A_1 = 2$ are of type N. Furthermore, we provide a class of Novikov superalgebras of type S with $A_0 = A_1A_1$ and $\dim A = 2n + 1$ for $n \geqslant 2$.

Throughout the paper we assume that the algebras are finite-dimensional over $\mathbb{C}$. Obvious proofs are omitted.

2. Novikov superalgebras with $A_0 = A_1A_1$

Let $A = A_0 + A_1$ be a Novikov superalgebra.

Lemma 2.1. *The subspace $A_1A_1 + A_1$ is an ideal of A.*

Proof. For any $x \in A_0$, $y, z \in A_1$, we have

$$(yz)x = (yx)z \in A_1A_1,$$
$$x(yz) = y(xz) + (xy)z - (yx)z \in A_1A_1.$$

It follows that $A_1A_1 + A_1$ is an ideal of A. □

Lemma 2.2. *For any $x \in A_1$, $y \in A$, we have $(xx)y = x(xy)$ and $(yx)x = 0$.*

Now we consider the Novikov superalgebras $A = A_0 + A_1$ with $A_0 = A_1A_1$, $\dim A_1 = 2$ and $A_0 \neq \{0\}$. Then one can easily see that $\dim A_0 \leqslant 4$.

Let f_1, f_2 be a basis of A_1. Set

$$e_1 = f_1f_1, \quad e_2 = f_2f_2, \quad e_3 = f_1f_2, \quad e_4 = f_2f_1. \tag{2.1}$$

Then $A_0 = \langle e_1, e_2, e_3, e_4\rangle$.

Assume that $xx = 0$ for any $x \in A_1$. Then $\dim A_0 = 1$ and $f_1f_2 = -f_2f_1$ is a basis of A_0. By [6], $A_0A_1 = A_1A_0 = A_0A_0 = 0$ and A is of type N.

In the following, assume that $e_1 = f_1f_1 \neq 0$.

Lemma 2.3. $A_0A_1 = 0$.

P r o o f. By Lemma 2.2, one has $f_1e_1 = e_1f_1 = 0$, $e_4f_1 = 0$, $f_2e_2 = e_2f_2 = 0$ and $e_3f_2 = 0$. The following is to show that $e_2f_1 = 0$.

If e_2 is a multiple of e_1, the assertion is trivial. Assume that e_1 and e_2 are linearly independent. It is easy to see that

$$(e_2f_1)f_1 = (e_2f_1)f_2 = 0. \tag{2.2}$$

Furthermore,

$$\begin{aligned} f_2(e_2f_1) = f_2(f_2(f_2f_1)) &= (f_2f_2)(f_2f_1) = (f_2(f_2f_1))f_2 \\ &= -((f_2f_2)f_2)f_1 = 0. \end{aligned} \tag{2.3}$$

Assume that $e_2f_1 = af_1 + bf_2$, then by eqs. (2.2) and (2.3) we have that

$$ae_1 + be_4 = ae_3 + be_2 = ae_4 + be_2 = 0.$$

It follows that $a = b = 0$. Similarly, $e_3f_1 = e_4f_1 = 0$ and $A_0f_2 = 0$. $\square$

Lemma 2.4. *The subalgebra A_0 is skew-commutative.*

P r o o f. For any f_i, f_j, e_k, we have

$$(f_if_j)e_k = (f_ie_k)f_j = (e_kf_i)f_j + f_i(e_kf_j) - e_k(f_if_j) = -e_k(f_if_j),$$

since $e_kf_i = e_kf_j = 0$ by Lemma 2.3. $\square$

Lemma 2.5.

1) $f_1e_2 = -f_2e_3$, $f_1e_4 = -f_2e_1$, $f_1e_1 = f_1e_3 = 0$, $f_2e_2 = f_2e_4 = 0$.
2) $e_1e_2 = e_3e_4$, $e_1e_3 = e_2e_4 = 0$.

P r o o f. 1) By eq. (1.1), $(f_1f_2)f_2 - f_1(f_2f_2) = -(f_2f_1)f_2 + f_2(f_1f_2)$. So $f_1e_2 = -f_2e_3$. Similarly, $f_1e_4 = -f_2e_1$. Also, one has $f_1e_3 = f_1(f_1f_2) = (f_1f_1)f_2 = 0$. Similarly, $f_2e_2 = f_2e_4 = 0$.

2) By eq. (1.2), $e_1e_2 = (f_1f_1)e_2 = (f_1e_2)f_1 = -(f_2e_3)f_1 = -(f_2f_1)e_3 = -e_4e_3 = e_3e_4$. By 1) $e_1e_3 = (f_1f_1)e_3 = f_1(f_1e_3) = 0$ and similarly $e_2e_4 = 0$ and $f_1e_2 \neq 0$. $\square$

Lemma 2.6.

1) *The elements* f_1e_2 *and* f_1e_4 *commute with* f_1 *and* f_2.

2) $A_1A_0 = 0$ *and* $A_0A_0 = 0$.

P r o o f. 1) For $x = e_2$ or e_4, $f_1(f_1x) = (f_1f_1)x = (f_1x)f_1$ and similarly f_2 commutes with f_2y for $y = e_1$ or e_3. Then the assertion follows from Lemma 2.5.

2) If $A_1A_0 \neq 0$, then assume that $f_1e_2 = a_1f_1 + a_2f_2$ and $f_1e_4 = a_3f_1 + a_4f_2$, where at least one a_i is not zero. Then by 1), $a_ie_3 = a_ie_4$ for $1 \leqslant i \leqslant 4$, so $e_3 = e_4$. Consequently, $f_1e_4 = f_1e_3 = 0$ and $f_1e_2 = -f_2e_3 = -f_2e_4 = 0$. So $A_1A_0 = 0$. Moreover, $A_0A_0 = 0$. □

Theorem 2.7. *Let* $A = A_0 + A_1$ *be a Novikov superalgebra with* $A_0 = A_1A_1$ *and* $\dim A_1 = 2$. *Then* A *is of type* N.

Example 2.8. Assume that $A = A_0 + A_1$ is a vector space of dimension $2n+1$, where $n \geqslant 2$. Let $e_1, \ldots, e_n$ be a basis of A_0 and $f_1, \ldots, f_n, f_{n+1}$ a basis of A_1. Set

$$\begin{gathered}
f_1f_i = e_i, \quad 1 \leqslant i \leqslant n, \\
e_if_{n+1-i} = \begin{cases} f_{n+1}, & 1 \leqslant i \leqslant [\frac{1}{2}n], \\ -f_{n+1}, & [\frac{1}{2}(n+1)] + 1 \leqslant i \leqslant n, \end{cases} \\
f_ie_{n+1-i} = \begin{cases} f_{n+1}, & 1 \leqslant i \leqslant [\frac{1}{2}n], \\ -f_{n+1}, & [\frac{1}{2}(n+1)] + 1 \leqslant i \leqslant n, \end{cases}
\end{gathered}$$

Then we have

$$\begin{gathered}
(f_1f_j)f_{n+1-j} = -(f_1f_{n+1-j})f_j, \\
(f_1f_1)f_n = f_1(f_1f_n), \\
(f_1f_j)f_{n+1-j} - f_1(f_jf_{n+1-j}) = -(f_jf_1)f_{n+1-j} + f_j(f_1f_{n+1-j}), \quad j \geqslant 2.
\end{gathered}$$

It follows that A is a Novikov superalgebra of type S with $A_0 = A_1A_1$.

References

[1] *A. A. Balinskii, S. P. Novikov*: Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. Sov. Math. Dokl. *32* (1985), 228–231.

[2] *I. M. Gel'fand, I. Ya. Dorfman*: Hamiltonian operators and algebraic structures related to them. Funct. Anal. Appl. *13* (1980), 248–262.

[3] *I. M. Gel'fand, I. Y. Dorfman*: The Schouten brackets and Hamiltonian operators. Funct. Anal. Appl. *14* (1981), 223–226.

[4] *I. M. Gel'fand, I. Y. Dorfman*: Hamiltonian operators and infinite-dimensional Lie algebras. Funct. Anal. Appl. *15* (1982), 173–187.

[5] *V. G. Kac*: Vertex Algebras for Beginners. University Lecture Series, 10. American Mathematical Society (AMS), Providence, 1998.
[6] *Y. F. Kang, Z. Q. Chen*: Novikov superalgebras in low dimensions. J. Nonlinear Math. Phys *16* (2009), 251–257.
[7] *X. P. Xu*: Quadratic conformal superalgebras. J. Algebra *231* (2000), 1–38.
[8] *X. P. Xu*: Introduction to Vertex Operator Superalgebras and Their Modules. Kluwer, Dordercht, 1998.
[9] *X. P. Xu*: Hamiltonian operators and associative algebras with a derivation. Lett. Math. Phys. *33* (1995), 1–6.
[10] *X. P. Xu*: Hamiltonian superoperators. J. Phys A. Math. Gen. *28* (1995), 1681–1698.

Authors' addresses: F. Zhu, School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China; Z. Chen (corresponding author), School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China, e-mail: `chenzhiqi@nankai.edu.cn`.