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THE GENERALIZED FGM DISTRIBUTION AND ITS APPLICATION TO STEREOLOGY OF EXTREMES*

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Abstract. The generalized FGM distribution and related copulas are used as bivariate models for the distribution of spheroidal characteristics. It is shown that this model is suitable for the study of extremes of the 3D spheroidal particles observed in terms of their random planar sections.

Keywords: generalized FGM distribution, extremes, stereology, maximum domain of attraction

MSC 2010: 62H05

Introduction

Recently much attention has been paid to the stereology of extremes. Generally we have here in mind a prediction of extremes of certain characteristics of higher dimensional objects by means of observations of their lower dimensional probes or sections. More specifically, we consider three-dimensional particles in a given volume of material where the observations are the sections of the particles created by a random planar section of the material. A typical application is the metal fatigue problem, see e.g. [17] or [5]. The classical approach is to view the particles as balls of a random size so that the sections become random circles, the so called *Wicksell's corpuscle problem*, see [25] and [26]. The main goal is to predict the largest size of the ball in the material. Substantial amount of literature is devoted to this problem ([8], [19], [20], [21], [22], [23]).

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In our previous work ([12], [13], [4]) we have dealt with oblate spheroids rather than balls. We have concentrated on extremely flat particles whose size is sufficiently large enough rather than just the largest particles. It turns out that when studying the bivariate random vector which characterizes the spheroids, namely the pair "*size* and *shape factor*" (X, S) (see Subsection 1.1), it is necessary to impose a stronger assumption on the joint distribution of the characteristics, and we need to assume a certain tail *uniformity* of our model. It is shown in [11] that this assumption is fulfilled when using the standard Farlie-Gumbel-Morgenstern (FGM) bivariate family of distributions provided the FGM parameter λ satisfies $|\lambda| < 1$.

In this paper we turn our attention to a more general class of distributions suitable for our analysis. We extend the FGM distribution utilizing the recent results ([2], [3], [1], [18]). The main reason for using these extensions is the well-known fact that the covariance for the standard FGM family is limited by $1/3$. Hence, an application of the distribution in practice may be questionable.

In Section 1 we recall some basic facts from stereology and extreme value theory and restate the "stability of MDA" theorem which involves the tail uniformity assumption. In Section 2 we introduce the generalized FGM distributions related to the stereological problem at hand.

1. Distribution of spheroids and MDA

Consider a population of oblate spheroids uniformly distributed and isotropically oriented in a given volume of an opaque material. Here one cannot observe the particles directly but it is possible to observe a sample of their *profiles*. Profiles of the particles are produced by a random planar section of the volume of the material. The profiles of spheroids are ellipses and the ellipses observed on the section constitute a random sample of the population of profiles. The oblate spheroids and ellipses are characterized by the following two features.

1.1. Characterization of spheroidal particles

Consider an oblate (lentil shaped) spheroid with (two equal) major semiaxes X and a minor semiaxis V. The spheroid is then fully characterized by the pair

$$(X, S), \quad \text{where } S = \frac{X^2}{V^2} - 1.$$

In this definition X is called the *size* and S is called the *shape factor* of the spheroid.

Consider an ellipse with the major and minor semiaxes Y and W, respectively. This ellipse is fully characterized by the pair

$$(Y, T), \quad \text{where } T = \frac{Y^2}{W^2} - 1.$$

Analogously to the above Y is called the *size* and T is called the *shape factor* of the ellipse.

Below all the characteristics are considered to be random variables independent of the position and orientation of a particle. We shall assume that a joint probability density function (p.d.f.) $g(x,s)$ of (X,S) is available. The joint p.d.f. $f(y,t)$ of (Y,T) is to be calculated. It is given for example in [7] (where the eccentricity instead of the shape factor is used). The joint p.d.f. is

$$f(y,t) = \frac{y\sqrt{1+t}}{2M} \int_y^{X_f}\int_t^{S_f} \frac{g(x,s)\,\mathrm{d}s\,\mathrm{d}x}{\sqrt{s}\sqrt{1+s}\sqrt{s-t}\sqrt{x^2-y^2}}, \tag{1.1}$$

where M is a population mean size of the particles (half of the mean caliper diameter) and X_f, S_f are the upper endpoints of the marginal distributions of X and S respectively, i.e.

$$X_f = \inf\{x\colon\ \mathrm{P}[X \leqslant x] = 1\}, \quad \text{and} \quad S_f = \inf\{s\colon\ \mathrm{P}[S \leqslant s] = 1\}. \tag{1.2}$$

1.2. Maximum domain of attraction

We shall now restate briefly the results presented in [12] and [13] concerning the stability of the maximum domain of attraction (MDA). We shall recall the basic tenant about the extreme value theory utilized in this paper (these results are well known for over 50 years and presumably are due to Gnedenko [10]). Consider a random sample of n iid random variables $X_1, X_2, \ldots, X_n$ with the cumulative distribution function (c.d.f.) H and denote the sample maximum by $X_{n:n}$. It is well known that the c.d.f. of $X_{n:n}$ is $H^n(\cdot)$ and that an affine transformation of $X_{n:n}$ may converge to one of the three extreme value distributions: Gumbel, Fréchet and Weibull.

Definition 1.1 (MDA). If there exist pairs of *normalizing constants* (a_n, b_n) such that

$$H^n(a_n x + b_n) \underset{n\to\infty}{\longrightarrow} \begin{cases} \Lambda(x) = \exp(-\mathrm{e}^{-x}), & x \in \mathbb{R},\ \text{(Gumbel distr.), or} \\ \Phi_\alpha(x) = \exp(-x^{-\alpha}), & x \geqslant 0,\ \text{(Fréchet distr.), or} \\ \Psi_\alpha(x) = \exp(-(-x)^{\alpha}), & x \leqslant 0,\ \text{(Weibull distr.)} \end{cases} \tag{1.3}$$

for some $\alpha > 0$ then the distribution H is said to belong to the *Maximum domain of attraction* of the Gumbel distribution (denoted as $H \in \mathrm{MDA}(\Lambda)$), Fréchet or Weibull extreme value distributions, respectively.

R e m a r k 1.2. There is a unifying approach to the limiting behaviour of the sample maxima—the so called *generalized extreme value distribution* (see, e.g., Coles [6]). We cite the three types mainly because for the proofs of the MDA stability in [12] or [13] it is required to distinguish between these three cases. It is also more suitable for the calculation of the normalizing constants.

Below the joint distribution of the spheroid characteristics (X, S) will always be assumed to be absolutely continuous with respect to the Lebesgue measure, hence a joint density function $g(x, s)$ is available. Denote by $g_x(s)$ and $G_x(s)$ the *conditional density and distribution function respectively of the shape factor given the size.* The maximum domain of attraction of $G_x(\cdot)$ is given by one of the following conditions (see Chapter 3 of [9]).

Lemma 1.3 (Sufficient condition for MDA). *Let $K(\cdot)$ and $k(\cdot)$ be the distribution function and the density function respectively of some univariate random variable S with the upper endpoint S_f. Assume that there exists an auxiliary function $b(\cdot)$ or a constant α such that one of the conditions*

$$\lim_{s \nearrow S_f} \frac{k(s + ub(s))}{k(s)} = \mathrm{e}^{-u}, \quad u \in \mathbb{R}, \tag{1.4}$$

$$\lim_{s \to \infty} \frac{k(us)}{k(s)} = u^{-(\alpha+1)}, \quad u > 0, \ S_f = +\infty, \tag{1.5}$$

$$\lim_{s \searrow 0} \frac{k(S_f - us)}{k(S_f - s)} = u^{\alpha-1}, \quad x > 0, \ S_f < +\infty \tag{1.6}$$

holds. Then

$$(1.4) \Rightarrow K \in \mathrm{MDA}(\Lambda), \ (1.5) \Rightarrow K \in \mathrm{MDA}(\Phi_\alpha), \ \textit{and} \ (1.6) \Rightarrow K \in \mathrm{MDA}(\Psi_\alpha).$$

Remark 1.4 (Uniformity of the conditions). Below we shall require the conditional density function $g_x(s)$ to be uniform for the limits (1.4), (1.5), (1.6) in the conditioning value. This property may be called the *tail equivalence/uniformity.* Here we have in mind that for the density function $g_x(s)$ the convergence in (1.4)–(1.6) is uniform with respect to x and that the auxiliary function $b(\cdot)$ in (1.4) and the constant α in (1.5), (1.6) could be chosen to be the same for all possible values of x.

The following notation will consistently be used throughout the paper. The joint distribution function (d.f.) of the spheroid characteristics (X, S) and the d.f. of the profile characteristics (Y, T) are denoted $G(x, s)$ and $F(y, t)$, respectively. The corresponding densities are denoted $g(x, s)$ and $f(y, t)$.

In applications we may require to study distributions of the shape factor both marginally and conditionally on the size of a spheroid. These one-dimensional distributions are denoted as follows. The marginal distribution functions of S and T are

$$G_S(s) = \mathrm{P}[S \leqslant s], \quad F_T(t) = \mathrm{P}[T \leqslant t]$$

the conditional distribution functions given the size are denoted

$$G_x(s) = \mathrm{P}[S \leqslant s | X = x], \quad F_y(t) = \mathrm{P}[T \leqslant t | Y = y]$$

and the conditional distribution functions given the size exceeding a threshold x (y) are expressed by

$$G_{>x}(s) = \mathrm{P}[S \leqslant s | X > x], \quad F_{>y}(t) = \mathrm{P}[T \leqslant t | Y > y].$$

For the corresponding density functions we shall use the lower cases g and f, respectively.

Proposition 1.5 (Stability of MDA). *The tail uniformity of the density function $g_x(\cdot)$ is sufficient for the stability of* MDA *in the following sense.*

(i) *Assume that the conditional density $g_x(s)$ obeys the condition* (1.4) *uniformly in x for some function $b(\cdot)$. Then the inclusions $\{G, G_x, G_{>x}\} \subset \mathrm{MDA}(\Lambda)$ and $\{F, F_y, F_{>y}\} \subset \mathrm{MDA}(\Lambda)$ are valid.*

(ii) *Assume that the conditional density $g_x(s)$ obeys the condition* (1.5) *uniformly in x for some $\alpha > 0$. Then the inclusions $\{G, G_x, G_{>x}\} \subset \mathrm{MDA}(\Phi_\alpha)$ and $\{F, F_y, F_{>y}\} \subset \mathrm{MDA}(\Phi_\alpha)$ are valid.*

(iii) *Assume that the conditional density $g_x(s)$ obeys the condition* (1.6) *uniformly in x for some $\alpha > 0$. Then the inclusions $\{G, G_x, G_{>x}\} \subset \mathrm{MDA}(\Psi_\alpha)$ and $\{F, F_y, F_{>y}\} \subset \mathrm{MDA}(\Psi_{\alpha+1/2})$ are valid.*

Proof. Proofs are given in [12] and [14]. □

2. Farlie-Gumbel-Morgenstern distribution and its generalization

We shall now analyze the tail equivalence (uniformity) condition presented in 1.5. In particular we shall prove that the generalized Farlie-Gumbel-Morgenstern (FGM) bivariate distribution satisfies (under some mild conditions) this assumption.

The standard FGM distribution introduced in [15] is a classical example of a copula with prescribed quadratic sections. See e.g. [16] for an introduction to copulas. Recall that a copula is a bivariate function $C\colon [0,1]^2 \to [0,1]$ which associates a bivariate distribution function with its marginals. If the marginal distribution functions are continuous (as they are in our case) the well-known Sklar's theorem provides us with the uniqueness of the copula. A copula $C(x, y) = xy$ represents a product of two independent variables.

Definition 2.1 (A general FGM family). A copula given by

$$C(x,y) = xy\{1 + \lambda A(x)B(y)\}, \tag{2.1}$$

where $A(\cdot)$ and $B(\cdot)$ are differentiable functions defined on $[0,1]$ such that $\lim\limits_{x\to 1} A(x) = 0$, $\lim\limits_{x\to 1} B(x) = 0$ and $C(\cdot,\cdot)$ is a bivariate d.f. with uniform marginals is called a general FGM copula. The real parameter λ is a "dependence parameter", if $\lambda = 0$ then C is clearly the independence copula. If a copula C is applied to a pair of absolutely continuous distribution functions $H_X(\cdot)$, $H_Y(\cdot)$ with densities $h_X(\cdot)$ and $h_Y(\cdot)$, respectively, the joint probability density given by C becomes

$$\begin{aligned} h(x,y) = h_X(x)h_Y(y)\bigl\{1 + \lambda\bigl[A(H_X(x)) + H_X(x)A'(H_X(x))\bigr] \\ \times \bigl[B(H_Y(y)) + H_Y(y)B'(H_Y(y))\bigr]\bigr\}. \end{aligned} \tag{2.2}$$

Remark 2.2. The main advantage of the general FGM copula for our purposes is the *factorization* of the two variables. In other words, for any bivariate function $\gamma(x,y)$, $\gamma(x,y) = \gamma_1(x)\gamma_2(y)$ is valid.

There are several choices of the functions A and B. In [1] the form

$$A(x) = (1 - x^q)^p = B(x) \tag{2.3}$$

is proposed. While the standard FGM distribution with a copula $C(x,y) = xy\bigl(1 + \lambda(1-x)(1-y)\bigr)$ allows a limited correlation between the components, which does not exceed $1/3$, the correlation for the proposed distribution (2.3) *may be greater than* $1/2$ which is useful for applications. Moreover, in [1] one could find an extension of the general FGM class for which the correlation between components exceeds 0.6.

Definition 2.3 (An extended FGM family of class I). The *extended Huang-Kotz FGM* distribution is a bivariate distribution given by the copula

$$C(x,y) = x^p y^p\bigl\{1 + \lambda(1 - x^q)^n(1 - y^q)^n\bigr\}, \quad p,q \geqslant 1,\ n > 1, \tag{2.4}$$

with the marginals x^p and y^p. The possible range for λ which determines the correlation between the components is

$$-\min\left\{1, \frac{p^2}{q^2}\Bigl[\frac{p+qn}{q(n-1)}\Bigr]^{2(n-1)}\right\} \leqslant \lambda \leqslant \frac{p}{q}\Bigl[\frac{p+qn}{q(n-1)}\Bigr]^{n-1}.$$

Differentiating one easily obtain that the density of the extended Huang-Kotz FGM distribution with the marginals x^p, y^p is:

$$\begin{aligned} h(x,y) = x^{p-1}y^{p-1}\bigl\{p^2 + \lambda(1 - x^q)^{n-1}[p - (p+qn)x^q] \\ \times (1 - y^q)^{n-1}[p - (p+qn)x^q]\bigr\}. \end{aligned}$$

Another extension of the FGM family of distributions is given in [18].

Definition 2.4 (An extended FGM family of class II). Consider a bivariate extended FGM family given by the copula

$$C(x,y) = xy\bigl\{1 + \lambda x^{a-1}(1-x)^c y^{b-1}(1-y)^d\bigr\}, \quad a,b,c,d \geqslant 1. \tag{2.5}$$

The possible range for λ for this copula (which is quite complex) is given in Example 4.1 in [18].

The construction of a bivariate distribution from the generalized FGM family is briefly as follows: Given two "marginal" density functions $h_X(x)$ and $h_Y(y)$ (these need not be the real marginals as follows from the Huang-Kotz extension [1]) and the corresponding distribution functions $H_X(x)$, $H_Y(y)$, the bivariate density function $h(x,y)$ is given by

$$\begin{aligned} h(x,y) = {} & h_X(x)h_Y(y)\xi_X(H_X(x))\xi_Y(H_Y(y)) \\ & \times \bigl[1 + \lambda\psi_X(H_X(x))\psi_Y(H_Y(y))\bigr], \end{aligned} \tag{2.6}$$

where $\xi(\cdot)\colon\ [0,1] \to \mathbb{R}$ and $\psi(\cdot)\colon\ [0,1] \to \mathbb{R}$ are appropriate functions of $H_X(x)$ and $H_Y(y)$ respectively, and λ is within its possible range. Moreover, to ensure that $h(x,y)$ in (2.6) is a bonafide bivariate density the conditions

$$\begin{aligned} \lim_{x\to1} \xi_X(x) &= \lim_{x\to1} \xi_Y(x) = 1, \\ \lim_{x\to1} \psi_X(x) &= \lim_{x\to1} \psi_Y(x) = 0 \end{aligned}$$

should be valid.

We are now ready to prove the tail equivalence/uniformity property for the family of distributions given in (2.6).

Theorem 2.5. *Consider a bivariate density function*

$$\begin{aligned} g(x,s) = {} & g_X(x)g_S(s)\xi_X(G_X(x))\xi_S(G_S(s)) \\ & \times \bigl[1 + \lambda\psi_X(G_X(x))\psi_S(G_S(s))\bigr] \end{aligned}$$

such that

(i) $|\lambda| < K$ *and* $|\psi_X(\cdot)| < K$ *for some finite constant* K, *and*
(ii) $1 + \lambda\psi_X(x)\psi_Y(y) > \varepsilon > 0$ *for all* $(x,y) \in [0,1]^2$, *where* ε *is a suitable positive constant.*

Then for the conditional density $g_x(s)$ *given in Proposition* 1.5 *the tail uniformity in* x *is valid.*

Remark 2.6. Condition (i) of the above theorem is usually satisfied. The crucial condition is the strict inequality in (ii) which for the standard FGM family means that $|\lambda| \neq 1$. Note that for any distribution in this class the condition $1 + \lambda\psi_X(x)\psi_Y(y) \geqslant 0$ must hold a.e. (to ensure the positivity of $h(x, y)$).

Proof. In the conditions (1.4)–(1.6) of Lemma 1.3 consider a transformed variable in the numerator. Denote the transform by $\varphi(\cdot)$. For example, (1.5) may be written as

$$\lim_{s\to\infty} \frac{g_x(\varphi(s))}{g_x(s)} = u^{-(\alpha+1)}$$

for some $\alpha > 0$, and $u > 0$ which is specified by $\varphi(s) = us$. We shall concentrate on the case of the Fréchet limit distribution; the other cases are completely analogous.

Suppose that the above equality holds for some x_0 and that the density $g(x, s)$ can be written in the form (2.6). It then follows that

$$R(x, x_0, s) = \frac{g_x(\varphi(s))}{g_x(s)} \frac{g_{x_0}(s)}{g_{x_0}(\varphi(s))} - 1 \underset{s\to\infty}{\longrightarrow} 0. \tag{2.7}$$

Indeed,

$$\begin{aligned} |R(x, x_0, s)| &= \left| \frac{[1 + \lambda\psi_X^G(x)\psi_Y^G(\varphi(s))]}{[1 + \lambda\psi_X^G(x)\psi_Y^G(s)]} \frac{[1 + \lambda\psi_X^G(x_0)\psi_Y^G(s)]}{[1 + \lambda\psi_X^G(x_0)\psi_Y^G(\varphi(s))]} - 1 \right| \\ &= \left| \frac{\lambda[\psi_X^G(x) - \psi_X^G(x_0)][\psi_Y^G(\varphi(s)) - \psi_Y^G(s)]}{[1 + \lambda\psi_X^G(x_0)\psi_Y^G(\varphi(s))][1 + \lambda\psi_X^G(x)\psi_Y^G(s)]} \right| \\ &< \frac{2K^2}{\varepsilon^2} |\psi_Y^G(\varphi(s)) - \psi_Y^G(s)|, \end{aligned} \tag{2.8}$$

where $\psi_X^G(\cdot)$ is an abbreviated notation for $\psi_X(G_X(\cdot))$. The last term in (2.8) does not depend on x, and since both $G_X(\varphi(s))$ and $G_X(s)$ tend to 1 as s goes to infinity, the convergence

$$|\psi_Y^G(\varphi(s)) - \psi_Y^G(s)| \underset{s\to\infty}{\longrightarrow} 0$$

is valid. This completes the proof. □

Our aim is to estimate the distribution function (d.f.) of the flattest spheroid in the volume based on the random profiles observed. We know the transformation of the joint probability density function (p.d.f.) of the spheroid characteristics to the joint p.d.f. of the profile characteristics. We also know that under mild assumptions the general FGM bivariate distribution is sufficient for preservation of the maximum domain of attraction as was shown in Proposition 1.5 and Theorem 2.5. Hence, under this model it is possible to use the profile shape factor observations to test MDA and to estimate the parameter α for the Fréchet and Weibull cases (see e.g. Chapter 3 in [6]). The limiting behavior for the spheroid shape factor may be determined in accordance with Proposition 1.5. We can therefore approximate the distribution of extreme shape factor by one of the limiting distributions, provided the normalizing constants are known. We shall now briefly discuss this concept.

Definition 3.1 (Normalizing constants). Let (a_n, b_n) be a sequence of real constants such that for the sample extreme $M_{n:n}$ of the random sample of iid's $M_1, \ldots, M_n$ the convergence

$$\frac{M_{n:n} - b_n}{a_n} \underset{n\to\infty}{\longrightarrow} L$$

holds, where L is a random variable whose distribution function belongs to the set of Gumbel, Fréchet, and Weibull distributions. The constants a_n and b_n are called *normalizing constants for the distribution of* M_1.

Usually, normalizing constants (n.c.) are not unique. Indeed, consider another sequence (a'_n, b'_n) such that $a'_n/a_n \to 1$ and $(b_n - b'_n)/a_n \to 0$ as $n \to \infty$. It is easy to verify that a'_n and b'_n are also n.c. Thus one could consider a class of equivalent normalizing constants. Below we shall refer to *the* normalizing constants having the previous observation in mind. To determine the n.c. we are usually required to analyse the tail behavior of the d.f. at hand. In this connection the following lemma is often useful.

Lemma 3.2 (Normalizing constants I). *Suppose that a distribution function K has an upper endpoint M_f. Then the following statements are valid.*

(i) *If $M_f = \infty$ the d.f. K belongs to the Gumbel domain of attraction and if there exist constants $\alpha > 0$, $\beta, \gamma > 0$, $\delta > 0$ such that*

$$\lim_{v\to\infty} \frac{1 - K(v)}{\alpha v^{\beta} \mathrm{e}^{-\gamma v^{\delta}}} = 1,$$

then the normalizing constants can be chosen as

$$a_n = \Big(\frac{\log n}{\gamma}\Big)^{1/\delta - 1} \frac{1}{\gamma\delta},$$
$$b_n = \Big(\frac{\log n}{\gamma}\Big)^{1/\delta} + \frac{(\beta/\delta)(\log\log n - \log\gamma) + \log\alpha}{(\log n/\gamma)^{1-1/\delta}\gamma\delta}.$$

(ii) *If the distribution function K belongs to the Fréchet domain of attraction and if there exist constants $\alpha > 0$, $\beta, \gamma > 0$ such that*

$$\lim_{v\to\infty} \frac{1 - K(v)}{\alpha v^{-\gamma}} = 1,$$

then the normalizing constants can be chosen as

$$a_n = (n\alpha)^{1/\gamma}, \quad b_n = 0.$$

(iii) *If the distribution function K belongs to the Weibull domain of attraction and if there exist constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that*

$$\lim_{v\to M_f} \frac{1 - K(v)}{\gamma(v/M_f)^{\beta}(M_f - v)^{\alpha}} = 1,$$

then the normalizing constants can be chosen as

$$a_n = (n\gamma)^{-1/\alpha}, \quad b_n = M_f.$$

Sketch of the proof. The proof is based on the possible choices of the normalizing constants. (See, e.g. the description in [9].) First, we need to find the $(1 - n^{-1})$th quantile of the distribution. Denoting the quantile by q the possible choices of normalizing constants are:

(i) for the Gumbel limit distribution $b_n = q$ and $a_n = b(q)$, where $b(\cdot)$ is the auxiliary function described in Lemma 1.3.

(ii) for the Fréchet limit distribution $b_n \equiv 0$ and $a_n = q$.

(iii) for the Weibull limit distribution $b_n \equiv M_f$ and $a_n = M_f - q$.

We shall briefly analyse the Gumbel distribution.

The $(1-n^{-1})$th quantile q is determined from

$$\alpha q^{\beta} \mathrm{e}^{-\gamma q^{\delta}} = \frac{1}{n},$$

$$q^{\delta} = \frac{1}{\gamma}(\log n + \beta \log q + \log \alpha),$$

$$q = \Big(\frac{\log n}{\gamma}\Big)^{1/\delta} \Big\{1 + \frac{\beta \log(\log n/\gamma)^{1/\delta} + \log(1 + \beta \log q + \log \alpha)^{1/\delta} + \log \alpha}{\log n}\Big\}^{1/\delta},$$

$$\Downarrow$$

$$b_n \doteq \Big(\frac{\log n}{\gamma}\Big)^{1/\delta} \Big\{1 + \frac{1}{\delta \log n}\Big(\frac{\beta}{\delta}(\log \log n - \log \gamma) + \log \alpha\Big)\Big\}. \tag{3.1}$$

The auxiliary function $b(\cdot)$ can be chosen as

$$b(q) = \frac{\int_q^{M_f} (1 - K(v))\,\mathrm{d}v}{1 - K(q)}.$$

(See [9], Chapter 3.3.) Hence, it is not difficult to verify that in the Gumbel case under consideration we have

$$b(q) \doteq \frac{(\alpha/\gamma^{\delta}) q^{\beta-\delta+1} \mathrm{e}^{-\gamma q^{\delta}}}{\alpha q^{\beta} \mathrm{e}^{-\gamma q^{\delta}}} = \frac{1}{\gamma\delta} q^{1-\delta}. \tag{3.2}$$

Finally, applying $b(\cdot)$ of (3.2) to b_n in (3.1), one obtains a_n. Recall the observation above Lemma 3.2. The parts omitted in the above approximations are negligible in the sense of this remark. $\square$

The following two cases are of interest. The former is to investigate the shape factor regardless of the size of the spheroid. Consequently, we need to study $1 - G_S(s)$ as $s \to S_f$. The latter goal is to relate the shape factor to the size. Here we are interested in the shape factor of those spheroids whose size exceeds the given threshold. Thus, $1 - G_{>x}(s)$ as $s \to S_f$ should be analysed.

Note that the only information available is the random sample of the *profiles* while we are interested in the *spheroids* characteristics. However, one can estimate normalizing constants for the shape factor (related and unrelated to the size) of the profiles, and then use Proposition 1.5. To this end we need also to study the limiting behaviour of $1 - F_T(t)$, and possibly that of $1 - F_{>y}(t)$. The normalizing constants estimated for the profiles will then be adjusted to the n.c. that are appropriate for the original particles.

We start with the general form of the density of the FGM class given in Definition 2.1. We are interested in the limiting behaviour of the shape factor S, and the marginal density of X is $g_X(x)$; hence we shall employ for the conditional p.d.f. $g_x(s)$ the form:

$$g_x(s) = g_S(s)\big\{1 + \lambda\psi_X^G(x)[B(G_S(s)) + G_S(s)B'(G_S(s))]\big\}, \tag{3.3}$$

where $B(u) \to 0$ as $u \to 1$, recall the definitions following Lemma 1.3. Hence, for an analysis of the tail behaviour of $1 - G_x(s)$ one can omit the term $B(G_S(s))$ and focus only on the term containing the derivative $B'(\cdot)$.

Example 3.3. Let $B(u) = 1 - u$ in the standard FGM family. Also let $g_S(s) = \exp\{-s\}$, and $S_f = \infty$. Then $B'(\cdot) = -1$ and

$$\begin{aligned} g_x(s) &= \mathrm{e}^{-s}\big\{1 + \lambda\psi_X^G(x)[1 - (1 - \mathrm{e}^{-s}) - (1 - \mathrm{e}^{-s})]\big\} \\ &= \mathrm{e}^{-s}(1 - \lambda\psi_X^G(x)) + \mathrm{e}^{-2s}2\lambda\psi_X^G(x). \end{aligned}$$

For large s the second summand is quite negligible compared to the first one and the density for large s is approximately:

$$g_x(s) \doteq \mathrm{e}^{-s}(1 - \lambda\psi_X^G(x)) = g_S(s)\{1 + \lambda\psi_X^G(x)B'(G_S(s))\}.$$

Example 3.4. Consider the same marginal density $g_S(s) = \exp\{-s\}$, where $S_f = \infty$ and the function B is (see [3])

$$B(u) = \frac{u^{p+1} - u}{(p+1)^2} - \frac{u^{p+1}\log u}{p+1}$$

for some $p > 0$. Evidently $B(u) \to 0$ as $u \to 1$ and $B'(u) = -u^p \log u - (p+1)^{-2} \to -(p+1)^{-2}$ as $u \to 1$. After some calculations we obtain for large s

$$\begin{aligned} g_x(s) &= \mathrm{e}^{-s}(1 - \lambda(p+1)^{-2}\psi_X^G(x)) + O(\mathrm{e}^{-2s}) \\ &\doteq \mathrm{e}^{-s}(1 - \lambda(p+1)^{-2}\psi_X^G(x)). \end{aligned}$$

Example 3.5. Consider once more the same marginal density $g_S(s) = \exp\{-s\}$, and $S_f = \infty$, but now let $B(u) = (1-u)^p$ for some $p > 1$. In this case $B'(u) = -p(1-u)^{p-1}$ and

$$\begin{aligned} g_x(s) &= \mathrm{e}^{-s}\big\{1 + \lambda\psi_X^G(x)[(\mathrm{e}^{-s})^p - p(\mathrm{e}^{-s})^{p-1}(1 - \mathrm{e}^{-s})]\big\} \\ &= \mathrm{e}^{-s} - \lambda p\psi_X^G(x)\mathrm{e}^{-ps} + \lambda p\psi_X^G(x)\mathrm{e}^{-2s} + \lambda\psi_X^G(x)\mathrm{e}^{-(p+1)s}. \end{aligned}$$

As above, the first summand dominates for large values of s. The second summand, however, may be negligible when p is close to 1. In that case we again approximate:

$$g_x(s) \doteq \mathrm{e}^{-s} - \lambda p\psi_X^G(x)\mathrm{e}^{-ps} = g_S(s)\{1 + \lambda\psi_X^G(x)B'(G_S(s))\}$$

(compare with Example 3.3). The last example and the outline of the proof of Lemma 3.2 indicate that a more precise version of Lemma 3.2 may be needed, since the $(1-n^{-1})$th quantile calculated from the less accurate approximation of $1-G_x(s)$ may be inadequate for moderate values of n (say 10^3–10^5).

Actually we are not particularly interested in the behavior of $1-G_x(s)$ for s close to S_f, since we need to analyse the d.f. $1-G_S(s)$ (which should not be too difficult, because the marginal distributions in the FGM class are quite simple) as well as $1-G_{>x}(s)$. However, the calculations of the d.f. $1-F_T(t)$ and $1-F_{>y}(t)$ may be quite challenging. Even so the form of $g_x(s)$ given above proves to be quite useful. We shall start with the marginal d.f. of the profile shape factor $F_T(\cdot)$. By the definition

$$
\begin{aligned}
(3.4)\quad F_T(t) &= \int_t^{S_f}\int_0^{X_f} f(y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_t^{S_f}\int_0^{X_f} \frac{y\sqrt{1+z}}{2M}\int_y^{X_f}\int_z^{S_f}\frac{g(x,s)\,\mathrm{d}s\,\mathrm{d}x}{\sqrt{s}\sqrt{1+s}\sqrt{s-z}\sqrt{x^2-y^2}}\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_t^{S_f}\left[\frac{\sqrt{1+t}\sqrt{z-t}+(1+z)\arctan\sqrt{\frac{z-t}{1+t}}}{\sqrt{z}\sqrt{1+z}}\right]\int_0^{X_f}\frac{xg(x,z)}{2M}\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_t^{S_f}\left[\frac{\sqrt{1+t}\sqrt{z-t}+(1+z)\arctan\sqrt{\frac{z-t}{1+t}}}{\sqrt{z}\sqrt{1+z}}\right] \\
&\quad\times\int_0^{X_f}\frac{xg_x(z)g_X(x)}{2M}\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}
$$

Thus for the general FGM family the corresponding survival function $1-F_T(t)$ can be written as

$$
\begin{aligned}
(3.5)\qquad 1-F_T(t) &= \int_0^{X_F}\frac{xg_X(x)\,\mathrm{d}x}{2M}\int_t^{S_F} g_S(s)\xi(s,t)\,\mathrm{d}s \\
&\quad+\lambda\int_0^{X_F}\frac{xg_X(x)\psi_X^G(x)\,\mathrm{d}x}{2M}\int_t^{S_F} g_S(s)\psi_S^G(s)\xi(s,t)\,\mathrm{d}s,
\end{aligned}
$$

where

$$
(3.6)\qquad \xi(s,t)=\left[\sqrt{\frac{(1+t)(s-t)}{s(1+s)}}+\sqrt{\frac{1+s}{s}}\arctan\sqrt{\frac{s-t}{1+t}}\right].
$$

We are now able to analyze the behaviour of $1-F_T(t)$ for different choices of $g_S(\cdot)$ and $\psi_S(\cdot)$. The integrals with respect to x do not change the tail behaviour as $t\to S_F$ for any choice of $g_X(\cdot)$ and $\psi_X(\cdot)$.

Example 3.6. We use the same setting as in Example 3.4 but now choose the marginal density to be $g_S(s) = \mu e^{-\mu s}$, where $\mu > 0$ is the parameter. In this case it follows that for large values of t the survival function is approximately:

$$\begin{aligned} 1 - F_T(t) &= \int_t^{\infty} \mu e^{-\mu s} \xi(s,t)\,\mathrm{d}s \int_0^{X_F} \frac{x g_X(x)}{2M} (1 - \lambda(p+1)^{-2} \psi_X^G(x))\,\mathrm{d}x \\ &= K_X \int_t^{\infty} \mu e^{-\mu s} \xi(s,t)\,\mathrm{d}s, \end{aligned}$$

where $\xi(s,t)$ is given in (3.6) and

$$K_X = \int_0^{X_F} \frac{x g_X(x)}{2M} (1 - \lambda(p+1)^{-2} \psi_X^G(x))\,\mathrm{d}x.$$

One can easily verify that

$$\lim_{t\to\infty} \frac{\int_t^{\infty} \mu e^{-\mu s} \xi(s,t)\,\mathrm{d}s}{\sqrt{\pi}(\mu t)^{-1/2} e^{-\mu t}} = 1,$$

and evidently

$$1 - G_S(s) = e^{-\mu s}.$$

According to Lemma 3.2 the two normalizing constants for the spheroid shape factor can be chosen as:

$$a_n = \frac{1}{\mu} \quad \text{and} \quad b_n = \frac{\log n}{\mu}$$

while the corresponding normalizing constants for the profile shape factor are

$$a_n^p = \frac{1}{\mu}, \quad b_n^p = \frac{1}{\mu}\Big[\log n - \frac{1}{2}\log\log n + \log(\sqrt{\pi} K_X)\Big].$$

It follows that in this particular case it is sufficient to estimate the normalizing constant a_n^p for the profiles or equivalently a_n for the shape factor (since all the other terms are constant). There are several methods of estimating the n.c.; in particular, the maximum likelihood estimator is given, e.g., in [24].

We now proceed to the shape factor conditioned on the size. All those particles (profiles) whose size is not sufficiently large will be omitted. We are required to evaluate

$$\begin{aligned} (3.7)\quad 1 - G_{>x}(s) &= \frac{1}{1 - G_X(x)} \int_x^{X_F} \int_s^{S_F} g(u,v)\,\mathrm{d}v\,\mathrm{d}u \\ &= 1 - G_S(s) + \frac{\lambda \int_x^{X_F} g_X(u)\psi_X^G(u)\,\mathrm{d}u}{1 - G_X(x)} \int_s^{S_F} g_S(v)\psi_S^G(v)\,\mathrm{d}v. \end{aligned}$$

Here we have used again the general FGM family structure. Similarly,

$$(3.8)\quad 1-F_{>y}(t)=\frac{\int_y^{X_F}\sqrt{x^2-y^2}g_X(x)\,\mathrm{d}x}{1-F_Y(y)}\int_s^{S_F} g_S(s)\xi(s,t)\,\mathrm{d}s + \lambda\frac{\int_y^{X_F}\sqrt{x^2-y^2}g_X(x)\psi_X(x)\,\mathrm{d}x}{1-F_Y(y)}\int_s^{S_F} g_S(s)\psi_S(s)\xi(s,t)\,\mathrm{d}s,$$

where $\xi(s,t)$ is defined in (3.6). Although the conditional survival functions seem to be more complicated than the unconditional ones we have also here disjoint "x" and "s" parts, namely, we can use the same ideas and procedures as above.

E x a m p l e 3.7. Let us proceed with Example 3.6 but now we shall calculate the conditional distribution functions. From (3.7) and (3.8) we obtain for large s

$$\begin{aligned}1-G_{>x}(s)&\doteq \mathrm{e}^{-\mu s}+\frac{\lambda\int_x^{X_F} g_X(u)\psi_X^G(u)\,\mathrm{d}u}{1-G_X(x)}\int_s^{\infty}\mu\mathrm{e}^{-\mu v}(-(p+1)^{-2})\,\mathrm{d}v\\&\doteq \mathrm{e}^{-\mu s}\left[1-\frac{\lambda\int_x^{X_F} g_X(u)\psi_X^G(u)\,\mathrm{d}u}{(p+1)^2(1-G_X(x))}\right],\end{aligned}$$

and $1-F_{>y}(t)$ becomes:

$$\begin{aligned}&1-F_{>y}(t)\\&\quad\doteq\frac{\int_y^{X_F}\sqrt{x^2-y^2}g_X(x)(1-\lambda(p+1)^{-2}\psi_X^G(x))\,\mathrm{d}x}{1-F_Y(y)}\int_s^{S_F} g_S(v)\xi(v,t)\,\mathrm{d}v\\&\quad= K_X(y)\int_s^{\infty}\mu\mathrm{e}^{-\mu v}\left[\sqrt{\frac{(1+t)(v-t)}{v(1+v)}}+\sqrt{\frac{1+v}{v}}\arctan\sqrt{\frac{v-t}{1+t}}\right]\mathrm{d}v\\&\quad\doteq\frac{\int_y^{X_F}\sqrt{x^2-y^2}g_X(x)(1-\lambda(p+1)^{-2}\psi_X^G(x))\,\mathrm{d}x}{1-F_Y(y)}\sqrt{\pi}(\mu t)^{-1/2}\mathrm{e}^{-\mu t}.\end{aligned}$$

Compare with the corresponding expression in Example 3.6. Here $\xi(v,t)$ is given in (3.6) and compare

$$K_X(y)=\frac{\int_y^{X_F}\sqrt{x^2-y^2}g_X(x)(1-\lambda(p+1)^{-2}\psi_X^G(x))\,\mathrm{d}x}{1-F_Y(y)}$$

with K_X given in Example 3.6. The normalizing constants for the shape factor conditioned on the size exceeding a threshold are

$$a_n=\frac{1}{\mu},\quad b_n=\frac{1}{\mu}\left[\log n+\log\left(1-\frac{\lambda\int_x^{X_F} g_X(u)\psi_X^G(u)\,\mathrm{d}u}{(p+1)^2(1-G_X(x))}\right)\right]$$

for the spheroids, and

$$a_n^p = \frac{1}{\mu}, \quad b_n^p = \frac{1}{\mu}\Big[\log n - \frac{1}{2}\log\log n + \log(\sqrt{\pi}K_X(y))\Big],$$
$$K_X(y) = \frac{\int_y^{X_F} \sqrt{x^2-y^2}\, g_X(x)(1-\lambda(p+1)^{-2}\psi_X^G(x))\,\mathrm{d}x}{1-F_Y(y)}$$

for the profiles. Note that the transition from b_n^p to b_n is here more difficult than in the "unconditional" Example 3.6.

We have observed in Example 3.5 that in some situations the normalizing constants ought to be calculated more precisely than simply using the approximation presented in Lemma 3.2 which may not be satisfactory for moderate sample sizes n. We now present a lemma which will provide a more accurate approximations involving "correction" terms.

Lemma 3.8 (Normalizing constants II). *Suppose that a distribution function K has an upper endpoint M_f. Then:*

(i) *If $M_f = \infty$ the d.f. K belongs to the Gumbel domain of attraction and if there exist constants $\alpha > 0$, $\beta, \gamma > 0$, $\delta > 0$ and $\alpha_1 > 0$, $\beta_1, \gamma_1 > 0$ such that $\gamma_1 > \gamma$ and*
$$\lim_{v\to\infty} \frac{1-K(v)}{\alpha v^{\beta}\mathrm{e}^{-\gamma v^{\delta}} + \alpha_1 v^{\beta_1}\mathrm{e}^{-\gamma_1 v^{\delta}}} = 1,$$
the normalizing constants may be then chosen as
$$a_n = \Big(\frac{\log n}{\gamma}\Big)^{1/\delta-1}\frac{1}{\gamma\delta}(1+\Delta(n)),$$
where
$$\Delta(n) = \frac{\gamma-\gamma_1}{\gamma}\Big(1+\frac{\alpha}{\alpha_1}\gamma^{(\beta_1-\beta)/\delta}(n\alpha)^{-1+\gamma_1/\gamma}(\log n)^{(\gamma_1\beta-\gamma\beta_1)/(\gamma\delta)}\Big)^{-1}$$
and
$$b_n = \Big(\frac{\log n}{\gamma}\Big)^{1/\delta} + \frac{\beta}{\delta}\frac{(\log\log n - \log\gamma) + \log\alpha + \delta(n)}{\big((\log n)/\gamma\big)^{1-1/\delta}\gamma\delta}$$
with
$$\delta(n) = \log\Big(1+\frac{\alpha}{\alpha_1}[(\log n)/\gamma]^{(\beta_1-\beta)/\delta}(n\alpha)^{1-\gamma_1/\gamma}\Big).$$

(ii) *If the distribution function K belongs to the Fréchet domain of attraction and if there exist constants $\alpha > 0$, $\gamma > 0$ and $\alpha_1 > 0$, $\gamma_1 > 0$ such that $\gamma_1 > \gamma$ and*
$$\lim_{v\to\infty}\frac{1-K(v)}{\alpha v^{-\gamma} + \alpha_1 v^{-\gamma_1}} = 1,$$

the normalizing constants may then be chosen as

$$a_n = (n\alpha)^{1/\gamma}\Big[1 + \frac{\alpha}{\alpha_1}(n\alpha)^{1-\gamma_1/\gamma}\Big]^{1/\gamma}, \quad b_n = 0.$$

(iii) *If the distribution function K belongs to the Weibull domain of attraction and if there exist constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\alpha_1 > 0$, $\beta_1 > 0$, $\gamma_1 > 0$ such that $\gamma_1 > \gamma$ and*

$$\lim_{v\to M_f} \frac{1-K(v)}{\alpha(v/M_f)^{\beta}(M_f - v)^{\gamma} + \alpha_1(v/M_f)^{\beta_1}(M_f - v)^{\gamma_1}} = 1,$$

the normalizing constants may then be chosen as

$$a_n = (n\alpha)^{-1/\gamma}\Big[1 + \frac{\alpha}{\alpha_1}(n\alpha)^{1-\gamma_1/\gamma}\Big]^{-1/\gamma}, \quad b_n = M_f.$$

R e m a r k 3.9. Comparing the above expressions for a_n and b_n, we note that the "correction" terms to the values of a_n and b_n given in Lemma 3.2 (in the three cases of Gumbel, Fréchet and Weibull MDAs for a_n and in the case of the Gumbel MDA for b_n) all vanish as $n \to \infty$. Nevertheless, this convergence may be extremely slow and we should not omit these terms even for large sample sizes n.

Conclusion

We have shown that the results obtained in [12] and [13] can be extended by employing a more general bivariate distribution than the standard FGM (which still retains the separation property). This may result in a more accurate models for the distribution of the spheroid characteristics; in particular, the correlation between the size and shape factor of the spheroids is now not limited by 1/3. It may be necessary to use more precise forms of the normalizing constants for the generalized class of distributions.

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