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HOMOGENIZATION WITH UNCERTAIN INPUT PARAMETERS

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Abstract. We homogenize a class of nonlinear differential equations set in highly heterogeneous media. Contrary to the usual approach, the coefficients in the equation characterizing the material properties are supposed to be uncertain functions from a given set of admissible data. The problem with uncertainties is treated by means of the worst scenario method, when we look for a solution which is critical in some sense.

Keywords: homogenization, uncertain input data, worst scenario

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1. Introduction

Modeling the real world phenomena usually exhibits a sort of uncertain behavior—there is a difference between reality and the solution of a mathematical model, a difference between an exact and numerical solution, we work with experimentally obtained input data that are loaded by errors, etc.

In this paper we focus on solving problems with uncertain input data. By inputs we understand coefficients in the equations, the right hand side, boundary values, etc. Our considerations are embedded in the framework of homogenization theory. Homogenization is a mathematical method designed for modeling highly heterogeneous media such as composite materials, porous media, etc. It enables us to compute macroscopic (effective) properties from the knowledge of the microstructure. The method provides a quite easy and powerful tool, however its practical use is restricted to the case of periodic structures.

As a model problem we choose a nonlinear monotone type elliptic boundary value problem. We assume that the coefficients in the equation characterize a composite

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material with a fine periodic structure. For the sake of simplicity we restrict ourselves to the case of the two-phase composite, where the inclusions form a periodic structure in the matrix. The shape of the inclusion is supposed to be uncertain, but controlled by a finite number of parameters determining e.g. a circle/ball, rectangle/box, etc. Some bounds for these parameters are given.

We adopt the worst scenario method, see [7], for solving the problems with uncertain input data. Considering a model problem (differential or integral equation, variational inequality, etc.), the main idea of the method consists in defining a functional over a suitable set of data (the so-called admissible data set). This functional can be dependent on both the data and the solution of the model problem and it is a criterion which evaluates a physical quantity from a certain point of view. In particular, it says which data are "bad" or "good". The maximization of the functional yields the "worst case". It means that the strategy of the method is to stay always on the safe side, looking for "dangerous" data. Here we are motivated by the effort to find some "critical" shapes of the inclusion in the composite with respect to the functional representing the value of the homogenized solution or its gradient in some crucial places of the domain.

The paper is a continuation of [12], where linear problems were studied, and is organized as follows. After some preliminaries, the model problem is introduced in Section 2. Section 3 discusses the related homogenized problem. These two sections present the known results and therefore the proofs are omitted here. Section 4 is devoted to the worst scenario method including the main result on the existence of a solution of the worst scenario problem with respect to the given functional. Some concluding remarks close the paper in Section 5.

2. Model problem

Throughout the paper, d is the dimension of the problem, Ω is a domain in $\mathbb{R}^d$ with Lipschitz boundary, $(\cdot,\cdot)$ denotes the scalar product on two elements from $\mathbb{R}^d$ and $|\cdot| = \sqrt{(\cdot,\cdot)}$ is the usual Euclidean norm. If S is a subset of $\mathbb{R}^d$, then $|S|$ means the d-dimensional Lebesgue measure. The Lebesgue space $L^2(\Omega)$ and its vector-valued analogue $L^2(\Omega;\mathbb{R}^d)$ equipped with the norms $\|u\|_{L^2(\Omega)} = (\int_\Omega u^2\,\mathrm{d}x)^{1/2}$ and $\|u\|_{L^2(\Omega;\mathbb{R}^d)} = (\int_\Omega |u|^2\,\mathrm{d}x)^{1/2}$ are used. We employ the Sobolev space $W_0^{1,2}(\Omega)$ of functions with zero traces on $\partial\Omega$. The norm is $\|u\|_{W_0^{1,2}(\Omega)} = (\int_\Omega u^2 + |\nabla u|^2)^{1/2}$. Let $Y = [0,1)^d$ be the unit cube. A function u defined on $\mathbb{R}^d$ is said to be Y-periodic if $u(y+k) = u(y)$, $\forall y \in Y$, $\forall k \in \mathbb{Z}^d$. The Banach spaces of Y-periodic functions are denoted by $X_\#(Y)$. A function $v \in X_\#(Y)$ is Y-periodic and, moreover, $v \in X_{\mathrm{loc}}(\mathbb{R}^d)$, i.e. $v \in X(Q)$ for every compact subset $Q \subset \mathbb{R}^d$. Here we use the Sobolev space $W_\#^{1,2}(Y)$ (a function from this space has the same traces almost everywhere on the opposite sides of Y). Additionally, we assume that every $u \in W^{1,2}_{\#}(Y)$ has the zero mean value over Y, i.e. $\int_Y u \, \mathrm{d}y = 0$. The norm is introduced as $\|u\|_{W^{1,2}_{\#}(Y)} = \|u\|_{W^{1,2}(Y)} = (\int_Y u^2 + |\nabla u|^2)^{1/2}$.

We consider a class of monotone-type nonlinear elliptic boundary value problems, where the domain Ω is filled by a two-phase composite (composed of the inclusion and the matrix) with a periodic structure such that the periodicity cell contains one fibre of the inclusion only. The geometry (shape) of the fibre is assumed to be uncertain in the sense that it can vary with a vector of parameters p, where some bounds on p are given (e.g. the cylinder can vary with $p = (p_1, p_2)$, where p_1 denotes the radius of the base and p_2 denotes the height).

More precisely, let $m \geqslant 1$ be a positive integer and $\mathcal{U}^{\mathrm{ad}}$ a set of admissible parameters defined as

$$\mathcal{U}^{\mathrm{ad}} = [p_1^l, p_1^u] \times [p_2^l, p_2^u] \times \ldots \times [p_m^l, p_m^u],$$

where $0 < p_i^l < p_i^u$, $i = 1, \ldots, m$ are suitable real constants (constraints). Let $p \in \mathcal{U}^{\mathrm{ad}}$ represent the geometrical parameters of a fibre and let Y_p denote the occupied set. Moreover, we assume that Y_p is a domain in $\mathbb{R}^d$ (i.e. an open and simply connected set) satisfying $\overline{Y}_p \subset \overline{Y}$, $\forall p \in \mathcal{U}^{\mathrm{ad}}$.

Further, we introduce a function $a_p(y, \xi)\colon\ \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:

$$\begin{aligned}
&a_p(y,\xi) = a_1(\xi) \text{ on } Y_p, \quad a_p(y,\xi) = a_2(\xi) \text{ on } Y \setminus Y_p, \quad a_1(\xi) \neq a_2(\xi),\\
&a_p(y+k,\xi) = a_p(y,\xi), \quad \forall y \in Y, \quad \forall k \in \mathbb{Z}^d, \quad \forall \xi \in \mathbb{R}^d,\\
&(a_i(\xi) - a_i(\eta), \xi - \eta) \geqslant \alpha_i |\xi - \eta|^2, \quad \forall \xi,\ \eta \in \mathbb{R}^d,\ i = 1,2,\\
&|a_i(\xi) - a_i(\eta)| \leqslant L_i |\xi - \eta|, \quad \forall \xi,\ \eta \in \mathbb{R}^d,\ i = 1,2,
\end{aligned}$$

where α_i, L_i are positive constants. Overall, the function a_p is constant in the first variable y on both Y_p and $Y \setminus Y_p$, is Y-periodic in y and satisfies the strong monotonicity and Lipschitz continuity conditions in the second variable ξ, i.e. we have

$$(a_p(y,\xi) - a_p(y,\eta),\ \xi - \eta) \geqslant \alpha |\xi - \eta|^2, \quad \forall y \in Y, \forall \xi,\ \eta \in \mathbb{R}^d, \tag{2.1}$$

$$|a_p(y,\xi) - a_p(y,\eta)| \leqslant L|\xi - \eta|, \quad \forall y \in Y,\ \forall \xi,\ \eta \in \mathbb{R}^d, \tag{2.2}$$

where $\alpha = \min\limits_i \alpha_i$ and $L = \max\limits_i L_i$.

The basic idea of the homogenization approach consists in considering a sequence of problems of the same type with diminishing period (one term is considered to be the original problem). This sequence is controlled by a sequence of small positive parameters $\varepsilon_n \to 0_+$ as $n \to \infty$ (as usual, we omit the subscript n in what follows).

The detailed explanation can be found e.g. in [3], [5]. Here we deal with the following sequence representing a nonlinear conservation law:

$$\begin{aligned} A_p^\varepsilon(u^\varepsilon) \equiv -\operatorname{div}(a_p^\varepsilon(x, \nabla u^\varepsilon)) &= f \quad \text{in } \Omega, \\ u^\varepsilon \in W_0^{1,2}(\Omega), & \end{aligned} \tag{2.3}$$

where $a_p^\varepsilon(x,\xi) \equiv a_p(x/\varepsilon, \xi)$ and $a_p(y,\xi)$ satisfies the above conditions. This kind of problem models many physical phenomena, e.g. in heat conduction, electrostatics, magnetostatics, etc. Typically, the vector a can have the form $a := \tilde{a}(|\nabla u|^2)\nabla u$ (a constitutive law based on the material properties).

The solvability of the problem (2.3) follows from the theory of monotone operators, see e.g. [14]. The properties (2.1), (2.2) guarantee the strong monotonicity and Lipschitz continuity of the operator A_ε which, in particular, implies the existence and uniqueness of the solution. More precisely, the following assertion holds.

Theorem 2.1. *Let $p \in \mathcal{U}^{\mathrm{ad}}$. Then there exists a unique solution $u^\varepsilon(p) \in W_0^{1,2}(\Omega)$ of the problem* (2.3) *for every $f \in L^2(\Omega)$ (or more generally, $f \in (W_0^{1,2}(\Omega))^\star \approx W^{-1,2}(\Omega)$). Moreover, we have an apriori estimate $\|u^\varepsilon\|_{W_0^{1,2}(\Omega)} \leqslant K$, where the constant K does not depend on ε and p.*

Although the existence and uniqueness of solution can be obtained also under weaker monotonicity and continuity assumptions, the introduced properties are employed in the proofs in Section 4.

3. Homogenized problem

In this section we introduce the homogenized problem to (2.3) and recall the corresponding convergence theorem. In homogenization, several concepts have been introduced so far. Besides the asymptotic expansion method [3], G, H and Γ-convergence [6], [10], the two-scale convergence method (and its generalizations) [1], [9], [13] seem to be the most powerful tools in homogenization theory. Using one of these approaches we can introduce:

Definition 3.1. The homogenized problem related to the sequence (2.3) is defined as

$$\begin{aligned} A_p^0(u^0) \equiv -\operatorname{div}(b_p(\nabla u^0)) &= f \quad \text{in } \Omega, \\ u^0 \in W_0^{1,2}(\Omega), & \end{aligned} \tag{3.1}$$

where the coefficient $b_p\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b_p(\xi) = \int_Y a_p(y, \xi + \nabla w_\xi(y))\,\mathrm{d}y \tag{3.2}$$

and the function $w_\xi \in W^{1,2}_{\#}(Y)$ is a solution of the so-called local problem

$$-\operatorname{div}(a_p(y, \xi + \nabla w_\xi(y))) = 0 \quad \text{in } Y,\ \forall \xi \in \mathbb{R}^d. \tag{3.3}$$

Note that the homogenized problem is not completely separated into a global and a local part as in the linear case, i.e. it is actually a two-scale problem (ξ acts as a parameter in (3.3)).

Theorem 3.1. *Let $p \in \mathcal{U}^{\rm ad}$ and let $u^\varepsilon(p)$ be the solution of the problem* (2.3). *Then there exist unique solutions $w_\xi(p)$ and $u^0(p)$ of the problems* (3.1) *and* (3.3) *such that*

$$\begin{gathered} u^\varepsilon \rightharpoonup u^0 \quad \text{in } W_0^{1,2}(\Omega), \\ a_p^\varepsilon(x,\xi) \rightharpoonup b_p(\xi) \quad \text{in } L^2(\Omega; \mathbb{R}^d), \end{gathered}$$

as $\varepsilon \to 0+$. Moreover, the coefficient $b\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the estimates

$$\begin{aligned} (b_p(\xi) - b_p(\eta), \xi - \eta) &\geqslant \alpha|\xi-\eta|^2, \quad \forall \xi,\eta \in \mathbb{R}^d, \\ |b_p(\xi) - b_p(\eta)| &\leqslant \frac{C_L^2}{\alpha}|\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^d, \end{aligned} \tag{3.4}$$

where α, L are the same constants as in (2.1) *and* (2.2).

In literature, the additional assumption $a(y,0) = 0$ is usually employed, which simplifies proofs a bit. Otherwise, we can follow the proof presented e.g. in [5].

The key step of the proof techniques deals with the fact that the product of two weakly converging sequences need not converge to the product of their limits, so we cannot immediately pass to the limit in the weak formulation of the problem. While classical approaches overcome this problem by a special choice of test functions, the two-scale convergence approach enables us to pass to the two-scale limit in the weak formulation (information on oscillations of the original functions is preserved via the second variable y). In other words, the two-scale convergence method is self-contained, as the output of the convergence analysis we obtain also the homogenized problem, however, it is in the non-separated (two-scale) form. At this place, it is worth mentioning that there is an alternative two-scale method, the so-called periodic unfolding method. It was originally introduced in [2] and later developed in [4], see also [11].

Some other variants of monotonicity assumptions have also been studied. An overview and guide to literature on homogenization of monotone operators can be found e.g. in [8].

4. Worst scenario method

In this section we formulate the worst scenario problem related to problem (3.1). The criterion functional is usually chosen with respect to the aim of interest and upon the expert decision. It can be defined quite arbitrarily, however, certain continuity assumptions must be satisfied, for details see [7]. In our considerations the following property is satisfactory.

Let $\Phi\colon \mathcal{U}^{\mathrm{ad}} \times W_0^{1,2}(\Omega) \to \mathbb{R}$ be a functional satisfying: taking arbitrary sequences $\{p_n\} \subset \mathcal{U}^{\mathrm{ad}}$, $\{v_n\} \subset W_0^{1,2}(\Omega)$ such that $p_n \to p$ in $\mathbb{R}^m$ (the limit p is in $\mathcal{U}^{\mathrm{ad}}$ since $\mathcal{U}^{\mathrm{ad}}$ is a compact set in $\mathbb{R}^m$) and $v_n \to v$ in $W_0^{1,2}(\Omega)$ as $n \to \infty$, we have

$$\Phi(p_n, v_n) \to \Phi(p, v). \tag{4.1}$$

A typical example of such functional is $\Phi(u^0(p)) = |\widetilde{\Omega}|^{-1} \int_{\widetilde{\Omega}} u^0(p)\,\mathrm{d}x$, i.e. the average value of the homogenized solution u^0 (representing e.g. the temperature) over $\widetilde{\Omega}$, where $\widetilde{\Omega}$ is a subdomain of Ω—usually a critical place in the material (e.g. the placement of a measuring probe). This choice is motivated by the question: what data yield the highest values of the solution (temperature) in the exposed places?

Now, we are in a position when the admissible data set and the criterion functional are given, so we can formulate the worst scenario problem:

$$\text{Find } \hat{p} \in \mathcal{U}^{\mathrm{ad}} \text{ such that } \Phi(p, u^0(p)) \leqslant \Phi(\hat{p}, u^0(\hat{p})),\ \forall p \in \mathcal{U}^{\mathrm{ad}}. \tag{4.2}$$

Let us note that similarly we can formulate the minimization analogue.

Before introducing the existence theorem to problem (4.2) we prove the following continuity property:

Lemma 4.1. *Let $\{p_n\} \subset \mathcal{U}^{\mathrm{ad}}$ be a sequence such that $p_n \to p$ in $\mathbb{R}^m$ as $n \to \infty$. Then $u^0(p_n) \to u^0(p)$ in $W_0^{1,2}(\Omega)$, where $u^0(p_n)$ and $u^0(p)$ are the solutions of the problem* (3.1) *for the parameter p_n and p, respectively.*

P r o o f. For lucidity, let us denote by $a_n(y,\xi) \equiv a_{p_n}(y,\xi)$, $w_n \equiv w_\xi(p_n)$ and $w \equiv w_\xi(p)$ the solutions of the local problem (3.3) with the parameters p_n and p, respectively. First, let us prove the boundedness of the sequence $\{w_n\}$ in $W_{\#}^{1,2}(Y)$. From Lipschitz continuity (2.2) we have

$$|a_p(y,\xi)| - |a_p(y,0)| \leqslant |a_p(y,\xi) - a_p(y,0)| \leqslant L|\xi| \Rightarrow |a_p(y,\xi)| \leqslant c + L|\xi| \tag{4.3}$$

for all $p \in \mathcal{U}^{\mathrm{ad}}$, where $c = \max_{y \in Y} |a_p(y,0)|$ (note that this maximum is indeed independent of p). Using the strong monotonicity condition (2.1), the weak formulation of the problem (3.3), the Cauchy-Bunjakovski inequality, (4.3) and the inequality $|ab| \leqslant \frac{1}{2}\theta^{-1}a^2 + \frac{1}{2}\theta b^2$, we may write

$$\begin{aligned}
\alpha\|\xi + \nabla w_n\|^2_{L^2_\#(Y;\mathbb{R}^d)} &\leqslant \left| \int_Y (a_n(y, \xi + \nabla w_n) - a_n(y, 0), \xi + \nabla w_n)\,\mathrm{d}y \right| \\
&= \left| \int_Y (a_n(y, \xi + \nabla w_n), \xi)\,\mathrm{d}y - \int_Y (a_n(y, 0), \xi + \nabla w_n)\,\mathrm{d}y \right| \\
&\leqslant \int_Y |(a_n(y, \xi + \nabla w_n), \xi)|\,\mathrm{d}y + \int_Y |(a_n(y, 0), \xi + \nabla w_n)|\,\mathrm{d}y \\
&\leqslant |\xi| \int_Y |a_n(y, \xi + \nabla w_n)|\,\mathrm{d}y + c\int_Y |\xi + \nabla w_n|\,\mathrm{d}y \\
&\leqslant |\xi|c + (|\xi|L + c)\int_Y |\xi + \nabla w_n|\,\mathrm{d}y \\
&\leqslant |\xi|c + \frac{\theta}{2}(|\xi|L + c)^2 + \frac{1}{2\theta}\|\xi + \nabla w_n\|^2_{L^2_\#(Y;\mathbb{R}^d)},
\end{aligned}$$

i.e.

$$\Big(\alpha - \frac{1}{2\theta}\Big)\|\xi + \nabla w_n\|^2_{L^2_\#(Y)} \leqslant |\xi|c + \frac{\theta}{2}(|\xi|C_L + c)^2,$$

where θ is a suitable large constant. Hence, with help of inequality $|a| \leqslant |a+b| + |b|$ and the Poincaré-Wirtinger inequality (we recall that the function w_n has the zero mean value over Y), we see that $\|w_n\|_{W^{1,2}_\#(Y)} \leqslant c_\xi$. It means that there exists an element $\tilde{w} \in W^{1,2}_\#(Y)$ such that, up to a subsequence, $w_{n'} \rightharpoonup \tilde{w}$ in $W^{1,2}_\#(Y)$. Using again (2.1) we have

$$\begin{aligned}
&\alpha\|\nabla w_{n'} - \nabla w\|^2_{L^2_\#(Y;\mathbb{R}^d)} \\
&\leqslant \left| \int_Y (a_{n'}(y, \xi + \nabla w_{n'}) - a_{n'}(y, \xi + \nabla w), \nabla w_{n'} - \nabla w)\,\mathrm{d}y \right| \\
&= \left| \underbrace{\int_Y (a_{n'}(y, \xi + \nabla w_{n'}), \nabla w_{n'} - \nabla w)\,\mathrm{d}y}_{=0} - \int_Y (a_{n'}(y, \xi + \nabla w), \nabla w_{n'} - \nabla w)\,\mathrm{d}y \right| \\
&= \left| \int_Y (a_{n'}(y, \xi + \nabla w), \nabla w_{n'} - \nabla \tilde{w})\,\mathrm{d}y + \int_Y (a_{n'}(y, \xi + \nabla w), \nabla \tilde{w} - \nabla w)\,\mathrm{d}y \right|.
\end{aligned}$$

Further, $a_n(y, \xi + v) \to a(y, \xi + v)$ in $L^2_\#(Y;\mathbb{R}^d)$, $\forall v \in L^2(Y;\mathbb{R}^d)$, $\forall \xi \in \mathbb{R}^d$. This is a consequence of the fact that every $a_n(y, \xi)$ differs from $a(y, \xi)$ in the variable y only on the set of a measure converging to zero as $n \to \infty$ while in the second variable the growth of this difference is controlled by the Lipschitz continuity conditions on functions a_1 and a_2. It means that the first integral on the right-hand side converges to zero, since it contains the product of a strongly and a weakly converging sequence in $L^2_\#(Y;\mathbb{R}^d)$. The second integral converges to $\int_Y (a(y,\xi+\nabla w),\nabla\tilde{w}-\nabla w)\,\mathrm{d}y$ which equals to zero due to the definition of a solution w. Hence, we have proved that $\nabla w_{n'} \to \nabla w$ in $L^2_\#(Y;\mathbb{R}^d)$. On the other hand, we know that $\nabla w_{n'} \rightharpoonup \nabla\tilde{w}$ in $L^2_\#(Y;\mathbb{R}^d)$ so that by uniqueness of the limit we have $\nabla w = \nabla\tilde{w}$ a.e. in $L^2_\#(Y;\mathbb{R}^d)$. Since w is the unique weak solution of the problem (3.3), the entire sequence converges.

Finally, let us denote by $u_n := u^0(p_n)$ and $u := u^0(p)$ the weak solutions of the homogenized problem (3.1). In a way similar to the above, we show that the sequence of solutions u_n is bounded with help of property (3.4), i.e.

$$\begin{aligned}
\alpha\|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^d)} &\leqslant \left|\int_\Omega (b_n(\nabla u_n)-b_n(0),\nabla u_n)\,\mathrm{d}x\right| \\
&= \left|\int_\Omega f u_n\,\mathrm{d}x - \int_\Omega (b_n(0),\nabla u_n)\,\mathrm{d}x\right| \\
&\leqslant (M\|f\|_{L^2(\Omega)}+c)\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^d)},
\end{aligned}$$

where M is a constant from Friedrich's inequality and c is a bound of coefficients b_n at the point 0. This proves that the solutions u_n are bounded in $W^{1,2}_0(\Omega)$ and thus there exists an element $\tilde{u} \in W^{1,2}_0(\Omega)$ such that, up to a subsequence, $u_{n'} \rightharpoonup \tilde{u}$ in $W^{1,2}_0(\Omega)$. It remains to prove that $\nabla u = \nabla\tilde{u}$ a.e. in $L^2(\Omega;\mathbb{R}^d)$. The definition (3.2) and an arguing analogous to the case of the sequence $\{a_n(y,\xi)\}$ yields that $b_n(v) \to b(v)$ in $L^2(\Omega;\mathbb{R}^d)$ for every $v \in L^2(\Omega;\mathbb{R}^d)$. Then by (3.4)

$$\begin{aligned}
&\alpha\|\nabla u_{n'}-\nabla u\|^2_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad\leqslant \left|\int_\Omega (b_{n'}(\nabla u_{n'})-b_{n'}(\nabla u),\nabla u_{n'}-\nabla u)\,\mathrm{d}x\right| \\
&\quad= \left|\int_\Omega f(u_{n'}-u)\,\mathrm{d}x - \int_\Omega (b_{n'}(\nabla u),\nabla u_{n'}-\nabla u)\,\mathrm{d}y\right| \\
&\quad= \left|\int_\Omega f(u_{n'}-u)\,\mathrm{d}x - \int_\Omega (b_{n'}(\nabla u),\nabla u_{n'}-\nabla\tilde{u})\,\mathrm{d}x - \int_\Omega (b_{n'}(\nabla u),\nabla\tilde{u}-\nabla u)\,\mathrm{d}x\right|.
\end{aligned}$$

The first integral converges to $\int_\Omega f(\tilde{u}-u)\,\mathrm{d}x$, the second converges to zero, since it contains the product of a strongly and a weakly convergent sequence, and the third converges to $-\int_\Omega (b(\nabla u),\nabla\tilde{u}-\nabla u)\,\mathrm{d}x = -\int_\Omega f(\tilde{u}-u)\,\mathrm{d}x$, which yields $\nabla\tilde{u} = \nabla u$ a.e. in $L^2(\Omega;\mathbb{R}^d)$. Since the solution u is unique, the entire sequence converges and the proof is complete. □

Now we are ready to prove the existence theorem for the worst scenario problem.

Theorem 4.1. *There exists a solution of the problem* (4.2).

P r o o f. Let $p \in \mathcal{U}^{\mathrm{ad}}$ and let us denote $J(p) \equiv \Phi(p, u^0(p))$, i.e. $J(p)$ is a function defined on $\mathcal{U}^{\mathrm{ad}}$. Taking an arbitrary sequence $\{p_n\} \subset \mathcal{U}^{\mathrm{ad}}$ such that $p_n \to p$ as $n \to \infty$, Lemma 4.1 yields $u^0(p_n) \to u^0(p)$ in $W_0^{1,2}(\Omega)$ and due to the property (4.1) we also have $J(p_n) \to J(p)$. It means that $J(p)$ is continuous and since $\mathcal{U}^{\mathrm{ad}}$ is a compact subset in $\mathbb{R}^m$, there exists a maximizing element $\hat{p} \in \mathcal{U}^{\mathrm{ad}}$. □

It should be emphasized that we have the existence of solution only. The uniqueness can certainly be achieved if $J(p)$ is strictly concave on $\mathcal{U}^{\mathrm{ad}}$, but intuitively, it is not clear under which conditions this is true (we recall that $J(p)$ is constructed via the functional Φ depending on the solution $u^0(p)$ of the homogenized problem and we do not have enough information on the behavior of u^0 with respect to p).

5. Remarks

The worst scenario method extends the solvability of mathematical models in the sense of taking uncertain inputs into account. Such situation is natural, the input data are usually mined by experiments and consecutive solving of an inverse (identification) problem. A certain amount of errors in both steps can be expected. The method expresses the requirement to stay on the safe side and therefore is sometimes too pessimistic. On the other hand, compared with stochastic methods, it does not require the probabilistic information about the data distribution. For comprehensive discussions on the worst scenario method and its comparison with stochastic methods, we refer to the monograph [7].

The aim of interest was to find some critical shapes of the inclusion in a composite material. This knowledge can help the designer to adjust the overall material properties properly. Of course, different configurations can be expected depending upon the choice of the criterion functional. Some numerical experiments are the subject of further research. Whether the method can be applicable also in the case of non-periodic structures remains an open question.

We have not discussed the finite dimensional approximations of the introduced problems yet. The standard procedure consists in formulation of the corresponding problems on the finite-dimensional subspaces $V_\delta \subset W_{\#}^{1,2}(Y)$ and $V_h \subset W_0^{1,2}(\Omega)$, where δ and h are the discretization parameters, e.g. those from the finite element method. Then also the solution of the worst scenario method depends on δ and h. We note that here the situation is simplified since we avoid the approximation of the set $\mathcal{U}^{\mathrm{ad}}$—it is already a subset of the finite-dimensional space $\mathbb{R}^m$. The last step consists in the convergence analysis of the approximate solutions $\hat{p}_{\delta,h} \to \hat{p}$ in $\mathbb{R}^m$, and $u^0_{\delta,h}(\hat{p}_{\delta,h}) \to u^0(\hat{p})$ in $W_0^{1,2}(\Omega)$ as $\delta, h \to 0+$.

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