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LOCALLY SPECTRALLY BOUNDED LINEAR MAPS

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Abstract. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We characterize locally spectrally bounded linear maps from $\mathcal{L}(\mathcal{H})$ onto itself. As a consequence, we describe linear maps from $\mathcal{L}(\mathcal{H})$ onto itself that compress the local spectrum.

Keywords: local spectrum, local spectral radius, linear preservers

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1. Introduction

Throughout this paper, X will denote a complex Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X with identity operator I. The local resolvent set of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$, $\varrho_T(x)$, is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U_λ of λ in $\mathbb{C}$ and an analytic function $f\colon U_\lambda \to X$ such that the equation $(\mu - T)f(\mu) = x$ holds for all $\mu \in U_\lambda$. The local spectrum of T at x, denoted by $\sigma_T(x)$, is given by

$$\sigma_T(x) := \mathbb{C} \setminus \varrho_T(x),$$

and is a compact subset of $\sigma(T)$. The local spectral radius of T at x is defined by

$$r_T(x) := \limsup_{n\to+\infty} \|T^n x\|^{1/n},$$

and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that T is said to have the single-valued extension property if for every open set U of $\mathbb{C}$, the equation

$$(T-\lambda)\varphi(\lambda) = 0 \quad (\lambda \in U),$$

has no nontrivial analytic solution on U. Evidently, every operator $T \in \mathcal{L}(X)$ with empty interior point spectrum enjoys this property. Our references on local spectral theory are the remarkable books of P. Aiena [1] and of K. Laursen and M. Neumann [11].

We will say that a linear map $\varphi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ is locally spectrally bounded at a fixed nonzero vector $e \in X$ if there is a positive constant M such that $r_{\varphi(T)}(e) \leqslant M r_T(e)$ for all $T \in \mathcal{L}(X)$. When X is an infinite dimensional Hilbert space, we prove the following.

Theorem 1.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let e be a fixed nonzero vector in $\mathcal{H}$. A continuous surjective linear map $\varphi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is locally spectrally bounded at e if and only if there are a nonzero scalar c and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $Ae = e$, and $\varphi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.*

This theorem is an extension of the result due to Bračič and Müller [7, Theorems 3.3 and 3.4], where they characterized continuous surjective linear maps from $\mathcal{L}(X)$ into itself that preserve the local spectrum (local spectral radius) at a fixed vector in X.

The following results show, unlike in the infinite dimensional case, that the additional assumption of continuity on φ can be omitted, and extend the main results from [4], [6], [9] to this more general scope.

Theorem 1.2. *Let $n \geqslant 3$ be a positive integer and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. Let $\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a surjective linear map. If φ is locally spectrally bounded at e, then there exist a nonzero scalar c and matrices $A, S \in M_n(\mathbb{C})$ with A invertible and $Ae = Se = e$ such that either $\varphi(T) = cATA^{-1} + c(S-I)\operatorname{tr}(T/n)$ or $\varphi(T) = cAT^{\mathrm{tr}}A^{-1} + c(S-I)\operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$. Here $\operatorname{tr}(\cdot)$ denotes the usual trace function on $M_n(\mathbb{C})$ and T^{tr} is the transpose of the matrix T.*

Corollary 1.3. *Let $n \geqslant 3$ be a positive integer and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. Let $\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a surjective linear map such that $\varphi(I)$ and I are linearly dependent. Then the map φ is locally spectrally bounded at e if, and only if, there are a nonzero scalar c and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ae = e$ and $\varphi(T) = cATA^{-1}$ for all $T \in M_n(\mathbb{C})$.*

2. Proof of the main results

We first fix some notation. The duality between the Banach spaces X and its dual X^* will be denoted by $\langle \cdot, \cdot \rangle$. For $x \in X$ and $f \in X^*$, as usual we denote by $x \otimes f$ the rank at most one operator on X given by $z \mapsto \langle z, f \rangle x$. For $T \in \mathcal{L}(X)$ we will denote by $\ker(T)$, T^*, $\sigma(T)$, $\sigma_{\mathrm{su}}(T)$, and $r(T)$, the null space, the adjoint, the spectrum, the surjectivity spectrum, and the spectral radius of T; respectively.

The proof of our results uses several auxiliary lemmas. The first is quoted in [3, lemma 2.1]. It concerns spectrally bounded linear maps from a purely infinite C^*-algebra with real rank zero onto a semi-simple Banach algebra. For our purposes, the only relevant example of an algebra having these properties is the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite-dimensional Hilbert space $\mathcal{H}$.

Recall that a linear map φ between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called spectrally bounded if there is a positive constant M such that

$$r(\varphi(a)) \leqslant M r(a) \quad (a \in \mathcal{A}),$$

where $r(\cdot)$ denotes the spectral radius function.

Lemma 2.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\varphi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be a surjective spectrally bounded linear map. Then there exist a nonzero scalar λ and an epimorphism or an anti-epimorphism $J\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ such that $\varphi(T) = \lambda J(T)$ for all $T \in \mathcal{L}(\mathcal{H})$.*

A few comments must be added to this statement. In [3, lemma 2.1], λ can be any central unitary element; however, since the centre of $\mathcal{L}(\mathcal{H})$ is trivial, λ must be a complex number in our setting. Further, the conclusion of [3, lemma 2.1] is that J is a Jordan epimorphism; since the algebra $\mathcal{C}(\mathcal{H})$ is prime, a well known theorem of Herstein [10] tells us that J must be an epimorphism or an anti-epimorphism in our setting.

The following lemma characterizes spectrally bounded linear maps from $M_n(\mathbb{C})$ onto itself.

Lemma 2.2. *A surjective linear map φ from $M_n(\mathbb{C})$ into itself is spectrally bounded if, and only if, there exist a nonzero scalar $c \in \mathbb{C}$ and an automorphism or an anti-automorphism φ on $M_n(\mathbb{C})$ such that $\varphi(T) = c\varphi(T) + (\varphi(I) - cI)\operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$.*

P r o o f. This is a consequence of [13, Remark 4]. $\square$

Lemma 2.3. *Let e be a fixed nonzero vector in X, and let $T \in \mathcal{L}(X)$. If $\lambda \in \sigma_{\mathrm{su}}(T)$, then for every $\varepsilon > 0$, there exists $T' \in \mathcal{L}(X)$ such that $\|T - T'\| < \varepsilon$ and $\lambda \in \sigma_{T'}(e)$.*

Proof. See [7, Lemma 2.2]. □

The following result is inspired by [7].

Lemma 2.4. *Let e be a fixed nonzero vector in X. If $\varphi\colon \mathcal{L}(X) \to \mathcal{L}(X)$ is a continuous surjective locally spectrally bounded linear map at e, then φ is spectrally bounded.*

Proof. Suppose that φ is locally spectrally bounded at e. Without loss of generality, we can assume that $r_{\varphi(T)}(e) \leqslant r_T(e)$ for every $T \in \mathcal{L}(X)$, and let us show that $r(\varphi(T)) \leqslant r(T)$ for all $T \in \mathcal{L}(X)$. To this end, let $T \in \mathcal{L}(X)$ and let $\lambda \in \sigma(\varphi(T))$ satisfy $|\lambda| = r(\varphi(T))$, which means $\lambda \in \sigma_{\mathrm{su}}(\varphi(T))$. By Lemma 2.3, for each integer $n \geqslant 1$ there exists an operator T'_n in $\mathcal{L}(X)$ such that $\|T'_n - \varphi(T)\| < n^{-1}$ and $\lambda \in \sigma_{T'_n}(e)$. Since φ is continuous and surjective, by the Banach open mapping theorem there exists $\eta > 0$ such that $\eta B(0,1) \subseteq \varphi(B(0,1))$, where $B(0,1)$ denotes the open unit ball of $\mathcal{L}(X)$. Therefore, for each n there exists $T_n \in \mathcal{L}(X)$ such that $\varphi(T_n) = T'_n$ and $\|T_n - T\| \leqslant \eta^{-1}\|T'_n - \varphi(T))\| \leqslant \eta^{-1}n^{-1}$. Thus $T_n \to T$ and $\lambda \in \sigma_{\varphi(T_n)}(e)$ for all $n \geqslant 1$. So, by the upper semi-continuity of the spectral radius function, we have

$$\begin{aligned} r(T) &\geqslant \limsup_{n\to\infty} r(T_n) \\ &\geqslant \limsup_{n\to\infty} r_{T_n}(e) \\ &\geqslant \limsup_{n\to\infty} r_{\varphi(T_n)}(e) \\ &\geqslant |\lambda| = r(\varphi(T)). \end{aligned}$$

Hence φ is spectrally bounded from $\mathcal{L}(X)$ onto itself. □

The next lemma is simple, and its proof is straightforward. We include it for the sake of completeness.

Lemma 2.5. *Assume that X is a complex Banach space of dimension at least two, and let $e \in X$ be a nonzero vector of X. If $A \in \mathcal{L}(X^*, X)$ is a bijective operator, then the anti-automorphism $\varphi\colon\ T \mapsto AT^*A^{-1}$ is not locally spectrally bounded at e.*

Proof. Assume, on the contrary, that φ is locally spectrally bounded at e, and let M be a positive constant such that $r_{AT^*A^{-1}}(e) \leqslant Mr_T(e)$ for all $T \in \mathcal{L}(X)$. Note that for every $T \in \mathcal{L}(X)$,

$$\begin{aligned} r_{T^*}(A^{-1}e) &= \limsup_{n\to\infty} \|T^{*n}A^{-1}e\|^{1/n} \\ &= \limsup_{n\to\infty} \|A^{-1}(AT^*A^{-1})^n e\|^{1/n} \\ &\leqslant \limsup_{n\to\infty} \|(AT^*A^{-1})^n e\|^{1/n} = r_{AT^*A^{-1}}(e). \end{aligned}$$

Similarly, we have $r_{AT^*A^{-1}}(e) \leqslant r_{T^*}(A^{-1}e)$, and so

$$r_{T^*}(A^{-1}e) = r_{AT^*A^{-1}}(e) \leqslant M r_T(e) \tag{2.1}$$

for all $T \in \mathcal{L}(X)$.

Now, let $x \in X$ be such that x and e are linearly independent and $\langle x, A^{-1}e\rangle = 1$, and let $f \in X^*$ be such that $\langle e, f\rangle = 0$ and $\langle x, f\rangle = 1$. The operator $T := x \otimes f$ satisfies $Te = 0$ and $T^{*n}A^{-1}e = f$ for all $n \geqslant 1$. Hence, $r_T(e) = 0$ and $r_{T^*}(A^{-1}e) = 1$. This contradicts the inequality (2.1) and completes the proof. □

R e m a r k 2.6. Just as in the proof of the above lemma one can see that when $X = \mathcal{H}$ is a Hilbert complex space and $A \in \mathcal{H}$ is a bijective operator, the anti-automorphism $\varphi\colon T \to AT^{\rm tr}A^{-1}$ is not locally spectrally bounded at a nonzero fixed vector $e \in \mathcal{H}$. Here $T^{\rm tr}$ denotes the transpose of the operator T relative to a fixed but arbitrary orthonormal basis.

Let us recall the following useful facts that will be often used in the sequel. It is well-known that $T \in \mathcal{L}(X)$ has the single-valued extension property if, and only if, for every $\lambda \in \mathbb{C}$ and every nonzero vector x in $\ker(\lambda - T)$ we have $\sigma_T(x) = \{\lambda\}$; see [1]. Furthermore, if $X = \mathbb{C}^n$ is a finite dimensional space, then for every $x \in \mathbb{C}^n$

$$\sigma_T(x) = \bigcup_{1\leqslant k\leqslant p} \{\lambda_k\colon\ 1 \leqslant k \leqslant p \text{ with } P_k(x) \neq 0\}. \tag{2.2}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the distinct eigenvalues of T, E_k the corresponding root spaces, and $P_k\colon\ \mathbb{C}^n \to E_k$ $(1 \leqslant k \leqslant p)$ the canonical projections; see [5].

We now have collected all the necessary ingredients and are therefore in a position to prove the main results of this paper.

P r o o f of Theorem 1.2. Suppose that φ is locally spectrally bounded at e, and let M be a positive constant such that $r_{\varphi(T)}(e) \leqslant M r_T(e)$ for all $T \in \mathcal{L}(X)$. By Lemma 2.4, the map φ is spectrally bounded; and so, by Lemma 2.2, there exist a nonzero scalar $c \in \mathbb{C}$ and an invertible matrix $A \in M_n(\mathbb{C})$ such that either $\varphi(T) = cATA^{-1} + (\varphi(I) - cI)\,\mathrm{tr}(T/n)$ or $\varphi(T) = cAT^{\rm tr}A^{-1} + (\varphi(I) - cI)\,\mathrm{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$.

Assume that $\varphi(T) = cATA^{-1} + (\varphi(I) - cI)\operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$, and note that e and Ae are linearly dependent. Indeed, suppose by the way of contradiction that e and $A^{-1}e$ are linearly independent. As $n \geqslant 3$, we can find a matrix $T_1 \in M_n(\mathbb{C})$ such that $T_1 e = 0$, $T_1 A^{-1} e = A^{-1} e$, and $\operatorname{tr}(T_1) = 0$. We have $\sigma_{T_1}(e) = \{0\}$ and $\sigma_{\varphi(T_1)}(e) = \{c\}$; and so $r_{T_1}(e) = 0$ and $r_{\varphi(T_1)}(e) = |c|$. This entails that $c = 0$ and contradicts the surjectivity of φ. Now, let us show by the way of contradiction that e and $\varphi(I)e$ are linearly dependent. So, assume on the contrary that e and $\varphi(I)e$ are linearly independent, and note that in this case e and $A^{-1}f$ are linearly independent too, where $f := \varphi(I)e - ce$. It is easy to see that we can find a matrix $T_2 \in M_n(\mathbb{C})$ such that $T_2 e = 0$, $T_2 A^{-1} f = c^{-1} A^{-1}((1+c)f - \varphi(I)f)$, and $\operatorname{tr}(T_2/n) = 1$. Thus, we have $\sigma_{T_2}(e) = \{0\}$, $r_{T_2}(e) = 0$, $\varphi(T_2)e = f$ and $\varphi(T_2)f = f$. From this together with the equality (2.2), we infer that $\sigma_{\varphi(T_2)}(e) = \{1\}$ and $r_{\varphi(T_2)}(e) = 1$, which contradicts the fact that φ is locally spectrally bounded at e. So, write $\varphi(I)e = ke$ for some nonzero constant $k \in \mathbb{C}$, and let us show that $k = c$. For every $\nu \in \mathbb{C}$ we can find a matrix T_ν such that $T_\nu e = e$ and $\operatorname{tr}(T_\nu/n) = \nu$. As e and $A^{-1}e$ are linearly dependent, it is easy to see that $\varphi(T_\nu)e = (c + (k-c)\nu)e$, and so $r_{\varphi(T_\nu)}(e) = |(c + (k-c)\nu)|$. Therefore, as $r_{T_\nu}(e) = 1$, we have $|(c + (k-c)\nu)| \leqslant M$ for all $\nu \in \mathbb{C}$, which implies that $k = c$ and $\varphi(I)e = ce$. Hence $\varphi(T) = cATA^{-1} + c(S - I)\operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$, with $Ae = \alpha e$ for some nonzero $\alpha \in \mathbb{C}$ and $Se = e$, where $S := c^{-1}\varphi(I)$. Dividing A by α if necessary, we may assume that $Ae = e$.

The case when φ takes the second form is dealt with similarly; and the proof is complete. □

Proof of Corollary 1.3. Checking the 'if' part is straightforward, so we will only deal with the 'only if' part. So assume that φ is locally spectrally bounded at e, and write $\varphi(I) = kI$ for some scalar k. By the proof of the above theorem there exist a nonzero scalar c and an invertible matrix A with $Ae = e$ and $\varphi(I)e = ce$ such that either $\varphi(T) = cATA^{-1} + (\varphi(I) - cI)\operatorname{tr}(T/n)$ or $\varphi(T) = cAT^{\mathrm{tr}}A^{-1} + (\varphi(I) - cI)\operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$. In particular, $k = c$ and so either $\varphi(T) = cATA^{-1}$ or $\varphi(T) = cAT^{\mathrm{tr}}A^{-1}$ for all $T \in M_n(\mathbb{C})$. Lemma 2.5 yields that φ takes only the first form; and the proof is therefore complete. □

Proof of Theorem 1.1. Note that, since the sufficiency condition is obvious, we only need to prove the necessity. So, assume that φ is locally spectrally bounded at e. By Lemma 2.4 the map φ is spectrally bounded, and therefore by Lemma 2.1 there exist a nonzero complex number c and an epimorphism or an anti-epimorphism J on $\mathcal{L}(\mathcal{H})$ such that $\varphi(T) = cJ(T)$ for all $T \in \mathcal{L}(\mathcal{H})$.

Next, let us show by way of contradiction that φ is injective. So, assume that φ is not injective and note that, in this case, $\ker(\varphi)$ is an ideal of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators on $\mathcal{H}$. So, pick an arbitrary $\varepsilon > 0$ and let

f_ε be a linear functional on $\mathcal{H}$ such that $f_\varepsilon(e) = \varepsilon - 1$. We have $(e \otimes f_\varepsilon + I)e = \varepsilon e$, $\sigma_{e\otimes f_\varepsilon+I}(e) = \{\varepsilon\}$, and $\sigma_{\varphi(e\otimes f_\varepsilon+I)}(e) = \sigma_{\varphi(I)}(e) = \{c\}$ since $e \otimes f_\varepsilon \in \mathcal{K}(\mathcal{H})$. In particular, $r_{e\otimes f_\varepsilon+I}(e) = \varepsilon$ and $r_{\varphi(e\otimes f_\varepsilon+I)}(e) = |c|$, and so there is a positive constant M such that $|c| \leqslant M\varepsilon$ for all $\varepsilon > 0$ since φ is locally spectrally bounded at e. This implies that $c = 0$, which is a contradiction. Thus J is an automorphism or an anti-automorphism. Now by the fundamental isomorphism theorem [12, Theorem 2.5.19] (see also [8]) there exists an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $J(T) = ATA^{-1}$ or $J(T) = AT^{\mathrm{tr}}A^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$. Lemma 2.5 ensures that the second form is excluded, and consequently φ takes only the first. Moreover, in this case, $Ae = \lambda e$ for some nonzero scalar λ; otherwise, we can find $T \in \mathcal{L}(X)$ such that $Te = 0$ and $TA^{-1}e = A^{-1}e$. This shows that $r_T(e) = 0$ and $r_{\varphi(T)}(e) = 1$, and gives a contradiction. Dividing A by λ if necessary, we may assume that $Ae = e$; and the proof is complete. □

3. Linear local spectrum compressors

This section is devoted to deriving some consequences of the main results of this paper. These consequences describe linear maps from $\mathcal{L}(\mathcal{H})$ onto itself compressing the local spectrum. A linear map φ from $\mathcal{L}(\mathcal{H})$ into itself is said to compress the local spectrum at a fixed nonzero vector $e \in \mathcal{H}$ if

$$\sigma_{\varphi(T)}(e) \subseteq \sigma_T(e)$$

for all $T \in \mathcal{L}(\mathcal{H})$.

The first consequence extends [7, Theorem 3.3] by replacing "preserves the local spectrum" by the weaker hypothesis "compresses the local spectrum" in the Hilbert space setting.

Theorem 3.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $e \in \mathcal{H}$ be a fixed nonzero vector. A continuous surjective linear map $\varphi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ compresses the local spectrum at e if and only if there is an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $Ae = e$, and $\varphi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.*

Proof. Assume that φ compresses the local spectrum at e. By using the fact that $r_T(x) \geqslant \max\{|\lambda|\colon \lambda \in \sigma_T(x)\}$ for all $T \in \mathcal{L}(\mathcal{H})$ (see for instance [1] or [11]) together with the same argument as in the proof of Lemma 2.4, one can see that φ is spectrally bounded. Now, in the same way as in the end of the proof of Theorem 1.1, one can see that φ is injective and there exist a nonzero scalar c and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that either $\varphi(T) = cATA^{-1}$ or $\varphi(T) = cAT^{\mathrm{tr}}A^{-1}$ for all

$T \in \mathcal{L}(\mathcal{H})$. By [9, Lemma 5], the second form is not possible, and $\varphi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$, from which one can see that A can be supposed to satisfy $Ae = e$. The desired conclusion follows from the fact that $\{c\} = \sigma_{\varphi(I)}(e) \subseteq \sigma_I(e) = 1$, which means that $c = 1$.

As the sufficiency condition is obvious, the proof is complete. $\square$

The second consequence describes linear maps on $M_n(\mathbb{C})$ which compress the local spectrum at a fixed nonzero vector, and extends the main result of [9].

Theorem 3.2. *Let n be a positive integer, and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. A linear map φ from $M_n(\mathbb{C})$ into itself compresses the local spectrum at e if and only if there is an invertible matrix $A \in M_n(\mathbb{C})$ such that*

$$Ae = e \text{ and } \varphi(T) = ATA^{-1} \quad \text{for all } T \in M_n(\mathbb{C}). \tag{3.3}$$

Proof. Evidently, the formula (3.3) defines a compressing local spectrum bijective linear map at e.

Conversely, if φ compresses the local spectrum, then in particular, it preserves at least one eigenvalue of each matrix, i.e., $\sigma(\varphi(T)) \cap \sigma(T) \neq \emptyset$ for all $T \in M_n(\mathbb{C})$. Therefore, by [2, Theorem 2], there is an invertible matrix $A \in M_n(\mathbb{C})$ such that $\varphi(T) = ATA^{-1}$ or $\varphi(T) = AT^{\mathrm{tr}}A^{-1}$ for all $T \in M_n(\mathbb{C})$. Moreover, by [9, Lemma 5] the second form is excluded, and consequently φ takes only the first, from which we may assume that $Ae = e$. This completes the proof. $\square$

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