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ASSOCIATIVE n–DIMENSIONAL COPULAS

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The associativity of n-dimensional copulas in the sense of Post is studied. These copulas are shown to be just n-ary extensions of associative 2-dimensional copulas with special constraints, thus they solve an open problem of R. Mesiar posed during the International Conference FSTA 2010 in Liptovský Ján, Slovakia.

Keywords: Archimedean copula, associativity in the sense of Post, n-dimensional copula

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1. INTRODUCTION

Copulas were introduced by Sklar [13] to capture the stochastic dependence structure of random variables. Recall that for $n \geq 2$, a function $C\colon [0,1]^n \to [0,1]$ is called an n-dimensional copula (n-copula, for short) whenever it is a restriction of an n-dimensional distribution function with all univariate margins uniformly distributed on $[0,1]$. Hence an n-copula is characterized by the properties:

(C1) $C(x_1, \ldots, x_n) = x_i$ whenever $\forall j \neq i,\ x_j = 1$;

(C2) $C(x_1, \ldots, x_n) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$;

(C3) the n-increasing property, i. e., $\forall\, \mathbf{x}, \mathbf{y} \in [0,1]^n$, $x_i \leq y_i$, $i = 1, \ldots, n$, it holds

$$\sum_{J \subset \{1,\ldots,n\}} (-1)^{|J|} C\left(u_1^J, \ldots, u_n^J\right) \geq 0, \text{ where } u_i^J = \begin{cases} x_i, & \text{if } i \in J, \\ y_i, & \text{if } i \notin J. \end{cases} \tag{1}$$

By the Sklar theorem [13], for any n-dimensional random vector $Z = (X_1, \ldots, X_n)$ there is an n-copula $C\colon [0,1]^n \to [0,1]$ such that for each $(z_1, \ldots, z_n) \in \mathbb{R}^n$

$$F_Z(z_1, \ldots, z_n) = C\left(F_{X_1}(z_1), \ldots, F_{X_n}(z_n)\right),$$

where $F_Z, F_{X_1}, \ldots, F_{X_n}$ are distribution functions of the corresponding random vectors.

There are two distinguished functions which are n-copulas for each $n \geq 2$: the so-called *minimum* n-copula M and the *product* n-copula Π, given by

$$\begin{aligned} M(x_1, \ldots, x_n) &= \min\{x_1, \ldots, x_n\}, \\ \Pi(x_1, \ldots, x_n) &= \prod_{i=1}^{n} x_i. \end{aligned}$$

The minimum n-copula M describes the comonotone dependence of random variables $X_1, \ldots, X_n$ and the product n-copula Π describes their independence. For more details we recommend monographs [4, 11].
For each n-copula C it holds

$$W \leq C \leq M,$$

where W is the so-called Fréchet-Hoeffding lower bound, given by

$$W(x_1, \ldots, x_n) = \max\left\{0, \sum_{i=1}^{n} x_i - (n-1)\right\}.$$

It is a well-known fact that this function is a copula only for $n = 2$, and in that case describes the countermonotone dependence of random variables X_1 and X_2.

All the three basic 2-copulas (copulas, for short) M, Π and W are associative, i.e., for all x_1, x_2, $x_3 \in [0,1]$ they satisfy the property

$$C\left(C(x_1, x_2), x_3\right) = C\left(x_1, C(x_2, x_3)\right). \tag{2}$$

Associativity as an algebraic property was originally introduced for binary functions only, see formula (2). Recently, based on ideas of Post [12], Couceiro [1] has studied the associativity of n-ary functions. Subsequently, during the open problem session at FSTA 2010, R. Mesiar has posed the problem of representation of associative n-copulas, see [8]. Recall that for $n = 2$ this problem was solved in seventies by Ling [6] and Moynihan [10].

The aim of this paper is to solve the above mentioned open problem for any fixed $n > 2$. The paper is organized as follows. In the next section, the representation of associative copulas is recalled. In Section 3 we study n-ary associative functions on $[0,1]$ possessing a neutral element and we show their relationship with binary associative functions. In Section 4, we introduce a representation theorem for associative n-copulas, together with some examples. Finally, some concluding remarks are added.

2. ASSOCIATIVE 2-DIMENSIONAL COPULAS

As mentioned above, 2-dimensional copulas will be referred to as copulas only. Let $C\colon [0,1]^2 \to [0,1]$ be an associative copula satisfying $C(x,x) < x$ for all $x \in]0,1[$. Then C is called an Archimedean copula. Moynihan [10] has proved the next representation theorem for Archimedean copulas.

Theorem 2.1. A function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous strictly decreasing convex function $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$, such that

$$C(x_1, x_2) = f^{(-1)}\left(f(x_1) + f(x_2)\right), \tag{3}$$

where $f^{(-1)}$ is the pseudo-inverse of f.

Recall that the pseudo-inverse $f^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}(u) = f^{-1}\left(\min(f(0), u)\right).$$

The function f in the above theorem is called a generator of an Archimedean copula C. It is unique up to a positive multiplicative constant.

Copulas W and Π are Archimedean, with generators f_{W} and f_{Π}, respectively, given by $f_{\mathrm{W}}(x) = 1 - x$ and $f_{\Pi}(x) = -\log x$. If we define the function $f_{(1)}\colon [0,1] \to [0,\infty]$ by $f_{(1)}(x) = \frac{1}{x} - 1$, it is also a generator and the corresponding Archimedean copula $C_{(1)}\colon [0,1]^2 \to [0,1]$ is given by

$$C_{(1)}(x_1, x_2) = \frac{x_1\, x_2}{x_1 + x_2 - x_1 x_2}$$

whenever $(x_1, x_2) \neq (0,0)$.

For a general associative copula C we have the next representation theorem [4, 11].

Theorem 2.2. A function $C\colon [0,1]^2 \to [0,1]$ is an associative copula if and only if there is a system $(\,]a_k, b_k[\,)_{k\in\mathcal{K}}$ of pairwise disjoint open subintervals of $[0,1]$ and a system $(C_k)_{k\in\mathcal{K}}$ of Archimedean copulas such that

$$C(x_1, x_2) = \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{x_1 - a_k}{b_k - a_k}, \frac{x_2 - a_k}{b_k - a_k}\right), & \text{if } (x_1, x_2) \in\,]a_k, b_k[^2 \\ & \text{for some } k \in \mathcal{K}, \\ M(x_1, x_2), & \text{else.} \end{cases} \tag{4}$$

Observe that if $\mathcal{K} = \emptyset$ then C in (4) is the strongest copula M. Archimedean copulas are linked to $\mathcal{K} = \{1\}$ and $]a_1, b_1[\,=\,]\,0,1\,[$. Copula C given by (4) is called an ordinal sum copula, with notation $(\langle a_k, b_k, C_k\rangle \mid k \in \mathcal{K})$.

Example 2.3. Let $C = \left(\langle 0, \frac{1}{2}, \Pi\rangle\right)$. Then

$$C(x_1, x_2) = \begin{cases} 2x_1 x_2, & \text{if } (x_1, x_2) \in\,]0, \frac{1}{2}[^2, \\ M(x_1, x_2), & \text{else.} \end{cases}$$

3. N-ARY ASSOCIATIVE FUNCTIONS WITH NEUTRAL ELEMENT

The associativity of n-ary functions was introduced by Post [12].

Definition 3.1. Let $n \geq 2$ and I be a real interval. A function $F\colon I^n \to I$ is said to be associative whenever for all $x_1, \dots, x_n, \dots, x_{2n-1} \in I$ it holds

$$\begin{aligned} &F\left(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}\right) = F\left(x_1, F(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}\right) \\ &= \dots = F\left(x_1, \dots, x_{n-1}, F(x_n, \dots, x_{2n-1})\right). \end{aligned} \tag{5}$$

Evidently, for $n = 2$, formulas (5) and (2) coincide, i. e., the Post n-ary associativity is a concept extending the standard notion of associativity for binary functions (operations). In the next definition, we recall the notion of neutral element, see [3].

Definition 3.2. Let $n \geq 2$ and I be a real interval. A function $F : I^n \to I$ is said to have neutral element $e \in I$ whenever $F(x_1, \dots, x_n) = x_i$ if $x_j = e$ for each $j \neq i$.

Evidently, property (C1) of n-copulas means that n-copulas have neutral element $e = 1$. We say that a function F is an n-ary extension of a binary function G if it holds

$$F(x_1, \dots, x_n) = G(G(\dots G(G(x_1, x_2), x_3) \dots), x_{n-1}), x_n)$$

for all n-tuples in I^n.

Example 3.3.

(i) Define a mapping $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x_1, x_2, x_3) = x_1 - x_2 + x_3$. Then F is a ternary associative function. Observe that there is no binary associative function whose ternary extension coincides with F. Moreover, F has no neutral element.

(ii) Let $C\colon [0,1]^3 \to [0,1]$ be given by $C(x_1, x_2, x_3) = x_1 \min\{x_2, x_3\}$. Then $e = 1$ is neutral element of C, but C is not associative. Note that C is a ternary copula.

Theorem 3.4. Consider $n \geq 2$. Let I be a real interval and $e \in I$. Then the following claims are equivalent:

(i) A mapping $F\colon I^n \to I$ is associative function with neutral element e.

(ii) There is a binary associative function $G\colon I^2 \to I$ with neutral element e whose n-ary extension is F.

Proof. If $n = 2$, the claim is trivial. Suppose that $n > 2$.

- (i) $\Leftarrow$ (ii) The proof is trivial.
- (i) $\Rightarrow$ (ii) Define a function $G\colon I^2 \to I$ by $G(x_1, x_2) = F(x_1, x_2, e, \dots, e)$. Then $G(x_1, e) = F(x_1, e, \dots, e) = x_1$ and $G(e, x_2) = F(e, x_2, e, \dots, e) = x_2$, i. e., e is a neutral element of G. Moreover, for any $x_1, x_2, x_3 \in I$ it holds

$$G(G(x_1, x_2), x_3) = F(F(x_1, x_2, e, \dots, e), x_3, e, \dots, e)$$
$$= F(x_1, x_2, F(\underbrace{e, \dots, e,}_{(n-2)\text{-times}}\ x_3, e), \underbrace{e, \dots, e}_{(n-3)\text{-times}}) = F(x_1, x_2, x_3, \underbrace{e, \dots, e}_{(n-3)\text{-times}}),$$

and

$$G(x_1, G(x_2, x_3)) = F(x_1, F(x_2, x_3, e, \dots, e), e, \dots, e)$$
$$= F(x_1, x_2, F(x_3, \underbrace{e, \dots, e}_{(n-1)\text{-times}}), \underbrace{e, \dots, e}_{(n-3)\text{-times}}) = F(x_1, x_2, x_3, \underbrace{e, \dots, e}_{(n-3)\text{-times}}),$$

which proves the associativity of G. From this proof it is also obvious that if $n = 3$, then $F(x_1, x_2, x_3) = G(G(x_1, x_2), x_3)$. For $n > 3$, $G(G(x_1, x_2), x_3) =$ $= F(x_1, x_2, x_3, \underbrace{e, \ldots, e}_{(n-3)\text{-times}})$ and similarly we can show that $G(G(G(x_1, x_2), x_3), x_4) = F(x_1, x_2, x_3, x_4, \underbrace{e, \ldots, e}_{(n-4)\text{-times}})$.
By induction on n it can be proved that for any $n > 2$,

$$G(G(\ldots G(G(x_1, x_2), \ldots), x_{n-1}), x_n) = F(x_1, \ldots, x_n).$$

□

Theorem 3.4 shows that under the neutral element existence, the associativity of n-ary functions is classically related to the associativity of binary functions.

4. ON THE STRUCTURE OF ASSOCIATIVE N-DIMENSIONAL COPULAS

Based on Theorems 2.1, 2.2, 3.4 and recent results on ordinal sum structure of n-copulas proved by Mesiar and Sempi [9], we have the next result.

Corollary 4.1. Let $n \geq 2$. A function $C\colon [0,1]^n \to [0,1]$ is an associative n-copula if and only if there is a system $(\,]a_k, b_k[\,)_{k \in \mathcal{K}}$ of pairwise disjoint open subintervals of $]\,0,1\,[$, and a system $(C_k)_{k \in \mathcal{K}}$ of associative n-copulas satisfying the diagonal inequality $C_k(x, \ldots, x) < x$ for all $x \in]\,0,1\,[$ and $k \in \mathcal{K}$ such that

$$C(x_1, \ldots, x_n) = \begin{cases} a_k + (b_k - a_k)\, \mathrm{C}_k \left(\frac{\min\{x_1, b_k\} - a_k}{b_k - a_k}, \ldots, \frac{\min\{x_n, b_k\} - a_k}{b_k - a_k} \right), \\ \qquad \text{if } \min\{x_1, \ldots, x_n\} \in]a_k, b_k[\quad \text{for some } k \in \mathcal{K}, \\ M(x_1, \ldots, x_n), \qquad \text{else.} \end{cases} \tag{6}$$

To complete the representation of associative n-copulas, the characterization of such copulas satisfying the diagonal inequality is necessary.

Theorem 4.2. Let $n \geq 2$. A function $C\colon [0,1]^n \to [0,1]$ is an associative n-copula satisfying the diagonal inequality $C(x, \ldots, x) < x$ for all $x \in]\,0,1\,[$ if and only if there is a generator f whose pseudo-inverse $f^{(-1)}$ is an $(n-2)$-times differentiable function with derivatives alternating the sign, such that $(-1)^n \frac{\mathbf{d}^{n-2} f^{(-1)}}{\mathbf{d}\, x^{n-2}}$ is a convex function, and

$$C(x_1, \ldots, x_n) = f^{(-1)} \left(\sum_{i=1}^{n} f(x_i) \right). \tag{7}$$

Proof. The sufficiency of conditions follows from [7].
By Theorem3.4, C is an n-ary extension of an associative copula G. Suppose that $G(x_0, x_0) = x_0$ for some $x_0 \in]\,0,1\,[$. Then

$$C(x_0, \ldots, x_0) = G(G(\ldots G(G(x_0, x_0), \ldots), x_0), x_0) = x_0,$$

which violates the diagonal inequality satisfied by C. Therefore G also satisfies the diagonal inequality, i. e., $G(x,x) < x$ for all $x \in]0,1[$. By Theorem 1, formula (7) is satisfied for some generator f. Moreover, C given by (7) is n-increasing and hence, according to the results of McNeil and Nešlehová in [7], the required properties of f are necessary. □

Example 4.3.

(i) As already mentioned, the product n-copula Π is associative for any $n \geq 2$. Evidently, $\Pi(x,\ldots,x) = x^n < x$ whenever $x \in]0,1[$. As the generator f_Π of the copula Π is given by $f_\Pi(x) = -\log x$, it holds $f_\Pi^{(-1)}(x) = f_\Pi^{-1}(x) = e^{-x}$, hence for any k, $\frac{\mathbf{d}^k f_\Pi^{-1}(x)}{\mathbf{d}\,x^k} = (-1)^k e^{-x}$. Derivatives alternate the sign and for any $n \geq 2$, $(-1)^n \frac{\mathbf{d}^{n-2} f_\Pi^{(-1)}(x)}{\mathbf{d}\,x^{n-2}} = e^{-x}$ is a convex function.

(ii) A similar result can be shown for the generator $f_{(1)}$ introduced in Section 2, given by $f_{(1)}(x) = \frac{1}{x} - 1$. It holds $f_{(1)}^{(-1)}(x) = f_{(1)}^{-1}(x) = (1+x),^{-1}$ which implies that
$(-1)^n \frac{\mathbf{d}^{n-2} f_\Pi^{(-1)}(x)}{\mathbf{d}\,x^{n-2}} = (n-2)!\,(1+x)^{-n+1}$ is convex. The corresponding n-copula $C_{(1)}$ is given by $C_{(1)}(x) = \left(\sum_{i=1}^n \frac{1}{x_i} - (n-1)\right)^{-1}$.

(iii) The weakest associative n-copula is the Clayton copula $C_{(-\frac{1}{n-1})}$ generated by the generator $f_{(-\frac{1}{n-1})}\colon [0,1] \to [0,\infty]$, $f_{(-\frac{1}{n-1})} = 1 - x^{\frac{1}{n-1}}$. The corresponding pseudo-inverse $f_{(-\frac{1}{n-1})}^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$f_{(-\frac{1}{n-1})}^{(-1)}(x) = \begin{cases} (1-x)^{n-1}, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then $(-1)^n \frac{\mathbf{d}^{n-2} f_{(-\frac{1}{n-1})}^{(-1)}(x)}{\mathbf{d}\,x^{n-2}} = (n-1)!\,\max\{1-x, 0\}$ is convex but not differentiable. For more details we recommend [7].

(iv) The function $C\colon [0,1]^n \to [0,1]$ given by

$$\mathrm{C}(x_1,\ldots,x_n) = \begin{cases} 2^{n-1} \prod\limits_{i=1}^{n} \min\left\{x_i, \frac{1}{2}\right\}, & \text{if } \min\{x_1,\ldots,x_n\} < \frac{1}{2}, \\ M(x_1,\ldots,x_n), & \text{else}, \end{cases} \tag{8}$$

is an n-ary extension of the ordinal sum copula $\left(\langle 0, \frac{1}{2}, \Pi\rangle\right)$ introduced in Example 2.3. As n-ary function Π is an associative n-copula for each $n \geq 2$, our function C given by (8) is also an associative n-copula for each $n \geq 2$.

5. CONCLUDING REMARKS

We have solved the Problem 2.1 posed in [8], showing that associative n-copulas are just n-ary extensions of appropriate associative copulas. Based on Theorem 3.4, similar results can be formulated for the representation of continuous n-ary triangular norms or triangular conorms [5], and also for n-ary uninorms [2] continuous up to the case when $\{0,1\} \subseteq \{x_1,\ldots,x_n\}$.

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