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TRANSFERRAL OF ENTAILMENT IN DUALITY THEORY: DUALISABILITY

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Abstract. A number of new results that say how to transfer the entailment relation between two different finite generators of a quasi-variety of algebras is presented. As their consequence, a well-known result saying that dualisability of a quasi-variety is independent of the generating algebra is derived. The transferral of endodualisability is also considered and the results are illustrated by examples.

Keywords: natural duality, dualisability, endodualisability, entailment, retraction

MSC 2010: 08C20, 08A35

1. Introduction

In 1936 M. H. Stone published a seminal work on duality theory, exhibiting a dual equivalence between the category of all Boolean algebras and the category of all Boolean spaces [20]. Almost at the same time L. S. Pontryagin showed that the category of abelian groups is dually equivalent to the category of compact topological abelian groups [13], [14]. The most important step toward the development of general duality theory was Priestley's duality for distributive lattices: the category of all distributive lattices was shown to be dually equivalent to the category of all compact totally-order disconnected ordered topological spaces [15], [16]. The general duality theory, called *natural duality theory*, grew out from these three dualities in a work by B. A. Davey and H. Werner [10] and by D. M. Clark and P. Krauss [2], and has been rapidly developed over the last twenty-five years. This culminated in a first-ever monograph of the field by D. M. Clark and B. A. Davey [1] which has since become

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a standard reference on natural dualities. More recently, another monograph by J. G. Pitkethly and B. A. Davey appeared [9]. The theory has already proved to be a valuable tool in algebra, algebraic logic and certain parts of computer science.

Generally speaking, the theory of natural dualities concerns the topological representation of algebras. The main idea of the theory is that, given a quasi-variety $\mathscr{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ of algebras generated by an algebra $\underline{\mathbf{M}}$, one can often find a topological relational structure $\underset{\sim}{\mathbf{M}}$ on the underlying set M of $\underline{\mathbf{M}}$ such that a dual equivalence exists between $\mathscr{A}$ and a suitable category $\mathscr{X}$ of topological relational structures of the same type as $\underset{\sim}{\mathbf{M}}$. Requiring the relational structure of $\underset{\sim}{\mathbf{M}}$ to be *algebraic over* $\underline{\mathbf{M}}$ (this is explained in Section 2), all the requisite category theory "runs smoothly" (we refer to Chapter 1 of [1]). A uniform way of representing each algebra $\underline{\mathbf{A}}$ in the quasi-variety $\mathscr{A}$ as an algebra of continuous structure-preserving maps from a suitable structure $\mathbf{X} \in \mathscr{X}$ into $\underset{\sim}{\mathbf{M}}$ can be obtained. In particular, the representation is relatively simple and useful for free algebras in $\mathscr{A}$.

The quasi-variety $\mathscr{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ of algebras generated by the algebra $\underline{\mathbf{M}}$ is said to *admit a natural duality* or to be *dualisable* if a natural duality based on $\underline{\mathbf{M}}$ exists. It is often simply said that in such case *the algebra* $\underline{\mathbf{M}}$ *is dualisable.* The main result proven by the first author in [18] is that, given a dualisable quasi-variety, each of its finite generating algebras is dualisable. Hence dualisability of a quasi-variety is *independent* of the generating algebra, which was also (independently) proved by B. A. Davey and R. Willard in [11]. As the number of dualising relations in all known dualities is finite and for a finite set S of relations, the *dualisability* via S is equivalent to the *entailment* of every algebraic relation by S, we seek here for a better understanding of the entailment process on different generators of a quasi-variety. We present a number of new results that say how to transfer the entailment relation and dualisability between two different generators of a quasi-variety.

Throughout the paper we assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that for the quasi-varieties $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and $\mathscr{M} := \mathbb{ISP}(\underline{\mathbf{M}})$ we have $\mathscr{D} \subseteq \mathscr{M}$. We also assume that there are homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one. In Section 3 we concentrate on the transferral of the entailment "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$, while in Section 4, where we assume that $\mathscr{D} = \mathscr{M}$ and that α is one-to-one, we concentrate on the transferral of the entailment "down" from $\underline{\mathbf{M}}$ to $\underline{\mathbf{D}}$. As a consequence of our results, we obtain a new proof of the main result of [18] on the transferral of dualisability at the end of Section 4. We also consider the transferral of endodualisability in Section 5, where our main result slightly generalizes similar results of [18] and [9]. We finally present an application of our results in two examples in Section 6.

We shall recall here the basic concepts of the theory of natural dualities. Those wanting further details related to these concepts are referred to the monograph D. M. Clark and B. A. Davey [1].

Let $\underline{\mathbf{M}}$ be a finite algebra, and let $\mathscr{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ be the quasi-variety generated by $\underline{\mathbf{M}}$. Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a structure on the same underlying set M which is *algebraic over* $\underset{\sim}{\mathbf{M}}$, meaning that:

- G is a set of finitary *algebraic operations on* $\underline{\mathbf{M}}$, that is, each $g \in G$ is a homomorphism $g\colon\ \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$ for some $n \geqslant 0$;
- H is a set of finitary *algebraic partial operations on* $\underline{\mathbf{M}}$, that is, each $h \in H$ is a homomorphism $h\colon\ \underline{\mathbf{N}} \to \underline{\mathbf{M}}$ for some subalgebra $\underline{\mathbf{N}}$ of $\underline{\mathbf{M}}^n$, $n \geqslant 1$;
- R is a set of finitary *algebraic relations on* $\underline{\mathbf{M}}$, that is, each $r \in R$ is the underlying set of a subalgebra of $\underline{\mathbf{M}}^n$ for $n \geqslant 1$;
- $\mathcal{T}$ is the discrete topology on M.

Given a closed substructure $\mathbf{X}$ of a non-zero power of $\underset{\sim}{\mathbf{M}}$, we define a morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ to be a map $\alpha\colon\ X \to \underset{\sim}{\mathbf{M}}$ that preserves the structure $G \cup H \cup R$ and is continuous. Let s be a finitary algebraic relation on $\underline{\mathbf{M}}$ (in particular, a graph of a finitary algebraic partial operation h on $\underline{\mathbf{M}}$). We say that $G \cup H \cup R$, or simply $\underset{\sim}{\mathbf{M}}$, *entails* s *on the structure* $\mathbf{X}$ (in particular, that $G \cup H \cup R$ or $\underset{\sim}{\mathbf{M}}$ *entails* h *on the structure* $\mathbf{X}$) if each morphism $\alpha\colon\ \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves s.

The dual category to $\mathscr{A}$ is defined to be the class $\mathscr{X} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ of all isomorphic copies of closed substructures of non-zero powers of the *alter ego* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ of $\underline{\mathbf{M}}$. The morphisms of the category $\mathscr{X}$ are the continuous structure preserving maps. A natural duality on $\mathscr{A}$ provides us with a representation of the algebras in $\mathscr{A}$ as algebras of structure-preserving maps from structures in $\mathscr{X}$ into $\underset{\sim}{\mathbf{M}}$.

More precisely, we consider a pair of contravariant functors $\mathrm{D}\colon\ \mathscr{A} \to \mathscr{X}$ and $\mathrm{E}\colon\ \mathscr{X} \to \mathscr{A}$ defined naturally as follows. For every $\mathbf{A} \in \mathscr{A}$, $\mathrm{D}(\mathbf{A})$ is the homset $\mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ regarded as a closed substructure of $\underset{\sim}{\mathbf{M}}^A$; the structure $\mathrm{D}(\mathbf{A}) \in \mathscr{X}$ is called the *dual of* $\mathbf{A}$. Similarly, for every $\mathbf{X} \in \mathscr{X}$, its *dual* $\mathrm{E}(\mathbf{X}) \in \mathscr{A}$ is defined to be the homset $\mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ regarded as a subalgebra of $\underline{\mathbf{M}}^X$. The functors D and E are naturally defined on morphisms, too: for $\varphi\colon\ \mathbf{A} \to \mathbf{B}$ in $\mathscr{A}$, $\mathrm{D}(\varphi)\colon\ \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ is given by $\mathrm{D}(\varphi)(\mathrm{x}) := \mathrm{x} \circ \varphi$, and for $\psi\colon\ \mathbf{X} \to \mathbf{Y}$ in $\mathscr{X}$, $\mathrm{E}(\psi)\colon\ \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ is given by $\mathrm{E}(\psi)(\alpha) := \alpha \circ \psi$.

For each $\mathbf{A} \in \mathscr{A}$, there is an embedding $e_{\mathbf{A}}\colon\ \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ defined by $e_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and $x \in \mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$; here $e_{\mathbf{A}}(a)$ is called the evaluation map. Similarly, for each $\mathbf{X} \in \mathscr{X}$, one can define an embedding $\varepsilon_{\mathbf{X}}\colon\ \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ by

$\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$ for all $x \in X$ and all $\alpha \in \mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$. If $e_{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \mathscr{A}$, then we say that the structure $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$ or that $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$.

Let $\mathscr{A}_{\mathrm{fin}}$ be the category of all finite algebras in $\mathscr{A}$. If for every $\underline{\mathbf{A}}$ in $\mathscr{A}_{\mathrm{fin}}$, $e_{\underline{\mathbf{A}}}$ is an isomorphism, then $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$ *at the finite level.*

The following Duality Compactness Theorem is due independently to Willard [22] and Zádori [23].

Theorem 2.1 ([1], Theorem 2.2.11). *If* $\underset{\sim}{\mathbf{M}}$ *is of finite type and yields a duality on* $\mathscr{A}_{\mathrm{fin}}$*, then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$*.*

In case $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$, we have got a representation for $\mathscr{A}$: each algebra $\mathbf{A} \in \mathscr{A}$ is isomorphic to the algebra $\mathrm{ED}(\mathbf{A})$ of all morphisms from its dual $\mathrm{D}(\mathbf{A}) \in \mathscr{X}$ into $\underset{\sim}{\mathbf{M}}$. If $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are isomorphisms for all $\mathbf{A} \in \mathscr{A}$ and $\mathbf{X} \in \mathscr{X}$, then $\underset{\sim}{\mathbf{M}}$ is said to yield a *full duality on* $\mathscr{A}$ or one says that $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$. In this case, the categories $\mathscr{A}$ and $\mathscr{X}$ are dually equivalent.

Entailment and duality are intimately connected. As far as duality is concerned we are interested only in entailment on the structures $\mathbf{X}$ of the form $\mathrm{D}(\underline{\mathbf{A}})$ for $\underline{\mathbf{A}} \in \mathscr{A}$. Thus we say (cf. [1], p. 55) that $\underset{\sim}{\mathbf{M}}$ *entails* s if it entails s on every structure of the form $\mathrm{D}(\underline{\mathbf{A}})$ for $\underline{\mathbf{A}} \in \mathscr{A}$. Let $\mathscr{B}_M$ be the class of all finitary algebraic relations on $\underline{\mathbf{M}}$. If a set R of relations in $\Omega \subseteq \mathscr{B}_M$ is such that R entails s for every $s \in \Omega$, then we say that R is *entailment-dense in* Ω. Later on, in Lemma 2.3(ii) we state that in case $G \cup H \cup R$ dualises $\underline{\mathbf{M}}$, it is entailment-dense in $\mathscr{B}_M$.

We denote by $\mathbf{s}$ the subalgebra of $\underline{\mathbf{M}}^n$ corresponding to the n-ary algebraic relation s on $\underline{\mathbf{M}}$, where $n \geqslant 1$. For each $i \in \{1, \dots, n\}$, we define $\varrho_i^s := \pi_i{\restriction_s} \colon s \to M$, where $\pi_i \colon M^n \to M$ is the natural projection. A formula in the language of $\underset{\sim}{\mathbf{M}}$ is called a *primitive positive formula* if it is an existential conjunct of atomic formulæ.

The following result is fundamental for the study of entailment (for its proof see [7], 2.3 or [1], 8.1.3, 9.1.2; cf. also [8], 1.4). It is usually stated and proved for finitary algebraic relations s. (We note that in [8], for the first time to our knowledge, the entailed relation s was considered to be infinitary; however, the concept of *structural entailment* introduced and studied in [8] is not considered in this paper.)

Theorem 2.2 (The Test Algebra Theorem). *Let* $\underline{\mathbf{M}}$ *be a finite algebra and let* $\underset{\sim}{\mathbf{M}}$ *be its alter ego. Let* s *be an* n*-ary algebraic relation on* $\underline{\mathbf{M}}$ *for some* $n \in \mathbb{N}$*. Then the following conditions are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}$ *entails* s*;*
(2) $\underset{\sim}{\mathbf{M}}$ *entails* s *on* $\mathrm{D}(\mathbf{s})$*;*
(3) *every morphism* $\alpha \colon \mathrm{D}(\mathbf{s}) \to \underset{\sim}{\mathbf{M}}$ *satisfies* $(\alpha(\varrho_1^s), \dots, \alpha(\varrho_n^s)) \in \mathbf{s}$*;*
(4) $s = \{\, (\alpha(\varrho_1^s), \dots, \alpha(\varrho_n^s)) \mid \alpha \in \mathrm{ED}(\mathbf{s}) \,\}$*;*

(5) *s may be obtained from $G \cup H \cup R$ via a primitive positive construct, that is, for some primitive positive formula $\Phi(x_1, \dots, x_n)$ in the language of $\underset{\sim}{\mathbf{M}}$,*

$$s = \{ (c_1, \dots, c_n) \in M^n \mid \underset{\sim}{\mathbf{M}} \models \Phi(c_1, \dots, c_n) \}$$

and $\mathrm{D}(\mathbf{s})$ satisfies $\Phi(\varrho_1^s, \dots \varrho_n^s)$.

As an immediate consequence we obtain the following result (see [8], Lemma 1.5).

Lemma 2.3. *Let $\underline{\mathbf{M}}$ be a finite algebra and $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$.*

(i) *Let s be an n-ary algebraic relation on $\underline{\mathbf{M}}$, for some $n \in \mathbb{N}$. In order to show that $\underset{\sim}{\mathbf{M}}$ entails s it suffices to prove that the embedding $e_{\underline{\mathbf{A}}} \colon \underline{\mathbf{A}} \to \mathrm{ED}(\underline{\mathbf{A}})$ is an isomorphism for some isomorphic copy $\underline{\mathbf{A}}$ of the algebra $\mathbf{s}$.*

(ii) *If $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then $\underset{\sim}{\mathbf{M}}$ entails every finitary algebraic relation on $\underline{\mathbf{M}}$, or equivalently, $G \cup H \cup R$ is entailment-dense in $\mathscr{B}_M$.*

The Brute Force Duality Theorem (cf. [1], Theorem 2.3.1) says that the set $\mathscr{B}_M$ of all finitary algebraic relations on $\underline{\mathbf{M}}$ (the brute force) yields a duality on $\mathscr{A}_{\mathrm{fin}}$. The following *Density Lemma* obviously holds also at the finite level.

Lemma 2.4 ([1], Lemma 8.2.2)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra, let $\Omega \subseteq \mathscr{B}_M$ yield a duality on $\mathscr{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ and let $R \subseteq \Omega$. The following conditions are equivalent:*

(1) *R yields a duality on $\mathscr{A}$;*

(2) *R is entailment-dense in Ω;*

(3) *R entails s for each $s \in \Omega \setminus R$;*

(4) *R entails s on $D(\mathbf{s})$ for each $s \in \Omega \setminus R$.*

Therefore, if a finite set $R \subseteq \mathscr{B}_M$ entails the brute force $\mathscr{B}_M$, then R yields a duality on $\mathscr{A}_{\mathrm{fin}}$, and by the Duality Compactness Theorem 2.1, R yields a duality on $\mathscr{A}$. Hence we have the following result.

Lemma 2.5. *Let $\underline{\mathbf{M}}$ be a finite algebra. A finite set $R \subseteq \mathscr{B}_M$ yields a duality on $\mathscr{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ if and only if R entails $\mathscr{B}_M$.*

So, studying when $\underline{\mathbf{M}}$ is dualisable via a finite set of relations $R \subseteq \mathscr{B}_M$ is equivalent to studying when R entails $\mathscr{B}_M$.

3. Transferring entailment up

Throughout this section we assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that for the quasi-varieties $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and $\mathscr{M} := \mathbb{ISP}(\underline{\mathbf{M}})$ we have $\mathscr{D} \subseteq \mathscr{M}$. We also assume that there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one. In this section we concentrate on studying the transferral of the entailment "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$.

For any algebra $\mathbf{A} \in \mathscr{D}$ we can without loss of generality assume that $\mathbf{A} \leqslant \underline{\mathbf{D}}^I$ for a set I. We denote by β the restriction to $\mathbf{A}$ of the embedding $\underline{\mathbf{D}}^I \to \underline{\mathbf{M}}^I$ that assigns to each element $\langle a_i \rangle_{i\in I}$ the element $\langle \beta(a_i)\rangle_{i\in I}$. By β^{-1} we denote the inverse of the isomorphism from $\mathbf{A}$ onto $\beta(\mathbf{A})$ given by β.

To every n-ary partial operation $h\colon \operatorname{dom} h \subseteq D^n \to D$ we assign the n-ary partial operation $h_\beta\colon \operatorname{dom} h_\beta \subseteq M^n \to M$ as follows:

$$\operatorname{dom} h_\beta := \{\, (\beta(a_1), \dots, \beta(a_n)) \mid (a_1, \dots, a_n) \in \operatorname{dom} h \,\}$$

and

$$h_\beta((\beta(a_1), \dots, \beta(a_n))) = \beta(h(a_1, \dots, a_n))$$

for every $(a_1, \dots, a_n) \in \operatorname{dom} h$. Then h_β is algebraic over $\underline{\mathbf{M}}$ whenever h is algebraic over $\underline{\mathbf{D}}$ (cf. [18], p. 199). Similarly, we assign to every m-ary relation r on D the relation

$$r_\beta := \{\, (\beta(a_1), \dots, \beta(a_m)) \mid (a_1, \dots, a_m) \in r \,\}$$

on M. Then r_β is algebraic over $\underline{\mathbf{M}}$ whenever r is algebraic over $\underline{\mathbf{D}}$.

Definition 3.1. For every $\mathbf{A} \in \mathscr{D}$ and every map $u\colon \mathscr{M}(\beta(\mathbf{A}), \underline{\mathbf{M}}) \to M$ that preserves $\beta(D)$, we define a map $u_D\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ by

$$u_D(x) := \beta^{-1}(u(\beta \circ x \circ \beta^{-1})).$$

We shall derive the following properties of the map u_D.

Lemma 3.2. *Let $\mathbf{A} \in \mathscr{D}$ and let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ be an embedding.*

(i) *If $u\colon \mathscr{M}(\beta(\mathbf{A}), \underline{\mathbf{M}}) \to M$ is a map that preserves $\beta(D)$ and r_β for some relation r on D, then the map u_D preserves r.*

(ii) *If $u\colon \mathscr{M}(\beta(\mathbf{A}), \underline{\mathbf{M}}) \to M$ is a map that preserves $\beta(D)$ and h_β for some (partial) operation h on D, then u_D preserves h.*

P r o o f. Let r be an m-ary relation on D and let $x_i \in \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}})$ for $i \in \{1, \dots, m\}$, be such that $(x_1, \dots, x_m) \in r$. This means that $(x_1(a), \dots, x_m(a)) \in r$ for all $a \in A$. Since we have

$$(\beta \circ x_1 \circ \beta^{-1}, \dots, \beta \circ x_m \circ \beta^{-1}) \in r_\beta,$$

where $\beta \circ x_i \circ \beta^{-1} \in \mathscr{M}(\beta(\mathbf{A}), \underline{\mathbf{M}})$, and since u preserves r_β, we have

$$(u(\beta \circ x_1 \circ \beta^{-1}), \ldots, u(\beta \circ x_m \circ \beta^{-1})) \in r_\beta,$$

and consequently $(u_D(x_1), \ldots, u_D(x_m)) \in \beta^{-1}(r_\beta) = r$. Thus (i) holds.

Now we prove (ii). Let h be an n-ary (partial) operation on D and assume that $(x_1, \ldots, x_n) \in \operatorname{dom} h$, with $x_i \in \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}})$ for all i. Since u preserves h_β and since $\operatorname{dom} h_\beta = (\operatorname{dom} h)_\beta$, we get $(u_D(x_1), \ldots, u_D(x_n)) \in \operatorname{dom} h$ by applying (i). From $(\beta \circ x_1 \circ \beta^{-1}, \ldots, \beta \circ x_n \circ \beta^{-1}) \in \operatorname{dom} h_\beta$ we also get

$$u(h_\beta(\beta \circ x_1 \circ \beta^{-1}, \ldots, \beta \circ x_n \circ \beta^{-1})) = h_\beta(u(\beta \circ x_1 \circ \beta^{-1}), \ldots, u(\beta \circ x_n \circ \beta^{-1})).$$

Therefore

$$\begin{aligned}
&\beta^{-1}(u(\beta \circ h(x_1 \circ \beta^{-1}, \ldots, x_n \circ \beta^{-1})))\\
&\quad = \beta^{-1}(u(h_\beta(\beta \circ x_1 \circ \beta^{-1}, \ldots, \beta \circ x_n \circ \beta^{-1})))\\
&\quad = \beta^{-1}(h_\beta(u(\beta \circ x_1 \circ \beta^{-1}), \ldots, u(\beta \circ x_n \circ \beta^{-1})))\\
&\quad = h(\beta^{-1}(u(\beta \circ x_1 \circ \beta^{-1})), \ldots, \beta^{-1}(u(\beta \circ x_n \circ \beta^{-1})))\\
&\quad = h(u_D(x_1), \ldots, u_D(x_n)).
\end{aligned}$$

We finally observe that

$$\begin{aligned}
&h(x_1 \circ \beta^{-1}, \ldots, x_n \circ \beta^{-1})(\beta(a))\\
&\quad = h(x_1 \circ \beta^{-1}(\beta(a)), \ldots, x_n \circ \beta^{-1}(\beta(a)))\\
&\quad = h(x_1(a), \ldots, x_n(a))\\
&\quad = h((x_1, \ldots, x_n)(a))\\
&\quad = (h(x_1, \ldots, x_n) \circ \beta^{-1})(\beta(a))
\end{aligned}$$

for every $a \in A$, which yields

$$\beta^{-1}(u(\beta \circ h(x_1, \ldots, x_n) \circ \beta^{-1})) = \beta^{-1}(u(\beta \circ h(x_1 \circ \beta^{-1}, \ldots, x_n \circ \beta^{-1}))).$$

We can conclude that $u_D(h(x_1, \ldots, x_n)) = h(u_D(x_1), \ldots, u_D(x_n))$. $\square$

Theorem 3.3. *Assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ be an embedding.*

If $G \cup H \cup R$ entails an m-ary algebraic relation s (the graph of a finitary algebraic partial operation h) on $\underline{\mathbf{D}}$, then $G_\beta \cup H_\beta \cup R_\beta \cup \{\,\beta(D)\}$ entails the relation s_β (the graph of h_β) on $\underline{\mathbf{M}}$, where
(a) $G_\beta = \{\, g_\beta \mid g \in G\}$;
(b) $H_\beta = \{\, h_\beta \mid h \in H\}$;
(c) $R_\beta = \{\, r_\beta \mid r \in R\}$.

Proof. Let $u\colon \mathscr{M}(\mathbf{s}_\beta, \underline{\mathbf{M}}) \to M$ be a map preserving $G_\beta \cup H_\beta \cup R_\beta \cup \{\,\beta(D)\}$. By Theorem 2.2, it suffices to prove that $(u(\varrho_1^{s_\beta}), \dots, u(\varrho_m^{s_\beta})) \in s_\beta$, where for each $\varrho_i^{s_\beta}\colon s_\beta \to M$ we have that $\varrho_i^{s_\beta} = \beta \circ \pi_i{\restriction_s} \circ \beta^{-1} = \beta \circ \varrho_i^s \circ \beta^{-1}$, $i \in \{1, \dots, m\}$ (we note that u is automatically continuous here). By Lemma 3.2, $u_D\colon \mathscr{D}(\mathbf{s}, \underline{\mathbf{D}}) \to D$ preserves $G \cup H \cup R$. Since the set $G \cup H \cup R$ entails s, the map u_D preserves s. We recall that $s_\beta \subseteq \beta(D)^m$ and hence, for all $i \in \{1, \dots, m\}$, $(\varrho_i \circ \beta)(s) \subseteq \beta(D)$. Therefore we have $\varrho_i^s = \beta^{-1} \circ \varrho_i^{s_\beta} \circ \beta \in \mathscr{D}(\mathbf{s}, \underline{\mathbf{D}})$. Since $(u_D(\varrho_1^s), \dots, u_D(\varrho_m^s)) \in s$, we obtain $(u(\beta \circ \varrho_1^s \circ \beta^{-1}), \dots, u(\beta \circ \varrho_m^s \circ \beta^{-1})) \in s_\beta$, so $(u(\varrho_1^{s_\beta}), \dots, u(\varrho_m^{s_\beta})) \in s_\beta$ as required. □

Now we shall show that the relation $\beta(D)$ is entailed from the set $\mathrm{End}(\underline{\mathbf{M}})$ of endomorphisms of $\underline{\mathbf{M}}$ whenever $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$. This is in particular true when $\underline{\mathbf{D}}$ is a retract of $\underline{\mathbf{M}}$.

Lemma 3.4. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one. Moreover, assume that $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$.*

Then $\mathrm{End}(\underline{\mathbf{M}})$ *entails* $\beta(D)$.

Proof. Let $u\colon \mathscr{M}(\beta(D), \underline{\mathbf{M}}) \to M$ preserve $\mathrm{End}(\underline{\mathbf{M}})$. We take $x \in \mathscr{M}(\beta(D), \underline{\mathbf{M}})$ and assume that $x \in \beta(D)$ on $\mathscr{M}(\beta(D), \underline{\mathbf{M}})$, that is for every $a \in \mathrm{dom}(x)$, $x(a) \in \beta(D)$. Hence for every $a \in \beta(D)$ there exists $b \in D$ such that

$$x(a) = \beta(b) = (\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ \beta)(b) = (\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(x(a)),$$

whence $x = \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ x$ and $u(x) = u(\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ x)$. As u preserves $\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \in \mathrm{End}(\underline{\mathbf{M}})$, we finally obtain $u(x) = (\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(u(x)) \in \beta(D)$ as required. □

If $k = 1$, and so α is a homomorphism from $\underline{\mathbf{M}}$ into $\underline{\mathbf{D}}$, then the finiteness of $\underline{\mathbf{D}}$ implies that $\alpha \circ \beta$ is an automorphism of $\underline{\mathbf{D}}$ and consequently $(\alpha \circ \beta)^n = \mathrm{id}_D$ for some $n \in \mathbb{N}$. Hence $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}$ is a retraction and $\beta \circ (\alpha \circ \beta)^{n-1}\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ is a co-retraction. In such a case we have the following consequence of the previous two results.

Corollary 3.5. *Let* $\underline{\mathbf{D}}$ *and* $\underline{\mathbf{M}}$ *be finite algebras and let* $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. *Assume that* $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ *is a co-retraction. If* $G \cup H \cup R$ *entails an algebraic relation* s *(the graph of a partial algebraic operation* h*) on* D*, then* $\mathrm{End}(\underline{\mathbf{M}}) \cup G_\beta \cup H_\beta \cup R_\beta$ *entails the relation* s_β *(the graph of the partial operation* h_β*) on* M.

Now let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ satisfy $\mathscr{D} = \mathbb{ISP}(\underline{\mathbf{D}}) = \mathbb{ISP}(\underline{\mathbf{M}}) = \mathscr{M}$ and assume there exist one-to-one homomorphisms $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, where $k \geqslant 1$. Instead of $\varrho_1^{\alpha(M)}, \dots, \varrho_k^{\alpha(M)}$ we shall briefly write $\varrho_1, \dots, \varrho_k$. For every n-ary partial operation $h\colon \operatorname{dom} h \subseteq M^n \to M$, let $\operatorname{dom} h_\alpha \subseteq D^{nk}$ be the set of all elements $(\varrho \circ \alpha)((a_1, \dots, a_n))$ of the form

$$((\varrho_1 \circ \alpha)(a_1), \dots, (\varrho_k \circ \alpha)(a_1), \dots, (\varrho_1 \circ \alpha)(a_n), \dots, (\varrho_k \circ \alpha)(a_n)),$$

where $(a_1, \dots, a_n) \in \operatorname{dom} h$. Let $h_\alpha\colon \operatorname{dom} h_\alpha \to D^k$ be the map defined so that for all $(a_1, \dots, a_n) \in \operatorname{dom} h$,

$$h_\alpha((\varrho \circ \alpha)((a_1, \dots, a_n))) = \alpha(h(a_1, \dots, a_n)).$$

We observe that h_α is a homomorphism from the subalgebra $\operatorname{dom} h_\alpha$ of $\underline{\mathbf{D}}^{nk}$ into $\underline{\mathbf{D}}^k$ whenever h is algebraic over $\underline{\mathbf{M}}$ (cf. [18], p. 201).

For every m-ary relation r on M, we define the mk-ary relation r_α on D as

$$r_\alpha := \{(\varrho \circ \alpha)((a_1, \dots, a_m)) \mid (a_1, \dots, a_m) \in r\}.$$

Again, r_α is algebraic over $\underline{\mathbf{D}}$ if r is algebraic over $\underline{\mathbf{M}}$ (cf. [18], p. 202).

For every $i \in \{1, \dots, k\}$, the homomorphism $\omega_i := \beta \circ \varrho_i \circ \alpha$ is an endomorphism of $\underline{\mathbf{M}}$. Let us denote

$$\Gamma_{\beta\alpha} := \{\, \omega_1, \dots, \omega_k\}.$$

We now define a homomorphism

$$\omega := \omega_1 \sqcap \dots \sqcap \omega_k\colon\ \underline{\mathbf{M}} \to \underline{\mathbf{M}}^k$$

by $\omega(a) := (\omega_1(a), \dots, \omega_k(a))$ for all $a \in M$. As the maps $\omega_1, \dots, \omega_k$ separate the points of M, ω is an embedding. Let

$$M_r := \omega(M) \subseteq M^k$$

and let $\sigma\colon M_r \to M$ be the inverse of ω regarded as a k-ary algebraic partial operation on $\underline{\mathbf{M}}$. It follows that for all $a \in M$,

$$\sigma(\omega_1(a), \dots, \omega_k(a)) = a.$$

The partial operation σ on $\underline{\mathbf{M}}$ introduced in Davey and Haviar [5] is known as the *schizophrenic operation* corresponding to $\omega_1, \dots, \omega_k$ and we shall apply it in the next results. The proof of the following lemma is easy and we leave it to the reader.

Lemma 3.6. *Assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are algebras such that there exist one-to-one homomorphisms $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$. For any m-ary relation r on M, the following conditions are equivalent:*

(i) $(\varrho \circ \alpha)((a_1, \dots, a_m)) \in r_\alpha$;

(ii) $(\omega_1(a_1), \dots, \omega_k(a_1), \dots, \omega_1(a_m), \dots, \omega_k(a_m)) \in (r_\alpha)_\beta$;

(iii) $(\sigma(\omega_1(a_1), \dots, \omega_k(a_1)), \dots, \sigma(\omega_1(a_m), \dots, \omega_k(a_m))) \in r$.

The following lemma will play an important role in our further investigations.

Lemma 3.7. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras such that there exist one-to-one homomorphisms $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$.*

For every m-ary relation r on M, r is entailed from $\Gamma_{\beta\alpha} \cup \{(r_\alpha)_\beta\}$.

Proof. Let $y_1, \dots, y_m \in \mathscr{D}(\mathbf{r}, \underline{\mathbf{M}})$ be such that $(y_1, \dots, y_m) \in r$ on $\mathscr{D}(\mathbf{r}, \underline{\mathbf{M}})$. Then

$$(\varrho_1 \circ \alpha \circ y_1, \dots, \varrho_k \circ \alpha \circ y_1, \dots, \varrho_1 \circ \alpha \circ y_m, \dots, \varrho_k \circ \alpha \circ y_m) \in r_\alpha.$$

Using Lemma 3.6 and the equality $\beta \circ \varrho_i \circ \alpha \circ y_j = \omega_i \circ y_j$, we obtain

$$(\omega_1 \circ y_1, \dots, \omega_k \circ y_1, \dots, \omega_1 \circ y_m, \dots, \omega_k \circ y_m) \in (r_\alpha)_\beta.$$

Let $u\colon \mathscr{D}(\mathbf{r}, \underline{\mathbf{M}}) \to M$ preserve $\Gamma_{\beta\alpha} \cup \{(r_\alpha)_\beta\}$. Then we obtain

$$(u(\omega_1 \circ y_1), \dots, u(\omega_k \circ y_1), \dots, u(\omega_1 \circ y_m), \dots, u(\omega_k \circ y_m)) \in (r_\alpha)_\beta$$

and

$$(\omega_1(u(y_1)), \dots, \omega_k(u(y_1)), \dots, \omega_1(u(y_m)), \dots, \omega_k(u(y_m))) \in (r_\alpha)_\beta.$$

Applying the schizophrenic operation σ and Lemma 3.6 again, we now have

$$(\sigma(\omega_1(u(y_1)), \dots, \omega_k(u(y_1))), \dots, \sigma(\omega_1(u(y_m)), \dots, \omega_k(u(y_m)))) \in r,$$

whence

$$(u(y_1), \dots, u(y_m)) \in r$$

as required. $\square$

We can now prove one of the main results of this section.

Theorem 3.8. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist one-to-one homomorphisms $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$.*

(i) *If S is entailment-dense in $\mathscr{B}_D$ then $\Gamma_{\beta\alpha} \cup S_\beta \cup \{\beta(D)\}$ is entailment-dense in $\mathscr{B}_M$.*

(ii) *If S is entailment-dense in $\mathscr{B}_D$ and $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$, then $\mathrm{End}(\underline{\mathbf{M}}) \cup S_\beta$ is entailment-dense in $\mathscr{B}_M$.*

P r o o f. Let $r \in \mathscr{B}_M$, $r \leqslant M^m$. By Lemma 3.7, r is entailed from $\Gamma_{\beta\alpha} \cup \{(r_\alpha)_\beta\}$. Because S entails r_α by assumption, from Theorem 3.3 we have that $(r_\alpha)_\beta$ is entailed from $S_\beta \cup \{\beta(D)\}$. Consequently, r is entailed from $\Gamma_{\beta\alpha} \cup S_\beta \cup \{\beta(D)\}$. This proves (i).

For (ii), if $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$, then $\beta(D)$ is entailed from $\mathrm{End}(\underline{\mathbf{M}})$ by Lemma 3.4 and thus r is entailed from $\mathrm{End}(\underline{\mathbf{M}}) \cup S_\beta$ (we recall that $\Gamma_{\beta\alpha} \subseteq \mathrm{End}(\underline{\mathbf{M}})$). $\square$

As a consequence of our results on entailment so far, we now obtain the first main result of [18] under the assumption that the dualising structure $G \cup H \cup R$ is finite, which, as already mentioned, is the case in all known dualities to date. Under this assumption, the result also generalizes Theorem 3.1 of [3].

Corollary 3.9 ([18], Proposition 2.1)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathscr{D} = \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. If $\underline{\mathbf{D}}$ is dualisable via a finite set of relations, then $\underline{\mathbf{M}}$ is dualisable.*

More specifically, assume that $\underset{\sim}{\mathbf{M}} = \langle D; G, H, R, \mathcal{T}\rangle$ yields a duality on $\mathscr{D}$ such that $G \cup H \cup R$ is a finite set of finitary algebraic partial operations and relations on D. Then $\underset{\sim}{\mathbf{M}} = \langle M; \Gamma_{\beta\alpha}, G_\beta \cup H_\beta, R_\beta, \mathcal{T}\rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{M}}) = \mathscr{D}$, where

(a) *$\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ is a one-to-one homomorphism;*

(b) *$\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ is a one-to-one homomorphism;*

(c) *$G_\beta := \{\, g_\beta \mid g \in G\,\}$;*

(d) *$H_\beta := \{\, h_\beta \mid h \in H\,\}$;*

(e) *$R_\beta := \{\, r_\beta \mid r \in R\,\} \cup \{\, \beta(D)\,\}$.*

P r o o f. If $\underline{\mathbf{D}}$ is dualisable via a finite set $G \cup H \cup R$, then $G \cup H \cup R$ is entailment-dense in $\mathscr{B}_D$ by Lemma 2.5. Hence by Theorem 3.8(i), $\Gamma_{\beta\alpha} \cup G_\beta \cup H_\beta \cup R_\beta$ is entailment-dense in $\mathscr{B}_M$. Since this set is finite, too, it dualises $\underline{\mathbf{M}}$ by applying Lemma 2.5 again. $\square$

4. Transferring entailment down

Now we concentrate on studying the transferral of entailment "down" from $\underline{\mathbf{M}}$ onto $\underline{\mathbf{D}}$.

Throughout this section we assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that for the quasi-varieties $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and $\mathscr{M} := \mathbb{ISP}(\underline{\mathbf{M}})$ we have that $\mathscr{D} = \mathscr{M}$. Moreover, we assume that there exist one-to-one homomorphisms $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$.

Definition 4.1. For every $\mathbf{A} \in \mathscr{D}$ and every map $u\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ that preserves $\alpha(M)$, we define a map $u_M\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ by

$$u_M(x) = \alpha^{-1}((u(\varrho_1 \circ \alpha \circ x), \ldots, u(\varrho_k \circ \alpha \circ x))),$$

where α^{-1} is the inverse of the isomorphism from $\underline{\mathbf{M}}$ onto $\alpha(\underline{\mathbf{M}})$ given by α.

The next lemma follows immediately from the proof of Proposition 2.2 of [18]. We note that the relations r_α and the partial operations h_α have been introduced in the previous section after Corollary 3.5.

Lemma 4.2. *Let $\mathbf{A} \in \mathscr{D}$ and let $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ be an embedding.*

(i) *If $u\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ is a map that preserves $\alpha(M)$ and r_α for some relation r on M, then the map u_M preserves r.*

(ii) *If $u\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ preserves $\{\varrho_1 \circ h_\alpha, \ldots, \varrho_k \circ h_\alpha\}$ and $\alpha(M)$, for some (partial) operation h on M, then the map u_M preserves h.*

We now prove our first result on the transferral of entailment "down" from $\underline{\mathbf{M}}$ onto $\underline{\mathbf{D}}$.

Theorem 4.3. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite generators of the same quasi-variety and let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, be one-to-one homomorphisms.*

If $G \cup H \cup R$ entails an m-ary relation s on M ($m \geqslant 1$), then $G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$ entails the relation s_α on D, where

(a) $G_\alpha := \{\, \varrho_1 \circ g_\alpha, \ldots, \varrho_k \circ g_\alpha \mid\ g \in G \,\}$;

(b) $H_\alpha := \{\, \varrho_1 \circ h_\alpha, \ldots, \varrho_k \circ h_\alpha \mid\ h \in H \,\}$;

(c) $R_\alpha := \{\, r_\alpha \mid r \in R\}$.

P r o o f. Let $u\colon \mathscr{D}(\mathbf{s}_\alpha, \underline{\mathbf{D}}) \to D$ preserve $G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$. Let us further assume that $(x_{11}, \ldots, x_{1k}, \ldots, x_{m1}, \ldots, x_{mk}) \in s_\alpha$ with $x_{ij} \in \mathscr{D}(\mathbf{s}_\alpha, \underline{\mathbf{D}})$.

By Lemma 4.2, the map $u_M\colon \mathscr{D}(\mathbf{s}_\alpha, \underline{\mathbf{M}}) \to M$ preserves $G \cup H \cup R$ and so it preserves s provided $G \cup H \cup R$ entails s.

For every $i \in \{1, \ldots, m\}$ we have $(x_{i1}, \ldots, x_{ik}) \in \alpha(M)$ and so we can define a map $y_i := \alpha^{-1} \circ \prod_{j \in \{1,\ldots,k\}} x_{ij} \in \mathscr{D}(\mathbf{s}_\alpha, \underline{\mathbf{M}})$. Since $(y_1, \ldots, y_m) \in s$, we have $(u_M(y_1), \ldots, u_M(y_m)) \in s$. But then

$$\begin{aligned}
&(u(x_{11}), \ldots, u(x_{1k}), \ldots, u(x_{m1}), \ldots, u(x_{mk})) \\
&\quad = (u(\varrho_1 \circ \alpha \circ y_1), \ldots, u(\varrho_k \circ \alpha \circ y_1), \ldots, u(\varrho_1 \circ \alpha \circ y_m), \ldots, u(\varrho_k \circ \alpha \circ y_m)),
\end{aligned}$$

which belongs to s_α as required. □

Proposition 4.4. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite generators of the same quasi-variety and let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, be one-to-one homomorphisms.*

If $\Gamma_{\beta\alpha} \cup \{s_\beta\}$ entails r_β on $\underline{\mathbf{M}}$ for finitary relations r, s on D, then $\mathrm{End}(\underline{\mathbf{D}}) \cup \{s, \alpha(M)\}$ *entails r on $\underline{\mathbf{D}}$.*

Proof. Let $r \subseteq D^m$ and $s \subseteq D^n$ be such that $\Gamma_{\beta\alpha} \cup \{s_\beta\}$ entails r_β on $\underline{\mathbf{M}}$. Let $u\colon \mathscr{D}(\mathbf{r}, \underline{\mathbf{D}}) \to D$ be a map that preserves $\mathrm{End}(\underline{\mathbf{D}}) \cup \{s, \alpha(M)\}$. We claim that the map $u_M\colon \mathscr{D}(\mathbf{r}, \underline{\mathbf{M}}) \to M$ preserves $\Gamma_{\beta\alpha} \cup \{s_\beta\}$.

For every $i \in \{1, \ldots, k\}$ and $x \in \mathscr{D}(\mathbf{r}, \underline{\mathbf{M}})$ we have that

$$\begin{aligned}
u_M(\omega_i(x)) = u_M(\omega_i \circ x) &= \alpha^{-1}(u(\varrho_1 \circ \alpha \circ \omega_i \circ x), \ldots, u(\varrho_k \circ \alpha \circ \omega_i \circ x)) \\
&= \alpha^{-1}(u(\varrho_1 \circ \alpha \circ \beta \circ \varrho_i \circ \alpha \circ x), \ldots, u(\varrho_k \circ \alpha \circ \beta \circ \varrho_i \circ \alpha \circ x)) \\
&= \alpha^{-1}(\varrho_1 \circ \alpha \circ \beta(u(\varrho_i \circ \alpha \circ x)), \ldots, \varrho_k \circ \alpha \circ \beta(u(\varrho_i \circ \alpha \circ x))) \\
&= \alpha^{-1} \circ \alpha \circ \beta(u(\varrho_i \circ \alpha \circ x)) = \beta(u(\varrho_i \circ \alpha \circ x)) \\
&= \omega_i(u_M(x))
\end{aligned}$$

and so u_M preserves $\Gamma_{\beta\alpha}$.

For every $(x_1, \ldots, x_n) \in s_\beta$ we have that

$$(\alpha \circ x_1, \ldots, \alpha \circ x_n) \in (\alpha \circ \beta)(s),$$

whence

$$((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1, \ldots, (\alpha \circ \beta)^{-1} \circ \alpha \circ x_n) \in s.$$

Since u preserves s, we obtain

$$(u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1), \ldots, u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_n)) \in s,$$

whence the tuple

$$\begin{aligned}
&(\varrho_1 \circ \alpha \circ \beta(u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1)), \ldots, \varrho_k \circ \alpha \circ \beta(u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1)) \ldots \\
&\varrho_1 \circ \alpha \circ \beta(u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_n)), \ldots, \varrho_k \circ \alpha \circ \beta(u((\alpha \circ \beta)^{-1} \circ \alpha \circ x_n)))
\end{aligned}$$

belongs to $(\alpha\circ\beta)(s)$. As u preserves End($\underline{\mathbf{D}}$), we have that

$$\begin{aligned}\varrho_i\circ\alpha\circ\beta(u((\alpha\circ\beta)^{-1}\circ\alpha\circ x_j)) &= u(\varrho_i\circ\alpha\circ\beta\circ(\alpha\circ\beta)^{-1}\circ\alpha\circ x_j)\\ &= u(\varrho_i\circ\alpha\circ x_j)\end{aligned}$$

and hence

$$u_M(x_j)=\alpha^{-1}(\varrho_1\circ\alpha\circ\beta(u((\alpha\circ\beta)^{-1}\circ\alpha\circ x_j)),\dots,\varrho_k\circ\alpha\circ\beta(u((\alpha\circ\beta)^{-1}\circ\alpha\circ x_j))).$$

Thus we finally obtain

$$(u_M(x_1),\dots,u_M(x_n))\in\alpha^{-1}((\alpha\circ\beta)(s))=s_\beta.$$

Hence u_M preserves s_β, and consequently, u_M preserves r_β.

Now we take $(y_1,\dots,y_m)\in r$ with $y_i\in\mathscr{D}(\mathbf{r},\underline{\mathbf{D}})$. Then

$$(\beta\circ y_1,\dots,\beta\circ y_m)\in r_\beta,$$

and as u_M preserves r_β,

$$(u_M(\beta\circ y_1),\dots,u_M(\beta\circ y_m))\in r_\beta.$$

Therefore the tuple

$$\begin{gathered}((u(\varrho_1\circ\alpha\circ\beta\circ y_1),\dots,u(\varrho_k\circ\alpha\circ\beta\circ y_1)),\dots\\ (u(\varrho_1\circ\alpha\circ\beta\circ y_m),\dots,u(\varrho_k\circ\alpha\circ\beta\circ y_m)))\end{gathered}$$

belongs to $\alpha(r_\beta)$, thus to $(\alpha\circ\beta)(r)$. This means that

$$\begin{gathered}((\varrho_1\circ\alpha\circ\beta(u(y_1)),\dots,\varrho_k\circ\alpha\circ\beta(u(y_1))),\dots\\ (\varrho_1\circ\alpha\circ\beta(u(y_m)),\dots,\varrho_k\circ\alpha\circ\beta(u(y_m))))\end{gathered}$$

belongs to $(\alpha\circ\beta)(r)$, whence

$$(\alpha\circ\beta(u(y_1)),\dots,\alpha\circ\beta(u(y_m)))\in(\alpha\circ\beta)(r).$$

Consequently, $(u(y_1),\dots,u(y_m))\in r$. □

Before moving to the main results, we need one technical lemma.

Lemma 4.5. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite generators of the same quasi-variety and let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, be one-to-one homomorphisms. The set* $\mathrm{End}(\underline{\mathbf{D}}) \cup \{\alpha(M)\}$ *entails* $\varrho_j \circ (\omega_i)_\alpha$ *for every* $i, j \in \{1, \dots, k\}$.

P r o o f. Let us take $i, j \in \{1, \dots, k\}$ and denote $\varrho_j \circ (\omega_i)_\alpha$ by h. Let $\mathbf{A} \in \mathscr{D}$ and let $u\colon \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ be a continuous map preserving $\mathrm{End}(\underline{\mathbf{D}}) \cup \{\alpha(M)\}$. Note that $\operatorname{dom} h = \alpha(M)$ and so for every $(x_1, \dots, x_k) \in \operatorname{dom} h$ with $x_1, \dots, x_k \in \mathscr{D}(\mathbf{A}, \underline{\mathbf{D}})$ we have $(u(x_1), \dots, u(x_k)) \in \operatorname{dom} h$. Also

$$\begin{aligned} h((x_1, \dots, x_k)) &= \varrho_j((\omega_i)_\alpha(x_1, \dots, x_k)) \\ &= \varrho_j(\alpha \circ \omega_i(\alpha^{-1}(x_1, \dots, x_k))) = \varrho_j \circ \alpha \circ \beta(x_i), \end{aligned}$$

and thus

$$\begin{aligned} u(h((x_1, \dots, x_k))) &= u(\varrho_j \circ \alpha \circ \beta(x_i)) \\ &= \varrho_j \circ \alpha \circ \beta(u(x_i)) = h(u(x_1), \dots, u(x_k)). \end{aligned}$$

□

Theorem 4.6. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite generators of the same quasi-variety and let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, be one-to-one homomorphisms.*

If $\Gamma_{\beta\alpha} \cup G \cup H \cup R$ is entailment-dense in $\mathscr{B}_M$, then $\mathrm{End}(\underline{\mathbf{D}}) \cup G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$ *is entailment-dense in $\mathscr{B}_D$.*

P r o o f. Let $r \in \mathscr{B}_D$. By assumption, $\Gamma_{\beta\alpha} \cup G \cup H \cup R$ entails r_β, so by Lemma 4.5 and Theorem 4.3, $\mathrm{End}(\underline{\mathbf{D}}) \cup G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$ entails the relation $(r_\beta)_\alpha$ (we note here that for each ω_i we have that $(\omega_i)_\alpha = \prod_{j \in \{1, \dots, k\}} \varrho_j \circ (\omega_i)_\alpha$). As, by Lemma 3.7, $\Gamma_{\beta\alpha} \cup \{((r_\beta)_\alpha)_\beta\}$ entails r_β, we have that $\mathrm{End}(\underline{\mathbf{D}}) \cup \{(r_\beta)_\alpha, \alpha(M)\}$ entails r, by Proposition 4.4. Consequently, $\mathrm{End}(\underline{\mathbf{D}}) \cup G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$ entails r as required. □

Corollary 4.7. *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underline{\mathbf{D}} \in \mathbb{ISP}(\underline{\mathbf{M}})$ be a finite algebra. Assume that $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ is a one-to-one homomorphism for some $k \geqslant 1$. If $\underline{\mathbf{M}}$ is dualisable via a finite set $\{\omega_1, \dots, \omega_k\} \cup G \cup H \cup R$, then $\underline{\mathbf{D}}$ is dualisable via the set* $\mathrm{End}(\underline{\mathbf{D}}) \cup G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$.

P r o o f. It follows from Theorems 4.6 and 2.1. □

From Corollary 3.9, Corollary 4.7 and the Duality Compactness Theorem we immediately obtain the following slight restriction of the important result of [18] (cf. [18], Theorem 2.3). (We again note that our restriction on the dualisability, considering it via a finite set of relations, is satisfied in all known dualities to date, so in practice it is no restriction.)

Theorem 4.8. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras such that $\underline{\mathbf{M}} \in \mathbb{ISP}(\underline{\mathbf{D}})$ and $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Then $\underline{\mathbf{M}}$ is dualisable via a finite set of relations if and only if $\underline{\mathbf{D}}$ is dualisable via a finite set of relations.*

Finally, as a consequence of Theorem 4.8, we obtain a new proof of the following result (cf. [18], Theorem 2.5) which can be interpreted so that dualisability of a quasi-variety is independent of the generating algebra.

Theorem 4.9. *Let $\underline{\mathbf{M}}$ be a finite algebra. If $\underline{\mathbf{M}}$ is dualisable via a finite set of relations then every finite algebra $\underline{\mathbf{D}}$ that generates $\mathbb{ISP}(\underline{\mathbf{M}})$ is dualisable via a finite set of relations, as well.*

Proof. We can assume that there exist one-to-one homomorphisms $\underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ and $\underline{\mathbf{D}} \to \underline{\mathbf{M}}^m$ for some $k, m \geqslant 1$. By Theorem 4.8 applied to the algebras $\underline{\mathbf{M}}$ and $\underline{\mathbf{M}}^m$, we get that $\underline{\mathbf{M}}^m$ is dualisable. But then, by applying Theorem 4.8 to the algebras $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}^m$, we conclude that $\underline{\mathbf{D}}$ is dualisable, as well. $\square$

5. Endodualisability

In Section 3 we showed that the relation $\beta(D)$ is entailed from $\operatorname{End}(\underline{\mathbf{M}})$ whenever $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$. Now we show that also $\operatorname{End}(\underline{\mathbf{D}})_\beta$ is entailed from $\operatorname{End}(\underline{\mathbf{M}})$ under the same assumptions.

Lemma 5.1. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one and $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$. Then $\operatorname{End}(\underline{\mathbf{M}})$ entails $\operatorname{End}(\underline{\mathbf{D}})_\beta$.*

Proof. As for every $g \in \operatorname{End}(\underline{\mathbf{D}})$, the graph of $g_\beta \in \operatorname{End}(\underline{\mathbf{D}})_\beta$ is isomorphic to its domain $\beta(\underline{\mathbf{D}})$, which is entailed from $\operatorname{End}(\underline{\mathbf{M}})$ by Lemma 3.4, it suffices to show that for any map $u\colon \mathscr{M}(\beta(\underline{\mathbf{D}}), \underline{\mathbf{M}}) \to M$ preserving $\operatorname{End}(\underline{\mathbf{M}})$ and for any $x \in \mathscr{M}(\beta(\underline{\mathbf{D}}), \underline{\mathbf{M}})$ such that $x \in \beta(D)$,

$$u(g_\beta(x)) = g_\beta(u(x)).$$

Similarly to the proof of Lemma 3.4 for every $a \in \beta(D)$ there exists $b \in D$ such that

$$\begin{aligned} g_\beta(x)(a) = g_\beta(x(a)) = g_\beta(\beta(b)) &= g_\beta((\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ \beta)(b)) \\ &= (g_\beta \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(x(a)), \end{aligned}$$

whence

$$g_\beta(x) = (g_\beta \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(x)$$

and

$$u(g_\beta(x)) = u((g_\beta \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(x)).$$

As u preserves $g_\beta \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \in \mathrm{End}(\underline{\mathbf{M}})$ and $u(x) \in \beta(D)$, we finally obtain

$$u(g_\beta(x)) = (g_\beta \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha)(u(x)) = g_\beta(u(x))$$

as required. □

From this and Theorem 3.8 (ii) we immediately obtain

Corollary 5.2. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ for some $k \geqslant 1$, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one and $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$.*

If $\mathrm{End}(\underline{\mathbf{D}})$ is entailment-dense in $\mathscr{B}_D$, then $\mathrm{End}(\underline{\mathbf{M}})$ is entailment-dense in $\mathscr{B}_M$.

The conditions of the above result are in particular satisfied if $\underline{\mathbf{D}}$ is, up to isomorphism, a retract of $\underline{\mathbf{M}}$. Therefore the following result generalizes a result obtained previously by the first author (see [18], Proposition 3.1), and independently by B. A. Davey and J. G. Pitkethly (see [9], Theorem 1.5 (i)), which says that if $\underline{\mathbf{D}}$ is a retract of $\underline{\mathbf{M}}$, then the endodualisability of $\underline{\mathbf{D}}$ yields the endodualisability of $\underline{\mathbf{M}}$.

Theorem 5.3. *Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathbb{ISP}(\underline{\mathbf{D}})$ such that there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one and $\alpha(\underline{\mathbf{M}}) = (\alpha \circ \beta)(\underline{\mathbf{D}})$.*

If $\underline{\mathbf{D}}$ is endodualisable then $\underline{\mathbf{M}}$ is endodualisable.

Proof. If $\underline{\mathbf{D}}$ is endodualisable, then $\mathrm{End}(\underline{\mathbf{D}})$ is entailment-dense in $\mathscr{B}_D$ by Lemma 2.5. By Corollary 5.2, $\mathrm{End}(\underline{\mathbf{M}})$ is entailment-dense in $\mathscr{B}_M$. Since $\underline{\mathbf{M}}$ is finite, Lemma 2.5 now implies that $\mathrm{End}(\underline{\mathbf{M}})$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{M}})$, thus $\underline{\mathbf{M}}$ is endodualisable. □

6. Examples

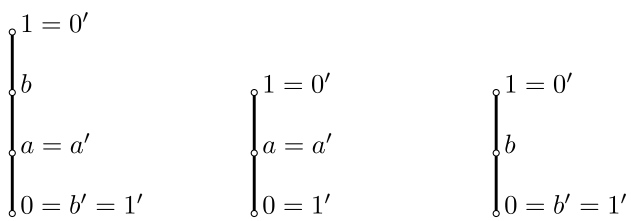
In this section we present two examples of lattice based algebras for which a dualising set can be obtained by using known dualising sets of some of their subalgebras and by applying the results of Section 3. The procedure we use in both examples can be applied to any quasi-variety generated by a finite lattice based algebra $\underline{\mathbf{M}}$ which admits a set of dualisable subalgebras as the set of the domains of the partial endomorphims of $\underline{\mathbf{M}}$. We begin by describing this general procedure.

Let $\mathscr{M} = \mathbb{ISP}(\underline{\mathbf{M}})$ be the quasi-variety generated by a finite lattice based algebra $\underline{\mathbf{M}}$. Assume that for every partial endomorphism h of $\underline{\mathbf{M}}$ a dualising set for its domain $\underline{\mathbf{D}}_h$ is already known. Let $\{\underline{\mathbf{D}}_1, \dots, \underline{\mathbf{D}}_n\}$ be the set of the domains of the partial endomorphisms of $\underline{\mathbf{M}}$. For each $i \in \{1, \dots, n\}$, let Ω_i be the dualising set of $\underline{\mathbf{D}}_i$. Since Ω_i entails every total or partial endomorphism of $\underline{\mathbf{D}}_i$, we can get a set of relations Ω_i^M on $\underline{\mathbf{M}}$ that entails the partial endomorphisms of $\underline{\mathbf{M}}$ having $\underline{\mathbf{D}}_i$ as their domain, by applying the results of Section 3. Then the union Ω_p^M of all those sets Ω_i^M entails the set of partial endomorphisms of $\underline{\mathbf{M}}$. Hence the union of a generating set Ω_e^M of $\operatorname{End}\underline{\mathbf{M}}$ with Ω_p^M entails all the endomorphisms and partial endomorphisms of $\underline{\mathbf{M}}$. From the general theory of failsets in [17] and [19] (see also [1], section 8.3) it follows that $\Omega_e^M \cup \Omega_p^M \cup T$ entails $\mathbb{S}(\underline{\mathbf{M}}^2)$, where T is the so-called transversal of the globally minimal failsets without partial endomorphisms. In the particular case of a distributive lattice based algebra, one can apply the algorithmic procedure given in [19], Section 3, and obtain T. Since $\Omega := \mathbb{S}(\underline{\mathbf{M}}^2)$ is known to yield a duality on $\mathscr{M}$ (cf. [1], p. 55), we apply Lemma 2.4 to conclude that the set $R := \Omega_e^M \cup \Omega_p^M \cup T \subseteq \Omega$ also yields a duality on $\mathscr{M}$.

Below we illustrate by two examples the process of finding the set $\Omega_e^M \cup \Omega_p^M \cup T$, and even a simpler set R that dualises $\underline{\mathbf{M}}$.

6.1. The subvariety of Ockham algebras generated by the four element chain $\underline{K}_2$.

Let $\mathscr{M} = \mathbb{ISP}(\underline{\mathbf{K}}_2)$ where $\underline{\mathbf{K}}_2$ is the Ockham four-element chain $\langle\{0, a, b, 1\}; \wedge, \vee, ', 0, 1\rangle$ such that $a' = a$ and $b' = 0$. Every partial endomorphism of $\underline{\mathbf{K}}_2$ is a total or partial endomorphism of one of its two three-element subalgebras, $\underline{\mathbf{K}} = \langle\{0, a, 1\}; \wedge, \vee, ', 0, 1\rangle$, which is the generating algebra of the variety of Kleene algebras, and $\underline{\mathbf{S}} = \langle\{0, b, 1\}; \wedge, \vee, ', 0, 1\rangle$, which is the generating algebra of the variety of Stone algebras. Let $\underline{\mathbf{D}}_1 := \underline{\mathbf{K}}$ and $\underline{\mathbf{D}}_2 := \underline{\mathbf{S}}$.



$\mathbf{K}_2$ $\qquad$ $\mathbf{K}$ $\qquad$ $\mathbf{S}$

It is well-known (cf. [1], Theorem 4.3.10), that

$$\Omega_1 := \{\{0, 1\}, \preccurlyeq_K, \sim_K\}$$

is a dualising set for $\underline{\mathbf{K}}$, where

$$\preccurlyeq_K = \{(0,0),(0,a),(a,a),(1,a),(1,1)\}$$

is a partial order on K and

$$\sim_K = \{(0,0),(0,a),(a,0),(a,a),(a,1),(1,a),(1,1))\}.$$

Further (cf. [1], Theorem 4.3.7), $\Omega_2 = \{f, \preccurlyeq_S\}$ is a dualising set for $\underline{\mathbf{S}}$, where f is the endomorphism of $\underline{\mathbf{S}}$ that maps b to 1, and $\preccurlyeq_S$ is the partial order $\{(0,0),(b,b),(1,b),(1,1)\}$ on S.

Now for $i \in \{1,2\}$ we want to obtain a set of relations $\Omega_i^{K_2}$ on K_2 that entails the partial endomorphisms of $\underline{\mathbf{K}}_2$ having $\underline{\mathbf{D}}_i$ as their domain, by applying the results of Section 3. First we take the embeddings $\beta_K\colon \underline{\mathbf{K}} \to \underline{\mathbf{K}}_2$ and $\beta_S\colon \underline{\mathbf{S}} \to \underline{\mathbf{K}}_2$ to be the inclusion maps and take the homomorphism $\alpha\colon \underline{\mathbf{K}}_2 \to \underline{\mathbf{K}}$ to be the retraction given by $\alpha(0) = 0, \alpha(a) = a$ and $\alpha(b) = \alpha(1) = 1$. By Corollary 3.5,

$$\Omega_1^{K_2} := \operatorname{End} K_2 \cup \{\{0,1\}_{\beta_K}, (\preccurlyeq_K)_{\beta_K}, (\sim_K)_{\beta_K}\}$$

entails $(\operatorname{End} K \cup \operatorname{End}_p K)_{\beta_K}$, and consequently it entails all the partial endomorphisms of $\underline{\mathbf{K}}_2$ whose domain is (a subalgebra of) $\underline{\mathbf{K}}$. By Theorem 3.3,

$$\Omega_2^{K_2} := \{f_{\beta_S}, (\preccurlyeq_S)_{\beta_S}, \beta_S(S)\}$$

entails $(\operatorname{End} S \cup \operatorname{End}_p S)_{\beta_S}$, so it entails all the partial endomorphisms of $\underline{\mathbf{K}}_2$ whose domain is (a subalgebra of) $\underline{\mathbf{S}}$. Then the set

$$\Omega_p^{K_2} := \operatorname{End} K_2 \cup \{\{0,1\}, (\preccurlyeq_K)_{\beta_K}, (\sim_K)_{\beta_K}, f_{\beta_S}, (\preccurlyeq_S)_{\beta_S}, S\}$$

entails $\operatorname{End}_p K_2$.

Obviously, $\operatorname{End} K_2 = \{\mathrm{id}, \alpha\}$, thus $\Omega_e^{K_2} = \{\alpha\}$.

In [19], the authors determined one of the transversals T by applying the algorithm developed there. They obtained $T = \{\preccurlyeq, \sim\}$, where $\preccurlyeq$ is the partial order

$$\{(0,0),(0,a),(a,a),(1,a),(1,1),(1,b),(b,b)\}$$

on K_2 and

$$\sim = \{(0,0),(0,a),(a,0),(a,a),(a,b),(a,1),(b,a),(b,b),(b,1),(1,a),(1,b),(1,1)\}.$$

We now conclude that

$$\{\alpha\} \cup \{\{0,1\}, (\preccurlyeq_K)_{\beta_K}, (\sim_K)_{\beta_K}, f_{\beta_S}, (\preccurlyeq_S)_{\beta_S}, S\} \cup \{\preccurlyeq, \sim\}$$

entails $\Omega = \mathbb{S}(\underline{\mathbf{K}}_2^2)$.

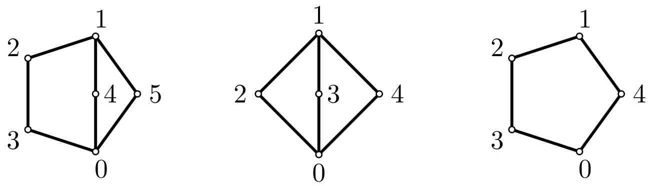
Now note that

- $\triangle_{\{0,1\}} = (\preccurlyeq_K)_{\beta_K} \cap (\preccurlyeq_S)_{\beta_S}$;
- $(\preccurlyeq_K)_{\beta_K} = \preccurlyeq \cap \beta_K(K)^2$;
- $(\sim_K)_{\beta_K} = \sim \cap \beta_K(K)^2$;
- $\operatorname{graph} f_{\beta_S} = (\operatorname{graph} \alpha) \cap \beta_S(S)^2$;
- $(\preccurlyeq_S)_{\beta_S} = \preccurlyeq \cap \beta_S(S)^2$.

Consequently, from the constructs for entailment (cf. [1]), p. 57), one can conclude that the set $R := \{\alpha, S, \preccurlyeq, \sim\}$ entails $\Omega = \mathbb{S}(\underline{\mathbf{K}}_2^2)$, and thus, by Lemma 2.4, R dualises $\underline{\mathbf{K}}_2$. The set R is in fact the set that yields the piggyback duality on $\mathscr{M} = \mathbb{ISP}(\underline{\mathbf{K}}_2)$ (for the piggyback dualities see Section 7 of [1]).

6.2. The irregular diamond $\underline{\mathbf{V}}_1$ and the diamond $\underline{\mathbf{M}}_3$.

Let $\mathscr{V} = \mathbb{ISP}(\underline{\mathbf{V}}_1)$ be the variety of lattices generated by the irregular diamond $\underline{\mathbf{V}}_1$. We will consider its subvarieties $\mathbb{ISP}(\underline{\mathbf{M}}_3)$, which is the variety of modular lattices, and $\mathbb{ISP}(\underline{\mathbf{N}}_5)$, which is the variety generated by the pentagon $\underline{\mathbf{N}}_5$.



$\mathbf{V}_1$ $\mathbf{M}_3$ $\mathbf{N}_5$

Every partial endomorphism of $\underline{\mathbf{V}}_1$ is entailed by a total or partial endomorphism of $\underline{\mathbf{M}}_3$ or $\underline{\mathbf{N}}_5$. So let $\underline{\mathbf{D}}_1 := \underline{\mathbf{M}}_3$ and $\underline{\mathbf{D}}_2 := \underline{\mathbf{N}}_5$. Let $\beta_3\colon \underline{\mathbf{M}}_3 \to \underline{\mathbf{V}}_1$ and $\beta_5\colon \underline{\mathbf{N}}_5 \to \underline{\mathbf{V}}_1$ be the embeddings such that $\beta_3(2) = 2, \beta_3(3) = 4, \beta_3(4) = 5$ and $\beta_5(4) = 4, \beta_5(2) = 2, \beta_5(3) = 3$. Then β_3 is a co-retraction. C. B. Wegener (cf. [21], p. 41) proved that

$$\Omega_1 := \{\{1,2\}, (23), (34)\cdot \geqslant_3, \diamondsuit_3\}$$

dualises $\underline{\mathbf{M}}_3$,and hence the set

$$\{\{1,2\}, (23), (34), \geqslant_3, \diamondsuit_3\}$$

entails $\mathscr{B}_{M_3}$, where (23) and (34) are automorphisms of $\underline{\mathbf{M}}_3$, $\geqslant_3 = \leqslant_3^{\smile}$ with $\leqslant_3$ the lattice order of $\underline{\mathbf{M}}_3$, $\diamondsuit_3 := (\{1,2\} \times M_3) \cup (M_3 \times \{2,0\})$ and $(34)\cdot \geqslant_3$ denotes the action by the automorphism (34) on the relation $\geqslant_3$ (cf. [1], p. 59). Then, by Corollary 3.5, the set

$$\Omega_1^{V_1} := \operatorname{End} \underline{\mathbf{V}}_1 \cup \{\{1,2\}, (23)_{\beta_3}, (34)_{\beta_3}, (\geqslant_3)_{\beta_3}, (\diamondsuit_3)_{\beta_3}\}$$

entails $(\operatorname{End} M_3 \cup \operatorname{End}_p M_3)_{\beta_3}$. Wegener (cf. [21], p. 46) also proved that the set

$$\Omega_2 := \{\{1,2\}, \{(1,1),(2,4),(3,4),(4,2),(0,0)\}, \geqslant_5, \diamondsuit_5\}$$

dualises $\underline{\mathbf{N}}_5$, where $\geqslant_5 = \leqslant_5^{\smile}$ with $\leqslant_5$ the lattice order of $\underline{\mathbf{M}}_5$, and

$$\diamondsuit_5 := (\{1,2\} \times N_5) \cup (N_5 \times \{3,0\}).$$

Then, by Theorem 3.3, the set

$$\Omega_2^{V_1} := \big\{\{1,2\}, \{(1,1),(2,4),(3,4),(4,2),(0,0)\}, (\geqslant_5)_{\beta_5}, (\diamondsuit_5)_{\beta_5}\big\}$$

entails $(\operatorname{End} N_5 \cup \operatorname{End}_p N_5)_{\beta_5}$. Consequently, we have that the set $\Omega_p^{V_1} := \operatorname{End} \underline{\mathbf{V}}_1 \cup \big\{\{1,2\},\ (23)_{\beta_3},\ (34)_{\beta_3},\ (\geqslant_3)_{\beta_3},\ (\diamondsuit_3)_{\beta_3},\ \{(1,1),(2,4),(3,4),(4,2),(0,0)\},\ (\geqslant_5)_{\beta_5},\ (\diamondsuit_5)_{\beta_5}\big\}$ entails $\operatorname{End}_p \underline{\mathbf{V}}_1$.

Since the globally minimal failsets without partial endomorphisms admit $\{\diamondsuit, \geqslant\}$ as a transversal (cf. [21], p. 49), where

$$\diamondsuit := (\{1,2\} \times V_1) \cup (V_1 \times \{3,0\})$$

and $\geqslant$ is the converse of the lattice order $\leqslant$ of V_1, we conclude that the set $\operatorname{End} \underline{\mathbf{V}}_1 \ \cup\ \big\{\{1,2\}, (23)_{\beta_3}, (34)_{\beta_3}, (\geqslant_3)_{\beta_3}, (\diamondsuit_3)_{\beta_3}, \{(1,1),(2,4),(3,4),(4,2),(0,0)\} \ \cup\ \{(\geqslant_5)_{\beta_5}, (\diamondsuit_5)_{\beta_5}\} \cup \{\diamondsuit, \geqslant\}\big\}$ entails $\Omega = \mathbb{S}(\underline{\mathbf{V}}_1^2)$.

We note that

- $(23)_{\beta_3}$ and $(34)_{\beta_3}$ are entailed by $\operatorname{End} \underline{\mathbf{V}}_1 \cup \{\beta_3(M_3)\}$;
- $\{(1,1),(2,4),(3,4),(4,2),(0,0)\} = f\restriction_{\beta_5(N_5)}$, where $f \in \operatorname{End} \underline{\mathbf{V}}_1$ maps 2 and 3 to 4, 4 to 2 and fixes the other elements;
- $(\geqslant_3)_{\beta_3} = \geqslant \cap\, \beta_3(M_3)^2$ and $(\geqslant_5)_{\beta_5} = \geqslant 1 \cap \beta_5(N_5)^2$;
- $(\diamondsuit_3)_{\beta_3} = (\beta_3 \circ \alpha)(\diamondsuit)$ and $(\diamondsuit_5)_{\beta_5} = \diamondsuit \cap \beta_5(N_5)^2$;
- Lemma 3.4 implies that $\beta_3(M_3)$ is entailed by $\operatorname{End} \underline{\mathbf{V}}_1$;
- $\beta_5(N_5) = \pi_1(\operatorname{graph} f_1 \cap \operatorname{graph} f_2)$, where $f_1, f_2 \in \operatorname{End} \underline{\mathbf{V}}_1$ are defined by $f_i(2) = f_i(3) = 5$, $f_i(4) = 4$, $f_1(5) = 2$ and $f_2(5) = 3$.

Consequently (cf. [1], p. 57 again), the set $R := \operatorname{End} \underline{\mathbf{V}}_1 \cup \{\{1,2\}, \geqslant, \diamondsuit\}$ entails $\Omega = \mathbb{S}(\underline{\mathbf{V}}_1^2)$ and therefore, by Lemma 2.4, R dualises $\underline{\mathbf{V}}_1$.

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