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ON AN INCLUSION BETWEEN OPERATOR IDEALS

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Abstract. Let $1 \leqslant q < p < \infty$ and $1/r := 1/p \max(q/2, 1)$. We prove that $\mathscr{L}^{(c)}_{r,p}$, the ideal of operators of Gel'fand type $l_{r,p}$, is contained in the ideal $\Pi_{p,q}$ of (p,q)-absolutely summing operators. For $q > 2$ this generalizes a result of G. Bennett given for operators on a Hilbert space.

Keywords: operator ideals, s-numbers

MSC 2010: 47L20, 47B06

1. Notation

Throughout this note standard definitions concerning the theory of operator ideals are taken from [9] and [10]. For the convenience of the reader we here collect some of them.

In the following E and F denote real or complex Banach spaces. $\mathscr{L}(E,F)$ is the Banach space of all (bounded linear) operators acting from E into F.

If $T \in \mathscr{L}(E,F)$ and $n = 1, 2, \ldots$, then the n-th *approximation number* and the *Gel'fand number* are defined by

$$a_n(T) := \inf\{\|T - L\| \colon \operatorname{rank}(L) < n\}$$

and

$$c_n(T) := \inf\{\|T J^E_M\| \colon \operatorname{codim}(M) < n\}$$

respectively, where J^E_M denotes the embedding map from M into E. Let $0 < r$, $w < \infty$. The quasi-Banach operator ideal $\mathscr{L}^{(a)}_{r,w}$ consists of all operators T such that

$$\|T \mid \mathscr{L}^{(a)}_{r,w}\| := \left(\sum_{n=1}^{\infty} [n^{1/r-1/w} a_n(T)]^w\right)^{1/w} < \infty.$$

The quasi-Banach operator ideal $\mathscr{L}^{(c)}_{r,w}$ determined by the Gel'fand numbers is defined in the same way.

Let $1 \leqslant q \leqslant p < \infty$. An operator $T \in \mathscr{L}(E, F)$ is called *absolutely* (p, q)*-summing* if there exists a constant c such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{1/p} \leqslant c \sup\Big\{\Big(\sum_{k=1}^{n} |\ \langle x_k, a\rangle\ |^q\Big)^{1/q} :\ \|a\| \leqslant 1\Big\}$$

for all finite families of elements $x_1, \dots, x_n \in E$. The class of these operators is denoted by $\Pi_{p,q}$. It follows that $\Pi_{p,q}$ becomes a Banach operator ideal if we define

$$\|T \mid \Pi_{p,q}\| := \inf c.$$

For $1 \leqslant q \leqslant s \leqslant \infty$, $[\mathtt{M}_{s,q}, \mu_{s,q}]$ denotes the Banach operator ideal of (s, q)-mixing operators (cf. also [9, (20.1.1)]).

Excellent references for the interpolation theory are [3] and [11]. We recall the so-called *real-interpolation method*. Let $0 < \theta < 1$ and $0 < w < \infty$. For every quasi-Banach interpolation couple (E_0, E_1) we denote by $(E_0, E_1)_{\theta,w}$ the collection of all elements $x \in E_0 + E_1$ such that the expression

$$\|x \mid (E_0, E_1)_{\theta,w}\| := \Big(\int_0^\infty [t^{-\theta} K(t, x, E_0, E_1)]^w \frac{\mathrm{d}t}{t}\Big)^{1/w}$$

is finite. Here $K(t, x, E_0, E_1)$ is the *Peetre K-functional* defined by

$$K(t, x, E_0, E_1) := \inf\{\|x_0 \mid E_0\| + t\|x_1 \mid E_1\| :\ x = x_0 + x_1\}.$$

Then $(E_0, E_1)_{\theta,w}$ becomes a quasi-Banach space with respect to the quasi-norm just defined.

2. Results

It was proved by Bennett [1] that on the Hilbert space l_2 the inclusion

$$\mathscr{L}^{(a)}_{2p/q,p}(l_2, l_2) \subseteq \Pi_{p,q}(l_2, l_2) \quad \text{for } 2 < q < p < \infty$$

holds. The converse inclusion is given in [2]. In order to study the above result for operators acting between arbitrary Banach spaces, we start with

Lemma. *Let* $1 \leqslant q \leqslant p < \infty$ *and* $1/r := 1/p \max(q/2,1)$. *Then*

$$\mathscr{L}^{(c)}_{r,1} \subseteq \Pi_{p,q}.$$

Proof. Given an operator $T \in \mathscr{L}(E,F)$ with $\operatorname{rank}(T) \leqslant n$ we write the factorization

$$T\colon\ E \xrightarrow{T_0} T(E) \xrightarrow{I} T(E) \xrightarrow{J} F$$

where T_0 is the restriction of T, I is the identity operator on $T(E)$ and J the natural injection. Put $1/s := 1/q - 1/p$. Since $\dim T(E) \leqslant n$, by [4] we have

$$\mu_{s,q}(I) \leqslant n^{1/p \max(q/2,1)}.$$

Hence

$$\mu_{s,q}(T) \leqslant n^{1/p \max(q/2,1)} \|T\|$$

and from [10, (2.3.10)] we obtain $\mathscr{L}^{(a)}_{r,1} \subseteq \mathtt{M}_{s,q}$. Using the inclusion $\mathtt{M}_{s,q} \subseteq \Pi_{p,q}$ (see [9, (20.1.11)]), then we have $\mathscr{L}^{(a)}_{r,1} \subseteq \Pi_{p,q}$. Since $\Pi_{p,q}$ is inyective the preceding inclusion is also valid for the operator ideal $\mathscr{L}^{(c)}_{r,1}$. □

Theorem 2.1. *Let* $1 \leqslant q < p < \infty$ *and* $1/r := 1/p \max(q/2,1)$. *Then*

$$\mathscr{L}^{(c)}_{r,p} \subseteq \Pi_{p,q}.$$

Proof. Choose p_0, p_1 and θ such that $1/p = (1-\theta)/p_0 + \theta/p_1,\ q < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$. If

$$1/r_i := 1/p_i \max(q/2,1) \quad \text{for} \quad i = 0,1,$$

we have $0 < r_0 < r_1 < \infty$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Then an interpolation result due to Peetre/Sparr [7] and König [5] (cf. also [6, (2.c.6)] and [10, (2.3.14)]) yields

$$(\mathscr{L}^{(a)}_{r_0,1}(E,F), \mathscr{L}^{(a)}_{r_1,1}(E,F))_{\theta,p} = \mathscr{L}^{(a)}_{r,p}(E,F).$$

Also, we know from [5] (see also [6, (2.c.10)] and [10, (1.2.6)]) that

$$(\Pi_{p_0,q}(E,F), \Pi_{p_1,q}(E,F))_{\theta,p} \subseteq \Pi_{p,q}(E,F).$$

The preceding Lemma yields

$$\mathscr{L}^{(a)}_{r_i,1}(E,F) \subseteq \Pi_{p_i,q}(E,F) \quad \text{for} \quad i = 0,1,$$

and from the above formulas we obtain $\mathscr{L}^{(a)}_{r,p}(E,F) \subseteq \Pi_{p,q}(E,F)$. Hence $\mathscr{L}^{(c)}_{r,p} \subseteq \Pi_{p,q}$, since $\Pi_{p,q}$ is injective. □

An immediate consequence of the preceding result is

Theorem 2.2. *If* $2 < q < p < \infty$, *then* $\mathscr{L}^{(c)}_{2p/q,p} \subseteq \Pi_{p,q}$.

Supplement. Now, we prove that the inclusions stated in the preceding results are strict. A well-known result (see [9, (6.5.4)]) says that the embedding map I from l_1 into l_2 is absolutely $(1,1)$-summing. Hence

$$I \in \Pi_{p,q}(l_1, l_2) \quad \text{for} \quad 1 \leqslant q \leqslant p < \infty.$$

However, I is not compact and consequently

$$I \notin \mathscr{L}^{(c)}_{r,w}(l_1, l_2) \quad \text{for} \quad 0 < r,\ w < \infty.$$

Remark. Taking $2 = q < p < \infty$ in Theorem 2.1 we have $\mathscr{L}^{(a)}_{p,p} \subseteq \Pi_{p,2}$, an inclusion proved by Pietsch in [8], from which he obtained, in the context of Weyl numbers, that $\mathscr{L}^{(x)}_{p,p} \subseteq \Pi_{p,2}$ (see also [10, (2.7.5)]).

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