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$n$ -FLAT AND  $n$ -FP-INJECTIVE MODULES

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*Abstract.* In this paper, we study the existence of the  $n$ -flat preenvelope and the  $n$ -FP-injective cover. We also characterize  $n$ -coherent rings in terms of the  $n$ -FP-injective and  $n$ -flat modules.

*Keywords:*  $n$ -flat module,  $n$ -FP-injective module,  $n$ -coherent ring, cotorsion theory

*MSC 2010:* 13D07, 13C11

1. INTRODUCTION

We use  $R\text{-Mod}$  (resp.,  $\text{Mod-}R$ ) to denote the category of all left (resp., right)  $R$ -modules. For any  $R$ -module  $M$ ,  $\text{pd}_R M$  (resp.,  $\text{id}_R M$ ,  $\text{fd}_R M$ ) denotes the projective (resp., injective, flat) dimension. The character module  $\text{Hom}_Z(M, Q/Z)$  is denoted by  $M^+$ .

Coherent rings have been characterized in various ways. The deepest result is the one due to Chase [2] which claims that the ring  $R$  is left coherent if and only if products of flat right  $R$ -modules are again flat if and only if products of copies of  $R$  are flat right  $R$ -modules. Lee [6] introduced the notions of left  $n$ -coherent and  $n$ -coherent rings and characterized them in various ways, using  $n$ -flat and  $n$ -FP-injective modules. In this paper we continue the study of  $n$ -coherent rings.

A ring  $R$  is called left  $n$ -coherent (for integers  $n > 0$  or  $n = \infty$ ) if every finitely generated submodule of a free left  $R$ -module whose projective dimension is  $\leq n - 1$  is finitely presented. Accordingly, all rings are left 1-coherent, and the left coherent rings are exactly those which are  $d$ -coherent ( $d$  denotes the left global dimension of  $R$ ). In particular, left  $\infty$ -coherent rings are left coherent. A right  $R$ -module  $M$  will

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be called  $n$ -flat if  $\text{Tor}_1^R(M, N) = 0$  holds for all finitely presented left  $R$ -modules  $N$  with  $\text{pd}_R N \leq n$ . A left  $R$ -module  $A$  is said to be  $n$ -FP-injective if  $\text{Ext}_R^1(N, A) = 0$  holds for all finitely presented left  $R$ -modules  $N$  of projective dimension  $\leq n$ .

Given a class  $\mathcal{C}$  of  $R$ -modules, let  ${}^\perp\mathcal{C}$  be the class of  $R$ -modules  $F$  such that  $\text{Ext}_R^1(F, C) = 0$  for every  $C \in \mathcal{C}$  and let  $\mathcal{C}^\perp$  be the class of  $R$ -modules  $F$  such that  $\text{Ext}_R^1(C, F) = 0$  for every  $C \in \mathcal{C}$ . A pair of classes of  $R$ -modules  $(\mathcal{F}, \mathcal{C})$  is called a cotorsion theory if  $\mathcal{F}^\perp = \mathcal{C}$  and  ${}^\perp\mathcal{C} = \mathcal{F}$ . A cotorsion theory is said to be complete if for every  $R$ -module  $M$  there is an exact sequence  $0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$  such that  $C \in \mathcal{C}$  and  $F \in \mathcal{F}$ . A cotorsion theory is said to be perfect if every  $R$ -module has an  $\mathcal{F}$ -cover and a  $\mathcal{C}$ -envelope. A cotorsion theory is said to be hereditary if  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  is exact with  $F, F'' \in \mathcal{F}$ , then  $F' \in \mathcal{F}$ .

We recall that given a class of  $R$ -modules  $\mathcal{T}$ , a morphism  $\varphi: T \rightarrow M$  where  $T \in \mathcal{T}$  is called a  $\mathcal{T}$ -cover of  $M$  if the following conditions hold:

- (1) For any linear map  $\varphi': T' \rightarrow M$  with  $T' \in \mathcal{T}$ , there exists a linear map  $f: T' \rightarrow T$  with  $\varphi' = \varphi f$ , or equivalently,  $\text{Hom}_R(T', T) \rightarrow \text{Hom}_R(T', M) \rightarrow 0$  is exact for any  $T' \in \mathcal{T}$ .
- (2) If  $f$  is an endomorphism of  $T$  with  $\varphi = \varphi f$ , then  $f$  must be an automorphism.

If (1) holds (and perhaps not (2)),  $\varphi: T \rightarrow M$  is called a  $\mathcal{T}$ -precover. A  $\mathcal{T}$ -envelope and  $\mathcal{T}$ -preenvelope are defined dually.

## 2. $n$ -FLAT AND $n$ -FP-INJECTIVE MODULES

Let  $n$  be a non-negative integer. In what follows,  $\mathcal{F}_n$  stands for the class of all  $n$ -flat right  $R$ -modules and  $\mathcal{F}\mathcal{I}_n$  denotes the class of all  $n$ -FP-injective left  $R$ -modules.

**Proposition 2.1.**  *$\mathcal{F}_n$  and  $\mathcal{F}\mathcal{I}_n$  are closed under pure submodules.*

*Proof.* Let  $B \in \mathcal{F}_n$  and let  $A \subseteq B$  be a pure submodule. Then  $0 \rightarrow (B/A)^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$  is split and  $B^+$  is  $n$ -FP-injective by [6, Lemma 5], and so  $A$  is  $n$ -flat by [6, Lemma 5].

Let  $M \in \mathcal{F}\mathcal{I}_n$ , let  $S$  be a pure submodule of  $M$  and let  $N$  be any finitely presented left  $R$ -module with  $\text{pd}_R N \leq n$ . Then we can get an induced exact sequence

$$0 \longrightarrow \text{Hom}_R(N, S) \longrightarrow \text{Hom}_R(N, M) \longrightarrow \text{Hom}_R(N, M/S) \longrightarrow 0,$$

and so  $\text{Ext}_R^1(N, S) = 0$  since  $\text{Ext}_R^1(N, M) = 0$ . It follows that  $S \in \mathcal{F}\mathcal{I}_n$ . □

**Lemma 2.1.** *The following conditions are equivalent:*

- (1)  $M$  is  $n$ -FP-injective if and only if  $\text{Ext}_R^1(R/I, M) = 0$  for any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$ ;

(2)  $N$  is  $n$ -flat if and only if  $\text{Tor}_1^R(N, R/I) = 0$  for any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$ .

**Proof.** (1) “ $\Rightarrow$ ” is trivial.

“ $\Leftarrow$ ” Let  $L$  be any finitely presented left  $R$ -module with  $\text{pd}_R L \leq n$ . Then there is an exact sequence  $0 \rightarrow A \rightarrow R^n \rightarrow L \rightarrow 0$  for some  $n \geq 0$  and  $A \subseteq R^n$  finitely generated with  $\text{pd}_R A \leq n - 1$ . Consider the following pullback of  $A \rightarrow R^n$  and  $R \rightarrow R^n$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & R^{n-1} \longrightarrow 0 \\
 & & \downarrow f & & \downarrow & & \parallel \\
 0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & R^{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & L & \xlongequal{\quad} & L & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Then  $L \cong R/\text{Im } f$  and  $\text{Im } f \cong B$  is finitely generated with  $\text{pd}_R \text{Im } f \leq n - 1$ . Thus  $\text{Ext}_R^1(L, M) \cong \text{Ext}_R^1(R/\text{Im } f, M) = 0$ , which gives that  $M$  is  $n$ -FP-injective.

(2) By analogy with the proof of (1). □

**Theorem 2.1.** Let  $n$  be a non-negative integer and  $R$  a ring. Then

- (1)  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is a perfect cotorsion theory;
- (2)  $({}^\perp \mathcal{F}_n, \mathcal{F}_n)$  is a complete cotorsion theory.

**Proof.** (1) Let  $\text{Card}(R) \leq \aleph_\beta$  and  $F \in \mathcal{F}_n$ . Then we can write  $F$  as a union of a continuous chain  $(F_\alpha)_{\alpha < \lambda}$  of pure submodules of  $F$  such that  $\text{Card}(F_0) \leq \aleph_\beta$  and  $\text{Card}(F_{\alpha+1}/F_\alpha) \leq \aleph_\beta$  whenever  $\alpha + 1 < \lambda$ . If  $N$  is a right  $R$ -module such that  $\text{Ext}_R^1(F_0, N) = 0$  and  $\text{Ext}_R^1(F_{\alpha+1}/F_\alpha, N) = 0$  whenever  $\alpha + 1 < \lambda$ , then  $\text{Ext}_R^1(F, N) = 0$  by [5, Theorem 7.3.4]. Since  $F_\alpha$  is a pure submodule of  $F$  for any  $\alpha < \lambda$ , we have  $F_\alpha \in \mathcal{F}_n$  by Proposition 2.1. On the other hand,  $F_\alpha$  is a pure submodule of  $F_{\alpha+1}$  whenever  $\alpha + 1 < \lambda$ , hence  $F_{\alpha+1}/F_\alpha \in \mathcal{F}_n$  by Proposition 2.1. Let  $X$  be a set of representatives of all modules  $G \in \mathcal{F}_n$  with  $\text{Card}(G) \leq \aleph_\beta$ . Then  $\mathcal{F}_n^\perp = X^\perp$ . So  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is a cotorsion theory by [1, Corollary 2.13]. Since  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is cogenerated by the set  $X$ ,  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is a complete cotorsion theory by [5, Theorem 7.4.1]. Moreover,  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is a perfect cotorsion theory by [5, Theorem 7.2.6] since  $\mathcal{F}_n$  is closed under direct limits.

(2) Let  $X \in (\perp \mathcal{F}\mathcal{I}_n)^\perp$  and let  $N$  be finitely presented with  $\text{pd}_R N \leq n$ . Then  $N \in \perp \mathcal{F}\mathcal{I}_n$ . So  $\text{Ext}_R^1(N, X) = 0$ , which gives that  $X \in \mathcal{F}\mathcal{I}_n$  and  $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is a cotorsion theory. By Lemma 2.1,  $M$  is  $n$ -FP-injective if and only if  $\text{Ext}_R^1(R/A, M) = 0$  for any finitely generated  $A \subseteq R$  with  $\text{pd}_R A \leq n - 1$ . Set  $X = \bigoplus R/A$ , where the sum is over all finitely generated left ideals  $A$  of  $R$  with  $\text{pd}_R A \leq n - 1$ . Then  $\mathcal{F}\mathcal{I}_n = X^\perp$ . So  $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is a complete cotorsion theory by [5, Theorem 7.4.1].  $\square$

### 3. $n$ -COHERENT RINGS

In this section we characterize  $n$ -coherent rings in terms of the  $n$ -FP-injective and  $n$ -flat modules. We obtain some characterizations of the situation when every  $R$ -module has a monic  $\mathcal{F}_n$ -preenvelope and an epic  $\mathcal{F}_n$ -preenvelope.

**Theorem 3.1.** *For a ring  $R$  and any  $n$  ( $0 < n \leq \infty$ ), the following conditions are equivalent:*

- (1)  $R$  is left  $n$ -coherent;
- (2) every right  $R$ -module has an  $\mathcal{F}_n$ -preenvelope;
- (3) any direct limit of  $n$ -FP-injective left  $R$ -modules is  $n$ -FP-injective;
- (4)  $\text{Ext}_R^1(N, \varinjlim M_i) \rightarrow \varinjlim \text{Ext}_R^1(N, M_i)$  is an isomorphism for any finitely presented left  $R$ -module  $N$  with  $\text{pd}_R N \leq n$  and any direct system  $(M_i)_{i \in I}$  of left  $R$ -modules;
- (5)  $\mathcal{F}\mathcal{I}_n$  is a coresolving subcategory;
- (6)  $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is a hereditary cotorsion theory.

*Proof.* (1)  $\Rightarrow$  (4) By [3, Lemma 2.9 (2)]; (4)  $\Rightarrow$  (3) and (5)  $\Rightarrow$  (6) are obvious.

(1)  $\Rightarrow$  (2) Let  $N$  be any right  $R$ -module. Then there is a cardinal number  $\aleph_\alpha$  such that for any homomorphism  $f: N \rightarrow L$  with  $L$   $n$ -flat, there is a pure submodule  $Q$  of  $L$  such that  $\text{Card}(Q) \leq \aleph_\alpha$  and  $f(N) \subseteq Q$ . Note that  $Q$  is  $n$ -flat by Proposition 2.1 and  $\mathcal{F}_n$  is closed under products by [6, Theorem 5], and so  $N$  has an  $\mathcal{F}_n$ -preenvelope by [5, Proposition 6.2.1].

(2)  $\Rightarrow$  (1) Let  $(F_i)_{i \in I}$  be a family of  $n$ -flat right  $R$ -modules and let  $\prod_{i \in I} F_i \rightarrow F$  be an  $\mathcal{F}_n$ -preenvelope. Then there are factorizations  $\prod_{i \in I} F_i \rightarrow F \rightarrow F_j$ , where  $\prod_{i \in I} F_i \rightarrow F_j$  is the canonical projection for each  $j$ . This gives rise to a map  $F \rightarrow \prod_{i \in I} F_i$  with the composition  $\prod_{i \in I} F_i \rightarrow F \rightarrow \prod_{i \in I} F_i$  being the identity. Hence  $\prod_{i \in I} F_i$  is isomorphic to a summand of  $F$ , and so  $\prod_{i \in I} F_i$  is  $n$ -flat, which implies that  $R$  is left  $n$ -coherent.

(3)  $\Rightarrow$  (1) Let  $K$  be a finitely generated submodule of a free left  $R$ -module  $F$  whose projective dimension is  $\leq n - 1$ . Consider the exact sequence  $0 \rightarrow K \rightarrow F \rightarrow$

$F/K \rightarrow 0$ . Then  $F/K$  is finitely presented and  $\text{pd}_R F/K \leq n$ . So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \text{Hom}_R(F/K, \varinjlim M_i) & \longrightarrow & \text{Hom}_R(F, \varinjlim M_i) & \longrightarrow & \text{Hom}_R(K, \varinjlim M_i) & \longrightarrow & 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ \varinjlim \text{Hom}_R(F/K, M_i) & \longrightarrow & \varinjlim \text{Hom}_R(F, M_i) & \longrightarrow & \varinjlim \text{Hom}_R(K, M_i) & \longrightarrow & 0 \end{array}$$

Since  $\alpha$  and  $\beta$  are isomorphisms,  $\gamma$  is an isomorphism by Five lemma. Thus  $K$  is finitely presented.

(1)  $\Rightarrow$  (5) Let  $N$  be a finitely presented left  $R$ -module with  $\text{pd}_R N \leq n$  and let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be exact in  $R\text{-Mod}$  with  $A, B \in \mathcal{F}\mathcal{S}_n$ . Then

$$0 = \text{Ext}_R^1(N, B) \longrightarrow \text{Ext}_R^1(N, C) \longrightarrow \text{Ext}_R^2(N, A) = 0$$

by [6, Theorem 1], and so  $C \in \mathcal{F}\mathcal{S}_n$ . Thus  $\mathcal{F}\mathcal{S}_n$  is a coresolving subcategory.

(6)  $\Rightarrow$  (1) Let  $S$  be a finitely generated submodule of a free left  $R$ -module  $F$  whose projective dimension is  $\leq n - 1$ . We need to prove that  $S$  is finitely presented. Let  $M$  be FP-injective and let  $0 \rightarrow M \rightarrow E \rightarrow C \rightarrow 0$  be exact with  $E$  injective. Then  $M \in \mathcal{F}\mathcal{S}_n$  and  $C \in \mathcal{F}\mathcal{S}_n$ , and so

$$\text{Ext}_R^1(S, M) \cong \text{Ext}_R^2(F/S, M) \cong \text{Ext}_R^1(F/S, C) = 0.$$

Thus  $S$  is finitely presented, which means that  $R$  is left  $n$ -coherent. □

**Proposition 3.1.** *The following conditions are equivalent:*

- (1)  $R$  is a left  $n$ -coherent ring;
- (2)  $\text{Ext}_R^1(I, N) = 0$  for any FP-injective left  $R$ -module  $N$  and any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$ ;
- (3)  $\text{Ext}_R^2(R/I, N) = 0$  for any FP-injective left  $R$ -module  $N$  and any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$ ;
- (4) if  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  is an exact sequence of left  $R$ -modules with  $N$  FP-injective and  $M$   $n$ -FP-injective, then  $L$  is  $n$ -FP-injective.

*Proof.* (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (3) Let  $N$  be an FP-injective left  $R$ -module and  $I$  a finitely generated left ideal with  $\text{pd}_R I \leq n - 1$ . Then the exact sequence  $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$  gives rise to the exact sequence

$$0 = \text{Ext}_R^1(I, N) \longrightarrow \text{Ext}_R^2(R/I, N) \longrightarrow \text{Ext}_R^2(R, N) = 0$$

by (2). Thus  $\text{Ext}_R^2(R/I, N) = 0$ .

(3)  $\Rightarrow$  (4) Let  $I$  be a finitely generated left ideal of  $R$  with  $\text{pd}_R I \leq n - 1$ . The exact sequence  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  induces the exactness of

$$0 = \text{Ext}_R^1(R/I, M) \longrightarrow \text{Ext}_R^1(R/I, L) \longrightarrow \text{Ext}_R^2(R/I, N) = 0$$

by (3), and hence  $\text{Ext}_R^1(R/I, L) = 0$ . That is,  $L$  is  $n$ -FP-injective by Lemma 2.1.

(4)  $\Rightarrow$  (1) Let  $I$  be a finitely generated left ideal with  $\text{pd}_R I \leq n - 1$ . For any FP-injective left  $R$ -module  $N$ , there is an exact sequence  $0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$  with  $E$  injective. Note that  $E/N$  is  $n$ -FP-injective by (4). Hence we get the exact sequence

$$0 = \text{Ext}_R^1(R/I, E/N) \longrightarrow \text{Ext}_R^2(R/I, N) \longrightarrow \text{Ext}_R^2(R/I, E) = 0,$$

and so  $\text{Ext}_R^1(I, N) \cong \text{Ext}_R^2(R/I, N) = 0$ . It follows that  $I$  is finitely presented. Therefore  $R$  is left  $n$ -coherent.  $\square$

**Lemma 3.1.** *Let  $R$  be a left  $n$ -coherent ring and let  $|M| = \lambda$  for a left  $R$ -module  $M$ . Let  $k$  be as in El Bashir's result. Then any map  $A \rightarrow M$  with  $A$   $n$ -FP-injective can be factored through an  $n$ -FP-injective left  $R$ -module  $B$  with  $|B| < k$ .*

*Proof.* Consider any homomorphism  $A \rightarrow M$  with  $A$   $n$ -FP-injective. If  $|A| < k$ , let  $B = A$ . So suppose  $|A| \geq k$ . Consider a submodule  $S \subseteq A$  maximal with respect to the two properties that  $S$  is pure in  $A$  and that  $S \subseteq \text{Ker}(A \rightarrow M)$ . Let  $B = A/S$ . Then  $B$  is  $n$ -FP-injective by Theorem 3.1. We wish to argue that  $|B| < k$ . Let  $K$  be the kernel of  $B \rightarrow M$ . Then  $|B/K| \leq |M| = \lambda$ . So if  $|B| \geq k$ , there is a nonzero pure submodule  $T/S$  of  $B$  contained in  $K$ . But then  $T$  is pure in  $A$  and is contained in the kernel of  $A \rightarrow M$ . This contradicts the choice of  $S$ .  $\square$

**Theorem 3.2.** *Let  $R$  be a left  $n$ -coherent ring. Then every left  $R$ -module has an  $\mathcal{F}\mathcal{I}_n$ -cover.*

*Proof.* By Lemma 3.1 and [5, Proposition 5.2.2 and Corollary 5.2.7].  $\square$

**Proposition 3.2.** *Let  $R$  be left  $n$ -coherent. Then the following conditions are equivalent:*

- (1) *every left  $R$ -module has an  $n$ -FP-injective cover with the unique mapping property (see [4]);*
- (2) *for every left  $R$ -modules exact sequence  $A \rightarrow B \rightarrow C \rightarrow 0$  with  $A$  and  $B$   $n$ -FP-injective,  $C$  is  $n$ -FP-injective.*

**Proof.** (1)  $\Rightarrow$  (2) Let  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be exact in  $R\text{-Mod}$  with  $A, B$   $n$ -FP-injective and  $\theta: H \rightarrow C$  an  $n$ -FP-injective cover with the unique mapping property. Then there exists a map  $\delta: B \rightarrow H$  such that  $g = \theta\delta$ . Thus  $\theta\delta f = gf = 0 = \theta 0$ , and hence  $\delta f = 0$ , which implies that  $\text{Ker } g = \text{Im } f \subseteq \text{Ker } \delta$ . Therefore there is a morphism  $\gamma: C \rightarrow H$  such that  $\gamma g = \delta$ , and so  $\theta\gamma g = \theta\delta = g$ , which gives that  $\theta\gamma = 1_C$ . Thus  $C$  is isomorphic to a direct summand of  $H$ , and so  $C$  is  $n$ -FP-injective.

(2)  $\Rightarrow$  (1) Let  $M$  be any left  $R$ -module. Then  $M$  has an  $n$ -FP-injective cover  $f: L \rightarrow M$  by Theorem 3.2. It is enough to show that for any  $n$ -FP-injective left  $R$ -module  $G$  and any homomorphism  $g: G \rightarrow L$  such that  $fg = 0$ , we have  $g = 0$ . In fact, there is a homomorphism  $\beta: L/\text{Im } g \rightarrow M$  such that  $\beta\pi = f$ , where  $\pi: L \rightarrow L/\text{Im } g$  is the natural map. Since  $L/\text{Im } g$  is  $n$ -FP-injective, there is a map  $\alpha: L/\text{Im } g \rightarrow L$  such that  $\beta = f\alpha$ , and so  $f\alpha\pi = f$ . Hence  $\alpha\pi$  is an isomorphism. Therefore  $\pi$  is monic and  $g = 0$ .  $\square$

**Proposition 3.3.** *The following conditions are equivalent:*

- (1)  $({}^\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is a hereditary cotorsion theory;
- (2)  $R$  is left  $n$ -coherent and  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is a hereditary cotorsion theory;
- (3)  $\text{Ext}_R^2(R/I, M) = 0$  for any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$  and any  $n$ -FP-injective left  $R$ -module  $M$ ;
- (4)  $R$  is left  $n$ -coherent and  $\text{Tor}_2^R(N, R/I) = 0$  for any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$  and any  $n$ -flat right  $R$ -module  $N$ .

**Proof.** (1)  $\Rightarrow$  (2) Since  $({}^\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is hereditary,  $R$  is left  $n$ -coherent by Theorem 3.1. On the other hand, let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be exact with  $B, C \in \mathcal{F}_n$ . Then  $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$  is exact with  $B^+, C^+ \in \mathcal{F}\mathcal{I}_n$  by [6, Theorem 3], and so  $A^+ \in \mathcal{F}\mathcal{I}_n$  by (1), which implies that  $A \in \mathcal{F}_n$ . That is,  $(\mathcal{F}_n, \mathcal{F}_n^\perp)$  is hereditary.

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (1) By [6, Theorem 1], (4)  $\Rightarrow$  (2). It is easy.

(2)  $\Rightarrow$  (4) Let  $N \in \mathcal{F}_n$  and let  $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$  be exact with  $P$  projective. Then  $K \in \mathcal{F}_n$  by (2), and hence  $\text{Tor}_2^R(N, R/I) \cong \text{Tor}_1^R(K, R/I) = 0$  for any finitely generated left ideal  $I$  with  $\text{pd}_R I \leq n - 1$ .  $\square$

**Proposition 3.4.** *The following conditions are equivalent for a left  $n$ -coherent ring  $R$ :*

- (1) every  $n$ -flat right  $R$ -module is flat;
- (2) every cotorsion right  $R$ -module belongs to  $\mathcal{F}_n^\perp$ ;
- (3) every  $n$ -FP-injective left  $R$ -module is FP-injective;
- (4) every finitely presented left  $R$ -module belongs to  $\mathcal{F}\mathcal{I}_n^\perp$ .



**Proof.** (1)  $\Leftrightarrow$  (2) and (3)  $\Leftrightarrow$  (4) follow from Theorem 2.1.

(1)  $\Rightarrow$  (3) Let  $M$  be any  $n$ -FP-injective left  $R$ -module. Then  $M^+$  is  $n$ -flat, and so  $M^+$  is flat by (1). On the other hand, for any finitely presented left  $R$ -module  $N$ , there is an exact sequence

$$\mathrm{Tor}_1^R(M^+, N) \longrightarrow \mathrm{Ext}_R^1(N, M^+) \longrightarrow 0$$

by [3, Lemma 2.7 (1)]. Thus  $\mathrm{Ext}_R^1(N, M) = 0$ , and so  $M$  is FP-injective.

(3)  $\Rightarrow$  (1) Let  $M$  be an  $n$ -flat right  $R$ -module. Then  $M^+$  is  $n$ -FP-injective, and so  $M^+$  is FP-injective by (3). Hence  $M$  is flat.  $\square$

Now we study when every right  $R$ -module has a monic  $\mathcal{F}_n$ -preenvelope and an epic  $\mathcal{F}_n$ -preenvelope.

**Proposition 3.5.** *The following conditions are equivalent:*

- (1) every right  $R$ -module has a monic  $\mathcal{F}_n$ -preenvelope;
- (2)  $R$  is left  $n$ -coherent and every flat left  $R$ -module is  $n$ -FP-injective;
- (3)  $R$  is left  $n$ -coherent and  ${}_R R$  is  $n$ -FP-injective;
- (4)  $R$  is left  $n$ -coherent and  $(\mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n^\perp)$  is a perfect cotorsion theory;
- (5)  $R$  is left  $n$ -coherent and every left  $R$ -module has an epic  $\mathcal{F}\mathcal{I}_n$ -cover.

**Proof.** (2)  $\Rightarrow$  (3) and (4)  $\Rightarrow$  (5) are obvious.

(1)  $\Rightarrow$  (2) Let  $M$  be a flat left  $R$ -module. Then  $M^+$  is injective and  $M^+$  has a monic  $\mathcal{F}_n$ -preenvelope  $\varphi: M^+ \rightarrow F$ . Set  $C = \mathrm{Coker} \varphi$ . Then  $0 \rightarrow M^+ \rightarrow F \rightarrow C \rightarrow 0$  is split, and so  $C \in \mathcal{F}_n$ , which gives that  $M^+ \in \mathcal{F}_n$  since  $R$  is left  $n$ -coherent. Thus  $M$  is  $n$ -FP-injective.

(3)  $\Rightarrow$  (4) By analogy with the proof of Theorem 2.1.

(5)  $\Rightarrow$  (1) Let  $M$  be any right  $R$ -module. Then  $M$  has an epic  $\mathcal{F}\mathcal{I}_n$ -cover  $E \rightarrow M^+$  by (5), and so there is a monomorphism  $M \rightarrow E^+$ . Thus every right  $R$ -module has a monic  $\mathcal{F}_n$ -preenvelope by Theorem 3.1.  $\square$

**Proposition 3.6.** *The following conditions are equivalent:*

- (1) every right  $R$ -module has an epic  $\mathcal{F}_n$ -preenvelope;
- (2)  $R$  is left  $n$ -coherent and every submodule of any  $n$ -flat right  $R$ -module is  $n$ -flat;
- (3) every quotient module of any  $n$ -FP-injective left  $R$ -module is  $n$ -FP-injective;
- (4) every left  $R$ -module has a monic  $\mathcal{F}\mathcal{I}_n$ -cover.

**Proof.** (1)  $\Rightarrow$  (2)  $R$  is left  $n$ -coherent by Theorem 3.1. Now suppose that  $N$  is a submodule of an  $n$ -flat right  $R$ -module  $L$  and  $\iota: N \rightarrow L$  is the inclusion.

By (1),  $N$  has an epic  $\mathcal{F}_n$ -preenvelope  $f: N \rightarrow F$ . Then there is a homomorphism  $g: F \rightarrow L$  such that the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{f} & F \\ \downarrow \iota & \searrow g & \\ L & & \end{array}$$

So  $gf = \iota$  is monic, and hence  $f$  is monic, which gives that  $f$  is an isomorphism and  $N \cong F$  is  $n$ -flat.

(2)  $\Rightarrow$  (3) Let  $M$  be any  $n$ -FP-injective left  $R$ -module and let  $M \rightarrow N \rightarrow 0$  be exact. Then  $0 \rightarrow N^+ \rightarrow M^+$  is exact and  $M^+$  is  $n$ -flat, and so  $N^+$  is  $n$ -flat by (2). Thus  $N$  is  $n$ -FP-injective by [6, Theorem 3].

(3)  $\Rightarrow$  (4) By [6, Theorem 2],  $R$  is left  $n$ -coherent, and hence every left  $R$ -module  $M$  has an  $\mathcal{F}\mathcal{I}_n$ -precover  $\varphi: C \rightarrow M$ . Note that  $\text{Im } \varphi$  is  $n$ -FP-injective by (3), so  $\text{Im } \varphi \rightarrow M$  is a monic  $\mathcal{F}\mathcal{I}_n$ -cover.

(4)  $\Rightarrow$  (1) Let  $E$  be an injective left  $R$ -module and  $S \subseteq E$  a pure submodule. Then  $E/S$  has a monic  $\mathcal{F}\mathcal{I}_n$ -cover  $f: C \rightarrow E/S$ . By analogy with the proof (1)  $\Rightarrow$  (2),  $f$  is an isomorphism and  $E/S$  is  $n$ -FP-injective, and hence  $R$  is left  $n$ -coherent by [6, Theorem 2], which means that every right  $R$ -module has an  $\mathcal{F}_n$ -preenvelope by Theorem 3.1. Let  $M$  be any right  $R$ -module. Then  $M^+$  has a monic  $\mathcal{F}\mathcal{I}_n$ -cover  $E \rightarrow M^+$ , and hence  $M^{++} \rightarrow E^+ \rightarrow 0$  is exact. Set  $K = \text{Ker}(M^{++} \rightarrow E^+)$ . Consider the following pullback of  $M \rightarrow M^{++}$  and  $K \rightarrow M^{++}$ :

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & K & \longrightarrow & M^{++}/M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & M^{++} & \longrightarrow & M^{++}/M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & E^+ & \xlongequal{\quad} & E^+ & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since  $E^+$  is  $n$ -flat,  $M$  has an epic  $\mathcal{F}_n$ -preenvelope. □

**Proposition 3.7.** *The following conditions are equivalent:*

- (1) every left  $R$ -module is  $n$ -FP-injective;
- (2) every cotorsion left  $R$ -module is  $n$ -FP-injective;
- (3) every right  $R$ -module is  $n$ -flat;
- (4) every cotorsion right  $R$ -module is  $n$ -flat;
- (5) every right  $R$ -module in  $\mathcal{F}_n^\perp$  is injective;
- (6) every left  $R$ -module in  ${}^\perp\mathcal{F}\mathcal{I}_n$  is projective;
- (7) every nonzero right  $R$ -module contains a nonzero  $n$ -flat submodule;
- (8)  $({}^\perp\mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$  is a hereditary cotorsion theory and every left  $R$ -module in  ${}^\perp\mathcal{F}\mathcal{I}_n$  is  $n$ -FP-injective.

*Proof.* (1)  $\Rightarrow$  (2), (3)  $\Rightarrow$  (4), (3)  $\Rightarrow$  (7) and (1)  $\Rightarrow$  (8) are obvious.

(2)  $\Rightarrow$  (3) Let  $M$  be any right  $R$ -module. Then  $M^+$  is  $n$ -FP-injective by (2), and so  $M$  is  $n$ -flat.

(4)  $\Rightarrow$  (1) Let  $M$  be any left  $R$ -module. Then  $M^+$  is  $n$ -flat by (4), and so  $M^{++}$  is  $n$ -FP-injective. Note that  $M$  is a pure submodule of  $M^{++}$ . So  $M$  is  $n$ -FP-injective.

(3)  $\Leftrightarrow$  (5) and (1)  $\Leftrightarrow$  (6) follow from Theorem 2.1.

(7)  $\Rightarrow$  (5) Assume that  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is any exact sequence. To simplify the notation, we think of  $A$  as a submodule of  $B$ . Let  $M \in \mathcal{F}_n^\perp$  and let  $f: A \rightarrow M$  be any homomorphism. By a simple application of Zorn's Lemma, we can find  $g: D \rightarrow M$ , where  $A \subseteq D \subseteq B$  and  $g|_A = f$ , such that  $g$  cannot be extended to any submodule of  $B$  properly containing  $D$ . We claim that  $D = B$ . Indeed, if  $D \neq B$ , then  $B/D \neq 0$ . By (7), there is a nonzero submodule  $N/D$  of  $B/D$  such that  $N/D$  is  $n$ -flat. Since  $M \in \mathcal{F}_n^\perp$ , there exists  $h: N \rightarrow M$  such that  $h|_D = g$ . It is obvious that  $h$  extends  $g$ , thus we get the desired contradiction, and so  $M$  is injective.

(8)  $\Rightarrow$  (1) Let  $M$  be a left  $R$ -module. By Theorem 2.1, there is an exact sequence  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  with  $F \in {}^\perp\mathcal{F}\mathcal{I}_n$ ,  $K \in \mathcal{F}\mathcal{I}_n$ . Then  $F \in \mathcal{F}\mathcal{I}_n$ , and hence  $M \in \mathcal{F}\mathcal{I}_n$  by (8).  $\square$

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