Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 2, 419–435

Persistent URL: http://dml.cz/dmlcz/141544

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CURVATURE FUNCTIONALS FOR CURVES IN THE EQUI-AFFINE PLANE

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(Received January 25, 2010)

Abstract. After having given the general variational formula for the functionals indicated in the title, the critical points of the integral of the equi-affine curvature under area constraint and the critical points of the full-affine arc-length are studied in greater detail.
Notice. An extended version of this article is available on `arXiv:0912.4075`.

Keywords: curvature functionals, variational problems, affine curves

MSC 2010: 49K05, 49K15, 53A15

1. Preface[2]

One of the many striking features of W. Blaschke's landmark book "*Vorlesungen II*" [4], being the first treatise on equi-affine differential geometry which also at present day remains in multiple aspects the best introduction to the subject, is the close analogy between the development of the main body of the equi-affine theory and the exposition of classical differential geometry [3]. Although Blaschke showed a great interest in isoperimetric and variational problems, a rare topic which breaks this similarity is precisely the question of the infinitesimal change of a planar curvature functional, for Radon's problem is indeed covered with some detail in [3] whereas in [4] only the variation of the equi-affine arc-length is considered. In fact, even after

[1] The author, who was employed at K. U. Leuven during the commencement of this work, while he was supported by Masaryk University (Brno) during its conclusion, is thankful to both these institutions. This research was partially supported by the Research Foundation Flanders (project G.0432.07) and the Eduard Čech Center for Algebra and Geometry (Basic Research Center no. LC505).

[2] "In Vorworten pflegen die Verfasser (wie Zahnärzte) die Lücken zu bohren, die sie später stopfen wollen." – W. Blaschke [6].

having asked many colleagues, to whom I extend my gratitude, I have not been able to find *any* article on equi-affine curvature functionals for planar curves (although centro-affine curvature functionals for planar curves are treated in [11] whereas [17] covers a variational problem w.r.t. the full-affine group).

For this reason, I propose to study some basic variational problems for curves in the equi-affine plane.

2. The general variational formula

Let γ be a curve in the equi-affine plane $\mathbb{A}^2$ (which is equipped with a fixed area form $|\cdot,\cdot|$). It will be assumed that γ is parameterised by *equi-affine arc-length* s, which means that $|\gamma',\gamma''| = 1$. A prime will always stand for a derivative w.r.t. the equi-affine arc-length parameter, *i.e.*, we adopt the shorthand notation $\mu' = (\partial/\partial s)\mu$ for any function μ. We will denote by $\{T, N\}$ the equi-affine Frenet frame consisting of the tangent vector $T = \gamma'$ and the Blaschke normal $N = \gamma''$. The *equi-affine curvature* κ is defined by means of the equation

$$N' = -\kappa T \qquad \text{(or, equivalently, } \gamma''' = -\kappa\gamma'\text{)}. \tag{1}$$

In the course of this article, some remarks concerning invariants of curves with respect to the Euclidean, the full-affine or the projective group will be made. These invariants will bear an index **E**, **F** or **P**.

The position vector field P (with respect to an arbitrarily chosen origin) can be written as $P = -\varrho N + \varphi T$, where the function ϱ is called the *equi-affine support function*. By expressing the fact that $T = P'$ one finds two first-order differential equations for ϱ and φ, which can be combined to give

$$\varrho'' + \kappa\varrho = 1. \tag{2}$$

It can easily be seen that for a deformation of a curve γ with deformation vector field $fN + gT$, the equi-affine tangent of the deformed curve γ_t is given by

$$T_t(s) = T(s) - \frac{t}{3}\big(f''(s) + \kappa(s)f(s)\big)T(s) + t\big(\dots\big)N(s) + \mathscr{O}(t^2).$$

Because the equi-affine curvature of the deformed curve γ_t is precisely the centro-affine curvature of its equi-affine tangent image T_t, the variational formula for equi-affine curvature functionals can immediately be found from eq. (2.6) of [11]. It is not necessary to calculate the component along N of the deformation vector field of the tangent image (which is indicated by dots in the above formula), because it merely represents an infinitesimal reparameterisation of the tangent image. Furthermore, it will be assumed that the deformation vector field is compactly supported such that no boundary terms occur. This requirement is automatically satisfied for closed curves. In this way the following variational formula is obtained, in which both f and κ should be evaluated in s:

$$\delta \int F(\kappa)\,\mathrm{d}s = \frac{1}{3}\int f\Big[\Big(\frac{\partial^2}{\partial s^2} + \kappa\Big)(F'''(\kappa)(\kappa')^2 + F''(\kappa)\kappa'' + 4F'(\kappa)\kappa - 2F(\kappa))\Big]\,\mathrm{d}s. \tag{3}$$

For every curve $\gamma = (x, y)$, the general solution of the differential equation $(\partial^2/\partial s^2 + \kappa)\xi = 0$ is given by $\xi = Ax' + By'$. Therefore the Euler-Lagrange equation can immediately be integrated twice, with the following result:

Theorem 1. *A curve $\gamma = (x, y)$ in the affine plane is a critical point of the equi-affine curvature functional $\int F(\kappa)\,\mathrm{d}s$ if and only if*

$$F'''(\kappa)(\kappa')^2 + F''(\kappa)\kappa'' + 4F'(\kappa)\kappa - 2F(\kappa) = Ax' + By'$$

for some A, B in $\mathbb{R}$.

We will now consider two special instances of the above variational formula, namely, the functionals $\int \kappa\,\mathrm{d}s$ (§ 3) and $\int \sqrt{\kappa}\,\mathrm{d}s$ (§ 4).

3. The total equi-affine curvature

Unlike in Euclidean differential geometry, the total equi-affine curvature $\int \kappa\,\mathrm{d}s$ of a closed curve is not topologically invariant[3], which makes the variational problem worthy of study. The variational formula

$$\delta \int \kappa\,\mathrm{d}s = \frac{2}{3}\int f\big(\kappa'' + \kappa^2\big)\,\mathrm{d}s, \tag{4}$$

can be derived as well from formula (4.1.12) of [15], p. 213 (see also [14]).

[3] In fact, no non-zero functional of the form $\int F(\kappa)\,\mathrm{d}s$ is a topological invariant, as can be deduced from the fact that for no non-zero function F the right-hand side of (3) vanishes identically.

Theorem 2. *A curve $\gamma = (x, y)$ in the affine plane is a critical point of $\int \kappa \, \mathrm{d}s$ if and only if*

$$\kappa = Ax' + By' \tag{5}$$

for appropriate constants A, and B, i.e., the centro-affine curvature of the tangent image described by T is a linear function of its position vector field.

Theorem 3. *A curve $\gamma = (x, y)$ in the affine plane is a critical point of $\int \kappa \, \mathrm{d}s$ under area constraint, without being an unconstrained critical point of the total equi-affine curvature, if and only if the origin can be chosen in such a way that the equi-affine support function becomes a non-zero multiple of the equi-affine curvature.*

Proof. We recall that, under a deformation with deformation vector field $fN + gT$, the area bounded by the curve changes according to $\delta\mathrm{Area} = -\int f \, \mathrm{d}s$. Consequently, the Euler-Lagrange equation which is satisfied by the curve reads

$$\kappa'' + \kappa^2 = C \tag{6}$$

for some non-zero constant C. Now consider the vector field

$$M = \kappa N - \kappa' T$$

along the curve. It follows from (6) that $M + CP$ is a constant vector field, which becomes zero after a suitable translation has been applied. We then have $\kappa = -|M, T| = C\,|P, T| = C\varrho$. □

Theorem 4 below, and perhaps Theorems 1, 2, 3 and 10 of this article as well, are reminiscent of the known result that a curve in $\mathbb{E}^2$ is a critical point of the Bernoulli-Euler bending energy $\int (\kappa_{\mathbf{E}})^2 \, \mathrm{d}s_{\mathbf{E}}$ under constrained area and arc-length (without being a critical point of this variational problem if the area constraint is relaxed) if and only if the Euclidean curvature can be written as $\kappa_{\mathbf{E}} = A + B\|\gamma\|^2$ for some choice of the origin and some A, B with $B \neq 0$ ([2], eq. (38)).

Theorem 4. *A curve $\gamma = (x, y)$ in the affine plane is a critical point of $\int \kappa \, \mathrm{d}s$ under constrained area and total equi-affine length (without being a critical point of $\int \kappa \, \mathrm{d}s$ under arc-length constraint only) if and only there holds $\kappa = C + A\varrho$ for some A, C (with $C \neq 0$) and some choice of the origin.*

Proof. This follows similarly from the equations $\kappa'' + \kappa^2 = C + A\kappa$ (with $C \neq 0$) and $M = \kappa N - \kappa' T - AN$. □

Theorem 5. *The ellipses are the only closed curves which are a critical point of the functional $\int \kappa\,\mathrm{d}s$ with respect to deformations under which the equi-affine arc-length is preserved.*

P r o o f. Let γ be a closed curve which solves the variational problem as stated in the theorem. From the Euler-Lagrange equation

$$\kappa'' + \kappa^2 = A\kappa \qquad (A \in \mathbb{R}) \tag{7}$$

it can be inferred that

$$\int \kappa^2\,\mathrm{d}s = \int \left(\kappa'' + \kappa^2\right)\mathrm{d}s = A\int \kappa\,\mathrm{d}s.$$

From (2) we deduce that

$$\int \kappa\,\mathrm{d}s = \int \kappa\left(\varrho'' + \kappa\varrho\right)\mathrm{d}s = \int \left(\kappa'' + \kappa^2\right)\varrho\,\mathrm{d}s = A\int \varrho\kappa\,\mathrm{d}s = A\int \mathrm{d}s.$$

A combination of the previous two equations results in

$$\int (\kappa - A)^2\,\mathrm{d}s = \int \kappa^2\,\mathrm{d}s - 2A\int \kappa\,\mathrm{d}s + \int A^2\,\mathrm{d}s = (A^2 - 2A^2 + A^2)\int \mathrm{d}s = 0,$$

which means that κ is equal to the constant A. $\square$

Remark 6. It has been shown in [4], §27.24 that every simple closed curve in the affine plane satisfies

$$\int \kappa\,\mathrm{d}s\,\sqrt[3]{\text{Area}} \leqslant 2\pi^{\frac{4}{3}},$$

and equality occurs precisely for the ellipses. By combining this equation with the Blaschke inequality, it follows that

$$\int \kappa\,\mathrm{d}s \int \mathrm{d}s \leqslant 4\pi^2, \tag{8}$$

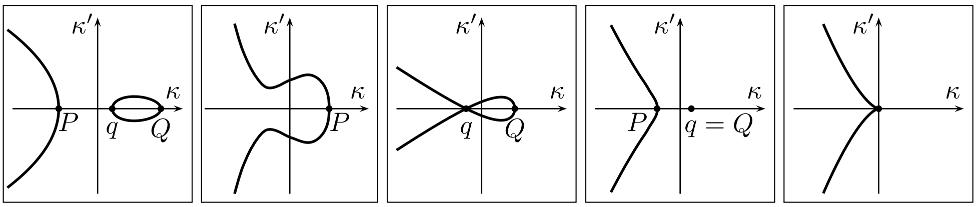
in which equality is also characteristic to the ellipses.

We now intend to give a full classification of all curves which are a critical point of $\int \kappa\,\mathrm{d}s$ under area constraint. Our first observation is that the Euler-Lagrange equation (6) can be integrated. Indeed, if the notation $f = -\frac{1}{6}\kappa$, $g_2 = \frac{1}{3}C$ is adopted, whereas g_3 stands for an integration constant, the following requirement on the function f and the constants g_2, $g_3 \in \mathbb{R}$ is found:

$$(f')^2 = 4f^3 - g_2 f - g_3 \qquad \left(\text{or yet, } (\kappa')^2 = -\tfrac{2}{3}\kappa^3 + 6g_2\kappa - 36g_3\right). \tag{9}$$

The constant g_2 vanishes if and only if the curve is a critical point of the total equi-affine curvature without constraint. Under the assumption $g_2 \neq 0$, the constant g_3 vanishes if and only if the curve traced out by the vector $\beta = \big(1/\sqrt{|\kappa|}\big)P$ (the position vector field P being chosen as in Theorem 3) is a straight line. This last fact can be immediately found from the formula $\beta'' = -27g_3\kappa^{-2}\beta$.

Furthermore, it may be instructive to have in mind the picture of the phase plane (κ, κ'), where the equation (9) determines a cubic $\mathcal{C}$ which is symmetric w.r.t. the horizontal axis which it intersects in precisely three distinct points if the cubic discriminant $\Delta = (g_2)^3 - 27(g_3)^2$ is strictly positive, and in precisely one point if this discriminant is strictly negative (see Figure 1). While a point in the affine plane describes a curve γ solving the variational problem, the corresponding point in the phase plane will describe a part of this cubic $\mathcal{C}$.



$\Delta > 0$. $\quad \Delta = 0,\ g_3 < 0$. $\quad \Delta < 0$. $\quad \Delta = 0,\ g_3 > 0$. $\quad g_2 = g_3 = 0$.

Figure 1. The different possibilities for the cubic $\mathcal{C}$ in the phase plane, depending on the sign of g_2, g_3 and the cubic discriminant $\Delta = (g_2)^3 - 27(g_3)^2$.

Generic Case. $g_2 \neq 0$ and $\Delta \neq 0$. Let us first consider this generic case, which will be split into Cases (A), (B) and (C) afterwards. It is known that complex constants ω_1 and ω_2 can be found, which are not proportional over $\mathbb{R}$, for which

$$g_2 = \sum \frac{60}{(2m_1\omega_1 + 2m_2\omega_2)^4} \quad \text{and} \quad g_3 = \sum \frac{140}{(2m_1\omega_1 + 2m_2\omega_2)^6},$$

the summations being taken for $m_1, m_2 \in \mathbb{Z}^2 \setminus \{(0,0)\}$ (see [20], §20.22; 21.73). Moreover if $\wp$ denotes the Weierstrass elliptic function with half-periods ω_1 and ω_2 and invariants g_2, g_3, then the only complex functions f which solve equation (9) are $z \mapsto \wp(z_0 \pm z)$ (for $z_0 \in \mathbb{C}$).

We remark that the restrictions of $\wp$ to the real and the imaginary line are periodic, and will denote their periods as $2\varpi_1$ (a strictly positive number) resp. $2\varpi_2$ (a strictly positive multiple of i).[4]

[4] Compare [1], Ch. 18, and [20], p. 444, Ex. 1. If $\Delta > 0$, the grid of poles of $\wp$ is rectangular and the standard choice of half-periods is $\omega_1 = \varpi_2$ and $\omega_2 = \varpi_1 + \varpi_2$. If the cubic discriminant Δ is strictly negative, this grid is rhombic and the standard half-periods are given by $\omega_1 = \frac{1}{2}(\varpi_1 + \varpi_2)$ and $\omega_2 = \varpi_1$.

Consider any parameter value s_0, and take a point $z_0 \in \mathbb{C}$ for which $\wp(z_0) = f(s_0)$. Since both f and $\wp$ satisfy the equation (9), we have $\wp'(z_0) = \pm f'(s_0)$. After possibly having reverted the equi-affine arc-length parameter, it can be assumed that the plus sign appears in this equation. Then, by integration of (9) for both $\wp$ and f, we find that $\wp(z_0 + s) = f(s_0 + s)$ for all real numbers s in a neighbourhood of the origin. We conclude that

$$f(s) = \wp(s - c_0)$$

holds for all real s in a certain interval and for a certain fixed number c_0, which can be assumed to be a pure imaginary. Since $\wp(s - c_0)$ is a real number whenever s is real, there are merely two different situations to consider: either $c_0 = \varpi_2$ or $c_0 = 0$.

If $c_0 = \varpi_2$ and $\Delta > 0$, the curvature function will oscillate between two values (say, $q = \min \kappa$ and $Q = \max \kappa$) and the image in the phase-plane describes the closed branch of the cubic $\mathcal{C}$. Whenever the equi-affine curvature has the value q or Q, the derivative κ' will vanish, and since this derivative is given by (9), the numbers g_2, g_3 can be expressed in terms of q and Q:

$$g_2 = \frac{1}{9}\big(q^2 + Q^2 + qQ\big) \qquad \text{and} \qquad g_3 = \frac{1}{54}\big(q^2Q + qQ^2\big). \tag{10}$$

If $c_0 = \varpi_2$ and $\Delta < 0$, then $\wp(s - c_0) = \wp(s - \varpi_1)$. After translation of the parameter s by the real number ϖ_1, this is already included in the next case.

If $c_0 = 0$, the curvature $\kappa(s) = -6\wp(s)$ will raise from $-\infty$ at $s = 0$, have a maximum P at ϖ_1, and fall again towards $-\infty$ for s approaching $2\varpi_1$.

Now that we have found the curvature function $\kappa(s) = -6\wp(s - c_0)$ of the curve, we have to find the co-ordinate functions x and y, as functions of s, which satisfy (1). The translates of these differentiated functions, which map s to $x'(s + c_0)$ resp. $y'(s + s_0)$, satisfy the homogeneous second-order ordinary differential equation

$$F'' = 6\wp F,$$

i.e., the equation of Lamé. Two independent complex solutions[5] of this differential equation are

$$\begin{cases} \varphi_1(z) = \dfrac{\partial}{\partial z}\Big(\dfrac{\sigma(z+c)}{\sigma(z)}\exp\Big(\Big(\dfrac{-\wp'(c)}{2\wp(c)} - \zeta(c)\Big)z\Big)\Big); \\ \varphi_2(z) = \displaystyle\int_{z_0}^{z} \frac{1}{\big(\varphi_1(v)\big)^2}\,\mathrm{d}v\varphi_1(z). \end{cases} \tag{11}$$

[5] Here φ_2 can be replaced by the second solution of Lamé's equation which is given in [20] p. 459, ex. 29, in case this function is not a multiple of φ_1.

Here c stands for a complex number[6] for which $\wp(c) = -g_3/g_2$, and which may be taken in one of the forms $\varpi_1 + d\mathrm{i}$, $\varpi_2 + d$, d or $d\mathrm{i}$ (for a real number d) if $\Delta > 0$, and in one of the forms d or $d\mathrm{i}$ (for a real number d) if $\Delta < 0$. Further, σ and ζ have their usual meaning as functions constructed from $\wp$ (see [20], Ch. XX). The curve (x, y) is now described by

$$(12)\qquad \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = B \begin{pmatrix} \int_{s_0}^{s} \varphi_1(t - c_0)\,\mathrm{d}t \\ \int_{s_0}^{s} \varphi_2(t - c_0)\,\mathrm{d}t \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for some numbers $x_0, y_0, s_0 \in \mathbb{R}$ and a non-degenerate complex matrix B. Because of the fact that $\varphi_1\varphi_2' - \varphi_2\varphi_1' = 1$, this curve will be parametrised by equi-affine arc-length if and only if $\det B = 1$.

The above construction can be simplified if the functions $\mathfrak{Re}\varphi_1(s - c_0)$ and $\mathfrak{Im}\varphi_1(s - c_0)$ are independent, for then we can write simply

$$(13)\quad \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = B \begin{pmatrix} \mathfrak{Re}\Big(\dfrac{\sigma(s - c_0 + c)}{\sigma(s - c_0)} \exp\Big(\Big(\dfrac{-\wp'(c)}{2\wp(c)} - \zeta(c)\Big)(s - c_0)\Big)\Big) \\ \mathfrak{Im}\Big(\dfrac{\sigma(s - c_0 + c)}{\sigma(s - c_0)} \exp\Big(\Big(\dfrac{-\wp'(c)}{2\wp(c)} - \zeta(c)\Big)(s - c_0)\Big)\Big) \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where the matrix B has now real entries, and should have a determinant appropriately chosen so as to obtain the equi-affine normalisation for the curve (x, y).

Case (A). $g_2 \neq 0$, $\Delta > 0$, and the closed component of $\mathcal{C}$ is described. – Since the image of the curve γ travels along the closed branch of $\mathcal{C}$ in the phase plane in Figure 1 (left), there necessarily holds $c_0 = \varpi_2$. We consider three subcases, depending on the sign of $q = \min \kappa$.

(A.1). $q > 0$. – In this case it is no restriction to write $q = 1$, which can always be achieved by a rescaling. Due to (10), the constant $Q > 1$ determines the invariants g_2, g_3, which will be strictly positive in our case, and consequently the numbers $\varpi_1 \in \mathbb{R}$ and $\varpi_2 \in \mathrm{i}\mathbb{R}$ can be found as well. There holds $c = \varpi_1 + d\mathrm{i}$, and formula (13) is valid as well.

Let us now search for a periodicity condition on γ. The period of γ has to be a multiple of $2\varpi_1$ which is the period of κ. From the quasi-periodicity of σ (see [20], §22.421) can be deduced that $X(s) = x(s) + \mathrm{i}y(s)$ satisfies

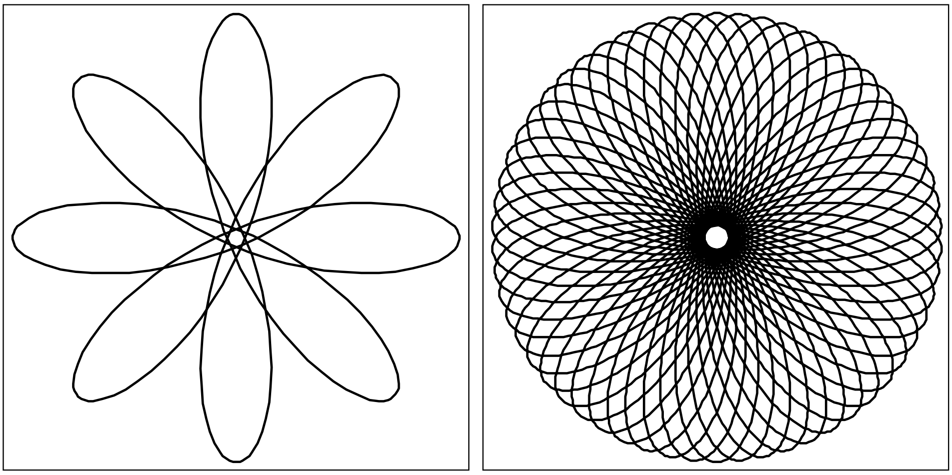
$$(14)\qquad X(s + 4m\varpi_1) = \exp\Big(-4m\Big(\Big(\frac{\wp'(c)}{2\wp(c)} + \zeta(c)\Big)\varpi_1 - \zeta(\varpi_1)c\Big)\Big)X(s).$$

[6] There are two possible ways to choose the constant c in each period parallelogram in such a way that the condition $\wp(c) = -g_3/g_2$ is satisfied, but this choice does not influence the geometry of the curve.

Thus the curve closes up if and only if

$$\left(\left(\frac{\wp'(c)}{2\wp(c)} + \zeta(c)\right)\varpi_1 - \zeta(\varpi_1)c\right)\frac{2\mathrm{i}}{\pi} = \frac{n}{m} \tag{15}$$

for some natural numbers n, m. It should be noticed that the imaginary part of the left-hand side of (15) vanishes automatically. A period for the curve is given by $4m\varpi_1$ but for m even, $2m\varpi_1$ will also be a period of the curve.



m	n	Q	ϖ_1	ϖ_2	c	Figure
4	5	8.947 960	1.009 840	(1.086 362)i	ϖ_1 − (1.058 687)i	Left.
29	37	6.926 543	1.129 313	(1.239 778)i	ϖ_1 − (1.194 744)i	Right.

Figure 2. Some closed curves from Case (A.1) and the corresponding data.

Two of such closed curves have been drawn in Figure 2. (We refer to [21, Figure 3] for a graphical representation of the closedness condition on the data d, m/n and Q.)

The critical observer will have noticed that these curves resemble the familiar hypotrochoid suspiciously well. It should be stressed that these are different curves, as follows from the fact that the hypotrochoids do not satisfy the Euler-Lagrange equation (9).

Let us now explain why these curves have such a "Euclidean" symmetry. The curve $s \mapsto \gamma(s - 2\varpi_1)$ has the same curvature function as γ, and consequently there exists an equi-affine congruence ψ under which this curve is sent to γ. Since γ closes up after $4m\varpi_1$, there necessarily holds $\psi^{2m} = \mathbf{1}$, which implies that the linear part of this mapping has two complex-conjugate eigenvalues. The points of the form $\psi^l(p)$, for a fixed point p, are then easily seen to lie on an ellipse.

We conclude that the group of orientation-preserving equi-affine symmetries of a closed curve consists of the orientation-preserving equi-affine transformations which preserve $2m$ points separated at equal equi-affine distance on an ellipse. With respect to the Euclidean metric for which this ellipse becomes a circle, this symmetry group becomes a finite group of rotations by an angle which is a multiple of π/m.

As such, intending to most appropriately visualise these affine curves on a sheet of paper which was endowed with a Euclidean metric anyway, I have opted to select this affine representative of the curve among all its affine transforms which solve the variational problem equally well, for which the ellipse joining the points of maximal equi-affine curvature becomes a circle. It is in this way that the geometrical properties of the curves under consideration are most enjoyed by an audience living in a Euclidean, and not an affine, world.

(A.2). $q = 0$. – From (10) it follows that $g_2 = \frac{1}{9}Q^2$ and $g_3 = 0$. The choice of Q merely influences the curve by an affine mapping, and therefore we will assume $Q = 1$. The curve can most conveniently be described by making use of the fact that (up to a rescaling of the parameter) $\beta = \gamma/\sqrt{\kappa}$ describes a centro-affinely parameterised straight line. In this way we find, up to a full-affine motion, $\big(x(s), y(s)\big) = \pm\sqrt{\kappa(s)}\big(1, s\big)$, where $\kappa(s) = -6\wp(s - \varpi_2)$. It should be remarked that the curve is not smooth at the points $s = (2l+1)\varpi_1$ (for $l \in \mathbb{Z}$), unless a change sign in the right-hand side is induced at these points. As is clear from Figure 3, this enforces the path described by β to be split in two parallel lines.

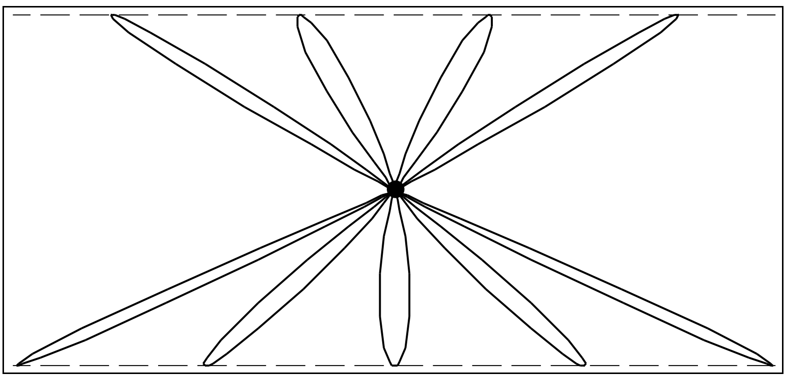


Figure 3. A part of the curve, arising from $q = 0$ and $Q > 0$, as in Case (A.2). Parts of the two dashed lines are described by β, and the dot in the middle represents the origin.

(A.3). $q < 0$. – The left-most intersection point of $\mathcal{C}$ with the axis $\kappa' = 0$, which has been denoted by P in Figure 1, is given by $P = -q - Q$ and is strictly smaller then q. Therefore, assuming $q = -1$, we necessarily have $Q > 2$, and from (10) follows that $g_2 > 0$ and $g_3 < 0$. The constant c can be taken in the form $c = \varpi_2 + d$ where

$0 \leqslant d \leqslant \varpi_1$. The x-co-ordinate of the curve, which is given by (12), satisfies (14). A simple calculation shows that

$$\Big(\Big(\frac{\wp'(c)}{2\wp(c)} + \zeta(c)\Big)\varpi_1 - \zeta(\varpi_1)c\Big)\frac{2\mathrm{i}}{\pi}$$
$$= 1 + \Big(\varpi_1\sqrt{\frac{-g_3}{g_2}} - \zeta(\varpi_1)d + \varpi_1\zeta(\varpi_2 + d) - \varpi_1\zeta(\varpi_2)\Big)\frac{2\mathrm{i}}{\pi},$$

and since the quantity between the brackets on the right-hand side never vanishes, as can be shown numerically, the curve can never be periodic. We refer to [21, Figure 5] for a figure displaying some of these curves.

A complete illustrated discussion of the remaining cases, which has been omitted here, can be found in [21] as well:

Case (B). $g_2 \neq 0$, $\Delta > 0$, and the non-closed branch of $\mathcal{C}$ is described.

Case (C). $g_2 \neq 0$ and $\Delta < 0$.

Case (D). $g_3 < 0$ and $\Delta = 0$.

Case (E). $g_3 > 0$ and $\Delta = 0$.

Case (F). $g_2 = 0$ and $g_3 \neq 0$.

Case (G). $g_2 = g_3 = 0$.

This finishes our description of the critical points of $\int \kappa \,\mathrm{d}s$ under area-constraint.

Remark 7. It seems noteworthy to mention that the curves of case (F) have already been studied in another context. We recall that an affine-orthogonal net of curves is a pair of one-parameter families of curves for which at each point the tangent of the curve of the first family is the affine normal of the curve of the second family and conversely. Now consider an affine-orthogonal net of curves which is invariant under the one-parameter family of translations generated by a vector v. Then the curves for which the tangent is harmonically conjugate to v w.r.t. the two tangents of the curves of the net precisely have natural equation $\kappa(s) = -6\wp(s)$ with $g_2 = 0$ (see [7], pp. 147–152, and [8]).

Remark 8. The integration of the Euler-Lagrange equation (7), which characterises the critical points of the total equi-affine curvature w.r.t. deformations under which the total equi-affine arc-length is preserved, reveals that the function $f = -\frac{1}{6}\kappa + \frac{1}{12}A$ satisfies the same equation $(f')^2 = 4f^3 - g_2 f - g_3$, where $g_2 = \frac{1}{12}A^2$ and g_3 stands for an integration constant. In the generic case when the corresponding cubic discriminant does not vanish, there holds $\kappa(s) = -6\wp(s - c_0) + \frac{1}{2}A$, and the derivatives of the co-ordinate functions satisfy an adaption of the equation of Lamé, which can be integrated with the result

$$\begin{cases} x(s) = -\zeta(s-c_0) + \frac{1}{12}As; \\ y(s) = \frac{1}{6}A\zeta(s-c_0)(s-c_0) + \wp(s-c_0) - \big(\zeta(s-c_0)\big)^2 - \big(\frac{1}{12}A(s-c_0)\big)^2, \end{cases}$$

up to an affine mapping. Some of such curves have been drawn in [21, Figure 10–11].

4. The functional $\int \sqrt{\kappa}\,\mathrm{d}s$

In this section, we will restrict our attention to *strictly positively curved* curves, *i.e.*, curves in the equi-affine plane with strictly positive equi-affine curvature. A functional which is of interest for such curves is given by $\int \sqrt{\kappa}\,\mathrm{d}s$.

A first motivation for the study of this functional is perhaps the fact that its Euclidean counterpart $\int \sqrt{\kappa_{\mathbf{E}}}\,\mathrm{d}s_{\mathbf{E}}$ has been studied already in [3] §27; in particular, the catenaries turned out to be the critical points of this functional.

But more importantly, the functional $\int \sqrt{\kappa}\,\mathrm{d}s$ is an invariant with respect to the *full affine* group. In the differential geometry of curves w.r.t. the full affine group, a *full-affine arc-length* element and a *full-affine curvature* have been defined ([10]; [16]; [18] §10.v; [19]). These are related to the equi-affine invariants by the formulae

$$\mathrm{d}s_{\mathbf{F}} = \sqrt{\kappa}\,\mathrm{d}s \qquad \text{and} \qquad \kappa_{\mathbf{F}} = \frac{\kappa'}{2\kappa^{3/2}}. \tag{16}$$

The class of curves for which the full-affine curvature is constant, which includes for instance the logarithmical spirals, has been described already by Blaschke.[7] These curves, which will be called *full-affine W-curves*, are precisely the orbits of a point under a one-parameter family of full-affine transformations.

In [21] it is recalled how these full-affine arc-length and curvature are classically described without relying on equi-affine notions (cf. [10], [16], [19]). Here we restrict ourselves to a new interpretation of the full-affine arc-length.

First, by a *pointed parabola* will be understood a parabola in $\mathbb{A}^2$ of which a special point has been singled out. Because every two such parabolas can be matched by means of a unique equi-affine orientation-preserving transformation bringing the special points into correspondence, the space of all pointed parabolas can be simply seen as the equi-affine group, once a standard pointed parabola has been singled out.

Next, we recall that a quadratic form φ of index q on $\mathbb{R}^{n+1}$ automatically endowes the hyperquadric $\varphi = -1$ with a pseudo-Riemannian metric of index $q-1$

[7] See [4, §10], but also [8], p. 91, footnote 2; [13], p. 52, note 1.3.1; [16], §5; [19], p. 17, ex. 5.1.

by restriction to its tangent spaces. When this is applied to the Lie group $\mathrm{SL}(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}\colon\ ad - bc = 1 \right\}$, where $\varphi = -\det$ is regarded as a quadratic form, we conclude that the metric g defined by $g(v_P, v_P) = -\det(v_P)$ (for $P \in \mathrm{SL}(2)$ and $v_P \in \mathrm{T}_P\mathrm{SL}(2) \subseteq \mathbb{R}^{2\times 2}$), which is obviously bi-invariant w.r.t. the group structure, makes the special linear group into a pseudo-Riemannian space $(\mathrm{SL}(2), g)$ of Lorentzian signature which is isometric to the pseudohyperbolic space H_1^3 of constant sectional curvature -1.

Finally, for an arbitrarily parameterised curve $\gamma\colon\ \mathbb{R} \to \mathbb{A}^2\colon\ t \mapsto \gamma(t)$, the osculating parabola of γ at $\gamma(t)$, endowed with the special point $\gamma(t)$, can be considered as an element of the equi-affine group. By omitting the translational part from this curve of osculating parabolas, a curve in a pseudo-Riemannian space results, which will be called the *osculating parabolic congruence* of γ:

$$\mathscr{P}_\gamma\colon\ \mathbb{R} \to (\mathrm{SL}(2), g).$$

We then have the following result, which is reminiscent of the interpretation of the Willmore integral of a surface in the conformal space as the area of its central sphere congruence (see [5], §69 and 74.21):

Theorem 9. *The full-affine arc-length of the curve γ is precisely the pseudo-Riemannian arc-length of its osculating parabolic congruence $\mathscr{P}_\gamma$.*

We refer to [21, Figure 13] for a visual representation of some geodesics of $(\mathrm{SL}(2), g)$ as one-parameter families of parabolas.

Let us now return to the full-affine arc length from the variational point of view. We remark that unlike the Euclidean and equi-affine variational formulae

$$\begin{cases} \delta \int \mathrm{d}s_{\mathbf{E}} \propto \int f\kappa_{\mathbf{E}}\,\mathrm{d}s_{\mathbf{E}} \text{ (w.r.t. variational vector field } fN_{\mathbf{E}} + gT_{\mathbf{E}}); \\ \delta \int \mathrm{d}s \ \propto \int f\kappa\,\mathrm{d}s \text{ (w.r.t. variational vector field } fN + gT), \end{cases}$$

a similar formula does not hold for the full-affine geometry. In fact, the following variational formula w.r.t. the variational vector field $fN + gT$ can be deduced from (3):

$$\delta \int \sqrt{\kappa}\,\mathrm{d}s = -\int \frac{f}{12}\Big(\Big(\frac{\kappa'}{\kappa^{3/2}}\Big)''' + \kappa\Big(\frac{\kappa'}{\kappa^{3/2}}\Big)'\Big)\,\mathrm{d}s. \tag{17}$$

Theorem 10. *Among the strictly positively curved curves in the affine plane, the critical points of the full-affine arc-length are precisely the full-affine W-curves and the curves for which the full-affine curvature is a non-zero linear function of the position vector w.r.t. a suitable origin.*

Proof. In view of the variational formula (17), the equi-affine curvature function κ and the full-affine curvature function $\kappa_{\mathbf{F}}$ of such a curve are related by

$$(\kappa_{\mathbf{F}})''' + \kappa(\kappa_{\mathbf{F}})' = 0. \tag{18}$$

According to (1), the functions x, y and 1 span the set of solutions of the homogeneous third-order differential equation $\xi''' + \kappa\xi' = 0$, and therefore we have $\kappa_{\mathbf{F}} = Ax + By + C$. If A or B is not equal to zero, a suitable translation of the co-ordinate system reduces this equation to the form $\kappa_{\mathbf{F}} = Ax + By$. □

Theorem 11.

(i) *The ellipses are the only simple closed strictly positively curved curves which are a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$.*

(ii) *The ellipses are the only simple closed strictly positively curved curves which are a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$ under area constraint.*

(iii) *The ellipses are the only simple closed strictly positively curved curves which are a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$ under equi-affine arc-length constraint.*

(iv) *The ellipses are the only simple closed strictly positively curved curves which are a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$ under total equi-affine curvature constraint.*

Proof. (i) Assume that a simple closed strictly positively curved curve is a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$ but is not an ellipse.

Because the ellipses are the only closed strictly positively curved full-affine W-curves, there holds $\kappa_{\mathbf{F}} = Ax + By$, where at least one of the constants A or B is non-zero. The equation $Ax + By = 0$ defines a straight line in the affine plane which cuts the curve in at most two points, and we conclude that there will be at most two critical points of the equi-affine curvature, *i.e.*, *sextactic points*, which contradicts an adaption of the four-vertex theorem (see [4], §19).

(ii) Assume that a simple closed strictly positively curved curve is a critical point of $\int \sqrt{\kappa}\,\mathrm{d}s$ under area constraint, but is not an ellipse. The curve $\gamma = (x, y)$ will be described w.r.t. a co-ordinate system which makes the support function ϱ strictly positive. The Euler-Lagrange equation expresses that

$$(\kappa_{\mathbf{F}})''' + \kappa(\kappa_{\mathbf{F}})' = Q \qquad \text{(for some constant } Q\text{)}, \tag{19}$$

where $\kappa_{\mathbf{F}} = \frac{1}{2}\kappa'/\kappa^{3/2}$ as in (16) and primes denote derivatives w.r.t. s. If R is a local primitive of the equi-affine support function ϱ then QR is a solution of this linear inhomogeneous differential equation for $\kappa_{\mathbf{F}}$, whereas the solution of the corresponding homogeneous differential equation was described in the proof of Theorem 10.

Therefore the full-affine curvature function $\kappa_{\mathbf{F}}$ of the curve has to satisfy

$$\kappa_{\mathbf{F}} + Ax + By + C = QR \tag{20}$$

for some constants A, B and C. The left-hand side of the above equation is the sum of four periodic functions, and hence the right-hand side should also be periodic. However $R' = \varrho$ does not change sign, and consequently there necessarily holds $Q = 0$, which means that we are in the above case (i).

(iii) This is similar to the previous case. Instead of the formulae (19) and (20) one should use, respectively,

$$(\kappa_{\mathbf{F}})''' + \kappa(\kappa_{\mathbf{F}})' = Q\kappa \qquad \text{and} \qquad \kappa_{\mathbf{F}} + Ax + By + C = Qs.$$

(iv) Here one should use, respectively,

$$(\kappa_{\mathbf{F}})''' + \kappa(\kappa_{\mathbf{F}})' = Q\left(\kappa'' + \kappa^2\right) \qquad \text{and} \qquad \kappa_{\mathbf{F}} + Ax + By + C = QK,$$

where K is a local primitive of κ (and hence non-periodic). □

Remark 12. From (8), we immediately obtain a *full-affine isoperimetric inequality*

$$\int \mathrm{d}s_{\mathbf{F}} \leqslant 2\pi$$

for simple closed strictly positively curved curves, in which equality is attained exactly for ellipses (cf. [12], §10). Furthermore, the total full-affine curvature vanishes for every closed curve:

$$\int \kappa_{\mathbf{F}}\,\mathrm{d}s_{\mathbf{F}} = \frac{1}{2}\int \frac{\kappa'}{\kappa}\,\mathrm{d}s = \frac{1}{2}\int \mathrm{d}\log\kappa = 0.$$

Remark 13. The Euler-Lagrange equation (18) can be written completely in full-affine invariants:

$$\frac{\partial^3\kappa_{\mathbf{F}}}{\partial(s_{\mathbf{F}})^3} + 3\kappa_{\mathbf{F}}\frac{\partial^2\kappa_{\mathbf{F}}}{\partial(s_{\mathbf{F}})^2} + \left(\frac{\partial\kappa_{\mathbf{F}}}{\partial s_{\mathbf{F}}}\right)^2 + (2(\kappa_{\mathbf{F}})^2 + 1)\frac{\partial\kappa_{\mathbf{F}}}{\partial s_{\mathbf{F}}} = 0. \tag{21}$$

This equation, which *should* have been obtained in [17, eq. (33)], has been contended to lack the simplicity of the Euler-Lagrange equation for the critical points of the projective arc-length functional which was derived by É. Cartan ([9], eq. (16)):

$$\frac{\partial^3\kappa_{\mathbf{P}}}{\partial(s_{\mathbf{P}})^3} + 8\kappa_{\mathbf{P}}\frac{\partial\kappa_{\mathbf{P}}}{\partial s_{\mathbf{P}}} = 0, \quad \text{or yet (for some } C \in \mathbb{R}), \quad \frac{\partial^2\kappa_{\mathbf{P}}}{\partial(s_{\mathbf{P}})^2} + 4(\kappa_{\mathbf{P}})^2 = C.$$

In this article we have seen in (6) as well as in (18) an Euler-Lagrange equation which is similar to Cartan's equation.

Example 14. Let us end with an example: The curve given by $\kappa_{\mathbf{F}}(s_{\mathbf{F}}) = \frac{3}{\sqrt{2}}\tanh(\sqrt{2}s_{\mathbf{F}})$ satisfies eq. (21). A figure of this curve can be found in [21] (Figure 14).

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