Jure Kališnik

Representations of étale Lie groupoids and modules over Hopf algebroids

Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 3, 653–672

Persistent URL: `http://dml.cz/dmlcz/141628`

Czechoslovak Mathematical Journal, 61 (136) (2011), 653–672

REPRESENTATIONS OF ÉTALE LIE GROUPOIDS AND MODULES OVER HOPF ALGEBROIDS

Jure Kališnik, Ljubljana

(Received March 17, 2010)

Abstract. The classical Serre-Swan's theorem defines an equivalence between the category of vector bundles and the category of finitely generated projective modules over the algebra of continuous functions on some compact Hausdorff topological space. We extend these results to obtain a correspondence between the category of representations of an étale Lie groupoid and the category of modules over its Hopf algebroid that are of finite type and of constant rank. Both of these constructions are functorially defined on the Morita category of étale Lie groupoids and we show that the given correspondence represents a natural equivalence between them.

Keywords: étale Lie groupoids, Hopf algebroids, representations, modules, equivalence, Morita category

MSC 2010: 16D40, 16D90, 19L47, 22A22, 58H05

1. Introduction

There are many phenomena in different areas of mathematics and physics that are most naturally described in the language of smooth manifolds and smooth maps between them. However, some natural constructions, coming from the theory of foliations or from Lie group actions, result in slightly more singular spaces and require a different approach. The Morita category of Lie groupoids and principal bundles [7], [12], [14], [21], [22], [25], [26], [27], [30] provides a natural framework in which to study many such singular spaces like spaces of leaves of foliations [7], [9], [10], [13], [24], [39], spaces of orbits of Lie group actions or for example orbifolds [1], [7], [23], [24] and [34]. A Lie groupoid can be viewed as an atlas for the given singular space. It turns out that different Lie groupoids represent the same geometric space precisely when they are Morita equivalent, i.e. when they are isomorphic in the Morita category. For this reason we are primarily interested in those algebraic invariants of Lie groupoids which are functorially defined on the Morita category of Lie groupoids.

The theory of representations of Lie groupoids naturally extends the notions of tangent bundles, bundles of higher order tensors and line bundles to the category of étale Lie groupoids [1], [24]. It entails well known constructions such as equivariant vector bundles [3], [35], orbibundles over orbifolds [1], foliated and transversal vector bundles over spaces of leaves of foliations [9], [10], [17], [24], as well as vector bundles over manifolds or representations of discrete groups. The construction of the category of representations of a Lie groupoid is invariant under the Morita equivalence and thus represents one of the basic algebraic invariants of Lie groupoids.

The Hopf algebroid of smooth functions with compact support [6], [7], [28], [29], [32] on an étale Lie groupoid is another example of such an invariant. Smooth functions with compact support on a smooth manifold, group Hopf algebras of discrete groups, matrix algebras and noncommutative tori are all examples of Hopf algebroids of functions. It turns out that the Hopf algebroid of smooth functions with compact support on an étale Lie groupoid completely determines the groupoid up to an isomorphism [29]. Moreover, the Morita category of étale Lie groupoids admits a precise algebraic description as the Morita category of locally grouplike Hopf algebroids and locally grouplike principal bimodules [16]. We are therefore interested in characterizing those modules over Hopf algebroids of functions which correspond to representations of étale Lie groupoids in the spirit of Serre and Swan [36], [37].

The paper is organized as follows. In Section 3 we first recall the definition of a natural action of the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of an étale Lie groupoid G on the space of sections $\Gamma_c^\infty(E)$ of any representation E of G. We then define the notion of a module of finite type and of constant rank, which characterizes modules of sections, and show that such modules form an additive monoidal category for any étale Lie groupoid in a functorial fashion and therefore represent a Morita invariant.

In Section 4 (Theorem 4.1) we naturally extend the Serre-Swan's correspondence to the category of étale Lie groupoids. The functor of smooth sections with compact support defines an equivalence between the category of representations of an étale Lie groupoid G and the category of modules over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ which are of finite type and of constant rank. These modules generalize finitely generated projective modules over algebras of functions and coincide with them if G is the unit groupoid of a compact connected manifold.

Categories of representations and of modules of finite type and of constant rank are both additive monoidal categories, and by passing to the sets of isomorphism classes they can be viewed as contravariant functors Rep and Mod from the Morita category of étale Lie groupoids to the category of semirings. Our main result in this paper (Theorem 5.1) shows that in the framework of étale Lie groupoids the

Serre-Swan's correspondence can be understood as a natural equivalence between these two functors with respect to generalized maps between Lie groupoids.

An alternative approach has been recently studied in [2], where representations up to homotopy of Lie groupoids are characterized in terms of modules over the differential graded algebra of smooth cochains on a groupoid.

2. Basic definitions and examples

2.1. The Morita category of Lie groupoids. For the convenience of the reader, and to fix the notation, we begin by summarizing some basic definitions concerning Lie groupoids that will be used throughout this paper. We refer the reader to one of the books [18], [24], [25] for a more detailed exposition and further examples.

A *Lie groupoid* over a smooth, second countable, Hausdorff manifold M is given by a smooth manifold of arrows $G_1 = G$ and a structure of a category on G_1 with objects $G_0 = M$, in which all the arrows are invertible and all the structure maps

$$G_1 \times^{s,t}_{G_0} G_1 \xrightarrow{\text{mlt}} G_1 \xrightarrow{\text{inv}} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{\text{uni}} G_1$$

are smooth. We allow the manifold G_1 to be non-Hausdorff, but we assume that the source map s is a submersion with Hausdorff fibers. If the space of arrows G_1 is Hausdorff, we call G a *Hausdorff* groupoid. For any $x, y \in G_0$ we denote by $G(x, y) = s^{-1}(x) \cap t^{-1}(y)$ the manifold of arrows from x to y.

A Lie groupoid is *étale* if all of its structure maps are local diffeomorphisms. A *bisection* of an étale Lie groupoid G is an open subset V of G such that both $s|_V$ and $t|_V$ are injective. Bisections of the groupoid G form a basis for the topology on G, so in particular they can be chosen arbitrarily small.

Generalized morphisms between Lie groupoids [7], [13], [14], [21], [25], [26], [27], [30] turn out to be the right notion of a map between Lie groupoids. They are closely connected to groupoid actions and principal bundles, which we briefly describe.

A smooth *left action* of a Lie groupoid G on a smooth manifold P along a smooth map $\pi\colon P \to G_0$ is a smooth map $\mu\colon G \times^{s,\pi}_{G_0} P \to P$, $(g, p) \mapsto g \cdot p$, which satisfies $\pi(g \cdot p) = t(g)$, $1_{\pi(p)} \cdot p = p$ and $g' \cdot (g \cdot p) = (g'g) \cdot p$ for all $g', g \in G$ and $p \in P$ with $s(g') = t(g)$ and $s(g) = \pi(p)$. Right actions of Lie groupoids on smooth manifolds are defined in a similar way.

A *principal H-bundle over G* is a manifold P, equipped with a left action μ of G along a smooth surjective submersion $\pi\colon P \to G_0$, and a right action η of H along a smooth map $\varphi\colon P \to H_0$, such that (i) φ is G-invariant, π is H-invariant and both actions commute: $\varphi(g\cdot p)=\varphi(p)$, $\pi(p\cdot h)=\pi(p)$ and $g\cdot(p\cdot h)=(g\cdot p)\cdot h$ for all $g\in G$, $p\in P$ and $h\in H$ with $s(g)=\pi(p)$ and $\varphi(p)=t(h)$, (ii) $\pi\colon P\to G_0$ is a principal right H-bundle: the map $(\mathrm{pr}_1,\eta)\colon P\times^{\varphi,t}_{H_0}H\to P\times^{\pi,\pi}_{G_0}P$ is a diffeomorphism.

Any smooth functor $\psi\colon G\to H$ defines a principal H-bundle $\langle\psi\rangle=G_0\times^{\psi,t}_{H_0}H$ over G with the actions given by the maps $g\cdot(x,h)=(t(g),\psi(g)h)$ for $g\in G(x,y)$ and $(x,h)\cdot h'=(x,hh')$ for $h,h'\in H$ such that $s(h)=t(h')$.

Principal H-bundles P and P' over G are isomorphic if there exists an equivariant diffeomorphism between them. A principal bundle P is isomorphic to one induced by a functor if and only if it is trivial [25], [26], [27].

If P is a principal H-bundle over G and if P' is a principal K-bundle over H for another Lie groupoid K, one defines the composition $P\otimes_H P'$ [25], [26], [27], which is a principal K-bundle over G. It is the quotient of $P\times_{H_0}P'$ with respect to the diagonal action of the groupoid H. So defined composition is associative up to a natural isomorphism, while Lie groupoids, viewed as principal bundles, act as units.

The *Morita category* GPD of Lie groupoids consists of Lie groupoids as objects and isomorphism classes of principal bundles as morphisms between them [25], [26]. The morphisms in GPD are sometimes referred to as Hilsum-Skandalis maps or generalized morphisms between Lie groupoids. Lie groupoids G and H are Morita equivalent if they are isomorphic in the category GPD. The *Morita category of étale Lie groupoids* EtGPD is the full subcategory of the category GPD with étale Lie groupoids as objects.

2.2. Representations of Lie groupoids. Let G be a Lie groupoid and let E be a smooth complex vector bundle over G_0. A *representation* of the groupoid G on E is a smooth left action $\varrho\colon G\times_{G_0}E\to E$, denoted by $\varrho(g,v)=g\cdot v$, of G on E along the bundle projection $p\colon E\to G_0$, such that for any arrow $g\in G(x,y)$ the induced map $g_*\colon E_x\to E_y$, $v\mapsto g\cdot v$, is a linear isomorphism [2], [4], [18], [24], [38]. We will restrict ourselves to smooth complex representations of constant rank, although similar formulas apply in other settings as well.

A *morphism* between representations E and E' of G is a G-equivariant morphism $\varphi\colon E\to E'$ of vector bundles over G_0. For any two representations E and E' of a Lie groupoid G one naturally defines their *direct sum* $E\oplus E'$ and *tensor product* $E\otimes E'$. These operations are associative up to natural isomorphisms, with the trivial representation $\mathbb{C}$ of the groupoid G acting as a unit for the tensor product. Representations of a Lie groupoid G, together with morphisms between them, form an additive monoidal category which will be denoted by $\mathsf{Rep}(G)$ [20], [38]. The direct sum and tensor product operations turn the set $\mathrm{Rep}(G)$ of isomorphism classes of representations of a Lie groupoid G into a semiring [3].

Example 2.1. (1) Let M be a smooth Hausdorff manifold and let Γ be a finite group. The *trivial bundle of finite groups* $G=M\times\Gamma$ has a natural structure of a Lie groupoid over M. For any vector bundle E over M and any representation V of Γ one obtains the tensor product representation $E \otimes \mathbf{V}$ of G, where G acts naturally on $\mathbf{V} = M \times V$ and trivially on E. It turns out that every representation of G can be decomposed as a direct sum of such representations [35].

(2) With any left action of a Lie group K on a manifold M one associates the *translation groupoid* $K \ltimes M$ [7], [24], [31] over M. Its space of arrows is equal to $K \times M$ while the structure maps are induced from the action of the group K on M. Representations of the groupoid $K \ltimes M$ then correspond to K-equivariant vector bundles over M [3], [35].

Generalized maps between groupoids can be used to pull back representations in the same sense as vector bundles can be pulled back along smooth maps. Let G and H be Lie groupoids and let P be a principal H-bundle over G. For any representation E of H one defines the *pullback representation* $P^*E = P \otimes_H E$ of G as follows (see [15] for details). The pullback bundle $P \times_{H_0} E$ has a natural structure of a vector bundle over P with projection onto the first factor as the projection map. The groupoid H acts diagonally from the right on $P \times_{H_0} E$, along the fibers of the projection onto G_0, and it is easy to see that the natural map $P \otimes_H E = P \times_{H_0} E/H \to G_0$ is well defined, smooth and makes $P \otimes_H E$ a vector bundle over G_0. Finally, the action of the groupoid G on the space P induces a representation of the groupoid G on the bundle $P \otimes_H E$ by acting on the first factor.

One can use an alternative description in the case of trivial bundles, i.e. when the principal bundle comes from a smooth functor. Suppose that $\psi \colon G \to H$ is a smooth functor between Lie groupoids and let E be a representation of H. One defines a representation ψ^*E of G on the vector bundle ψ_0^*E over G_0 with the action $g \cdot (x, v) = (t(g), \psi(g)v)$ for $g \in G(x, y)$ and $v \in E_{\psi_0(x)}$. So defined representation is naturally isomorphic to the representation $\langle\psi\rangle \otimes_H E$ of G via the isomorphism $\langle\psi\rangle \otimes_H E \to \psi^*E$ which sends an element $(x, h) \otimes v$ to the element $(x, h \cdot v)$.

The construction of pulling back representations along a principal bundle P extends to a covariant functor $\mathsf{Rep}(P) \colon \mathsf{Rep}(H) \to \mathsf{Rep}(G)$ between the categories of representations. Isomorphic principal bundles induce naturally equivalent functors, so one obtains a well defined map $\mathrm{Rep}(P) \colon \mathrm{Rep}(H) \to \mathrm{Rep}(G)$, which depends only on the isomorphism class of P. By noting that the pullback representation is locally just a pullback along a smooth map, it follows that $P^*(E \oplus E') \cong P^*E \oplus P^*E'$ and $P^*(E \otimes E') \cong P^*E \otimes P^*E'$, which shows that the map $\mathrm{Rep}(P)$ is a homomorphism of semirings. We thus obtain a contravariant functor

$$\mathrm{Rep} \colon \mathsf{GPD} \to \mathsf{Rng}$$

from the Morita category of Lie groupoids to the category of semirings.

2.3. Hopf algebroids and principal bimodules. For any Hausdorff étale Lie groupoid G one can naturally define a convolution product on the space $\mathcal{C}_c^\infty(G)$ of smooth functions with compact support on the space of arrows of G by the formula $(a * a')(h) = \sum a(g)a'(g')$ for any $a, a' \in \mathcal{C}_c^\infty(G)$, where the sum is over all pairs $g, g' \in G$ with $h = gg'$. The vector space $\mathcal{C}_c^\infty(G)$ with this multiplication is an associative algebra called the *convolution algebra* of the étale Lie groupoid G [6], [7]. It is in general noncommutative but contains the algebra $\mathcal{C}_c^\infty(G_0)$ of functions on the space of objects of G with pointwise multiplication as a commutative subalgebra.

In the case of a general étale Lie groupoid a suitable notion of a smooth function with compact support on a non-Hausdorff manifold is needed [8]. Considering that smooth functions on a Hausdorff manifold M correspond precisely to the continuous sections of the sheaf of germs of smooth complex valued functions on M, it makes sense to use this alternative approach to define smooth functions with compact support on an arbitrary manifold P. One first considers the vector space of all (not-necessarily continuous) sections of the sheaf of germs of smooth functions on P. The trivial extension of any smooth function with compact support in a Hausdorff open subset of P naturally represents a section of that sheaf. The vector space $\mathcal{C}_c^\infty(P)$ of smooth functions with compact support on P is then defined to be the subspace of the space of all sections, generated by such sections. This definition of the vector space $\mathcal{C}_c^\infty(P)$ agrees with the classical one if P is Hausdorff. The support, i.e. the set where the values of the section are nontrivial, of any function in $\mathcal{C}_c^\infty(P)$ is always a compact subset of P, but not necessarily closed if P is a non-Hausdorff manifold. If G is a (non-Hausdorff) étale Lie groupoid, then $\mathcal{C}_c^\infty(G)$ is generated as a vector space by functions with supports in bisections of G. Any $a \in \mathcal{C}_c^\infty(G)$ with support in a bisection V can be written in the form $a = a_0 \circ t|_V$ for a unique $a_0 \in \mathcal{C}_c^\infty(t(V))$. If $b \in \mathcal{C}_c^\infty(G)$ is another function, with support in a bisection W of G, one defines $a * b = (a_0 \circ t|_V)(b_0 \circ t|_W) = \big(a_0 \cdot \big(b_0 \circ s|_V \circ (t|_V)^{-1}\big)\big) \circ t|_{V\cdot W}$, where $V \cdot W = \{gh \,|\, g \in V,\, h \in W,\, s(g) = t(h)\}$ is the product bisection of bisections V and W, to obtain the convolution algebra $\mathcal{C}_c^\infty(G)$ for any non-Hausdorff étale Lie groupoid G [29]. This definition of the convolution algebra coincides with the original definition due to Connes [6], [7] in the case of a Hausdorff étale Lie groupoid. In general, however, the space $\mathcal{C}_c^\infty(G)$ as defined above contains more functions than the Connes's one.

Example 2.2. (1) The convolution algebra $\mathcal{C}_c^\infty(M \times \Gamma)$ of a trivial bundle of finite groups is naturally isomorphic to the algebra $\mathcal{C}_c^\infty(M) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$, where $\mathbb{C}[\Gamma]$ is the group algebra of the group Γ.

(2) *Noncommutative tori* or irrational rotation algebras are perhaps the most known examples of noncommutative spaces [5], [7], [33]. Let the group $\mathbb{Z}$ act on

S^1 by $k \cdot \mathrm{e}^{\mathrm{i}\varphi} = \mathrm{e}^{2\pi k\theta \mathrm{i}} \cdot \mathrm{e}^{\mathrm{i}\varphi}$ for $k \in \mathbb{Z}$, $\mathrm{e}^{\mathrm{i}\varphi} \in S^1$ and some fixed irrational number $\theta \in (0,1)$. We will denote by $G_\theta = \mathbb{Z} \ltimes S^1$ the associated translation groupoid. Any function $a_0 \in \mathcal{C}_c^\infty(S^1)$ can be expanded into its Fourier series $a_0 = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\varphi}$ where the sequence $\{c_k\}_{k\in\mathbb{Z}}$ is rapidly decaying since a_0 is smooth. First denote by $U = \mathrm{const}_1 \cdot 1$ the function which is equal to 1 on the bisection corresponding to the generator $1 \in \mathbb{Z}$ and is zero elsewhere. It then follows that $U^k = \mathrm{const}_1 \cdot k$ for all $k \in \mathbb{Z}$. Furthermore, let $V = \mathrm{e}^{\mathrm{i}\varphi} \cdot 0$ be the generator of the algebra $\mathcal{C}_c^\infty(S^1)$ in the sense of the Fourier expansion as above. An arbitrary function $a \in \mathcal{C}_c^\infty(G_\theta)$ can now be written as $a = \sum_{k,l} c_{kl} U^k * V^l$, where c_{kl} are nonzero for only finitely many $k \in \mathbb{Z}$ and such that the sequence $\{c_{kl}\}_{l\in\mathbb{Z}}$ decays rapidly for each $k \in \mathbb{Z}$. One checks the equalities $U * V = \mathrm{e}^{\mathrm{i}\varphi} \cdot 1$ and $V * U = \mathrm{e}^{2\pi\theta\mathrm{i}} \mathrm{e}^{\mathrm{i}\varphi} \cdot 1$ and thus $V * U = \mathrm{e}^{2\pi\theta\mathrm{i}} U * V$. The convolution algebra $\mathcal{C}_c^\infty(G_\theta)$ is therefore generated by two elements U and V that satisfy the above relation. The case $\theta = 0$ resembles the usual commutative algebra of functions on a torus which motivates us to call the algebra $\mathcal{C}_c^\infty(G_\theta)$, or a C^*-algebra completion of it, a noncommutative torus.

The convolution algebra $\mathcal{C}_c^\infty(G)$ of an étale Lie groupoid G admits an additional structure of a coalgebra over the commutative subalgebra $\mathcal{C}_c^\infty(G_0)$, which turns the space $\mathcal{C}_c^\infty(G)$ into a *Hopf algebroid* over $\mathcal{C}_c^\infty(G_0)$ [28], [29]. The construction of the Hopf algebroid of an étale Lie groupoid naturally extends to a functor from the Morita category of étale Lie groupoids to the Morita category of Hopf algebroids [27], [28]. More precisely, if G and H are étale Lie groupoids and if P is a principal H-bundle over G, then the space $\mathcal{C}_c^\infty(P)$ of smooth functions with compact support on P has a natural structure of a $\mathcal{C}_c^\infty(G)$-$\mathcal{C}_c^\infty(H)$-bimodule and a structure of a coalgebra over $\mathcal{C}_c^\infty(G_0)$ which is compatible with both actions. Composition of principal bundles is reflected as the tensor product of the corresponding bimodules. This functor induces an equivalence $\mathcal{C}_c^\infty \colon \mathsf{EtGPD} \to \mathsf{LgHoALGD}$ between the Morita category of étale Lie groupoids and the Morita category of locally grouplike Hopf algebroids [16]. A *locally grouplike* Hopf algebroid is given by a pair (A, M), where M is a smooth manifold and A is a Hopf algebroid over $\mathcal{C}_c^\infty(M)$ such that for every $x \in M$ the localized coalgebra A_x (over the algebra $\mathcal{C}_c^\infty(M)_x$ of germs of smooth functions at x) is freely generated by the subset of grouplike elements of A_x which consists of arrows of A at x [29]. Morphisms between locally grouplike Hopf algebroids (A, M) and (B, N) are isomorphism classes of locally grouplike principal A-B-bimodules. An A-B-bimodule $\mathcal{M}$ is *principal* if it is equipped with a compatible $\mathcal{C}_c^\infty(M)$-coalgebra structure (ε, Δ) such that $\varepsilon \colon \mathcal{M} \to \mathcal{C}_c^\infty(M)$ is surjective and such that the induced map $\overline{\Delta} \colon \mathcal{M} \otimes_{\mathcal{C}_c^\infty(N)} B \to \mathcal{M} \otimes_{\mathcal{C}_c^\infty(M)} \mathcal{M}$ is an isomorphism. A principal A-B-bimodule $\mathcal{M}$ is *locally grouplike* if for every $x \in M$ the localized coalgebra $\mathcal{M}_x$ is freely generated by grouplike elements.

A module over a Hopf algebroid of functions on an étale Lie groupoid will mean for us in the sequel just a module over the convolution algebra. Although coalgebra structures are used to define tensor products of modules over such Hopf algebroids, we can describe all constructions in terms of convolution algebras. However, due to the above correspondence we prefer to think of modules coming from representations of Lie groupoids as modules over Hopf algebroids rather than convolution algebras.

We will restrict ourselves to Hausdorff groupoids and manifolds for simplicity, although essentially the same formulas apply in the non-Hausdorff case as well.

3. Modules over Hopf algebroids

For any vector bundle E over a manifold M the vector space $\Gamma_c^\infty(E)$ of smooth sections of E with compact support admits a natural structure of a left module over the algebra $\mathcal{C}_c^\infty(M)$. We recall in this section how to extend that structure to a left action of the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of an étale Lie groupoid G on the vector space $\Gamma_c^\infty(E)$ of sections of an arbitrary representation E of the groupoid G. We describe modules of sections by being of finite type and of constant rank and show that such modules form an additive monoidal category for any étale Lie groupoid.

We will assume all our vector bundles to be of globally constant rank. This is automatically satisfied if for example the manifold of objects of the groupoid is connected. Similar constructions hold however in the case of vector bundles of globally bounded rank as well.

3.1. Module of sections of a representation. We first recall the construction of a left action of the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of an étale Lie groupoid G on the vector space $\Gamma_c^\infty(E)$ of sections of a representation E of G. Define a bilinear map

$$\mathcal{C}_c^\infty(G) \times \Gamma_c^\infty(E) \to \Gamma_c^\infty(E)$$

by the formula

$$(au)(x) = \sum_{t(g)=x} a(g)(g \cdot u(s(g)))$$

for $a \in \mathcal{C}_c^\infty(G)$ and $u \in \Gamma_c^\infty(E)$. Since the function $a \in \mathcal{C}_c^\infty(G)$ has a compact support, there are only finitely many $g \in t^{-1}(x)$ with $a(g) \neq 0$ for each $x \in G_0$, hence au is a well defined section of the vector bundle E. By decomposing a function $a \in \mathcal{C}_c^\infty(G)$ as a sum $a = \sum a_j$ of functions each of which has support contained in some bisection, it follows that $au \in \Gamma_c^\infty(E)$ and we have the following result.

Proposition 3.1. *The space of sections $\Gamma_c^\infty(E)$ has a natural structure of a left module over the Hopf algebroid $\mathcal{C}_c^\infty(G)$, given by the above formula.*

Example 3.1. (1) Let $M \times \Gamma$ be a trivial bundle of finite groups with fiber Γ over a manifold M and let $E \otimes \mathbf{V}$ be its representation as in Example 2.1 (1). The action of the Hopf algebroid $\mathcal{C}_c^\infty(M \times \Gamma) \cong \mathcal{C}_c^\infty(M) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$ on the module $\Gamma_c^\infty(E \otimes \mathbf{V}) \cong \Gamma_c^\infty(E) \otimes_{\mathcal{C}_c^\infty(M)} \Gamma_c^\infty(\mathbf{V})$ is then the natural tensor product action. In particular, if M is a point, this action corresponds to the action of the group algebra $\mathbb{C}[\Gamma]$ on the vector space V, extending the representation of Γ to V. On the other hand, if Γ is trivial, we recover the usual action of $\mathcal{C}_c^\infty(M)$ on $\Gamma_c^\infty(E)$.

(2) Now let $G_\theta = \mathbb{Z} \ltimes S^1$ be the translation groupoid from Example 2.2 (2). Unitary irreducible representations of the group $\mathbb{Z}$ are all one dimensional and classified by the group S^1. Explicitly, for any $\mathrm{e}^{\mathrm{i}\alpha} \in S^1$ let us denote by V_α the vector space $\mathbb{C}$ with the action of $\mathbb{Z}$ given by $m \cdot z = \mathrm{e}^{im\alpha} z$ for $m \in \mathbb{Z}$ and $z \in \mathbb{C}$. This representation induces a representation $\mathbf{V}_\alpha = S^1 \times V_\alpha$ [35] of the groupoid G_θ such that the vector space $\Gamma_c^\infty(\mathbf{V}_\alpha)$ of sections of $\mathbf{V}_\alpha$ is naturally isomorphic to the space $\mathcal{C}_c^\infty(S^1)$. With this identification in mind we can expand an arbitrary section $u \in \Gamma_c^\infty(\mathbf{V}_\alpha)$ as $u = \sum_{k\in\mathbb{Z}} u_k e_k$ where $u_k \in \mathbb{C}$ and $e_k = \mathrm{e}^{\mathrm{i}k\varphi}$. Recall now from Example 2.2 (3) that $\mathcal{C}_c^\infty(G_\theta)$ is generated by the elements $U = \mathrm{const}_1 \cdot 1$ and $V = \mathrm{e}^{\mathrm{i}\varphi} \cdot 0$. The action of the Hopf algebroid $\mathcal{C}_c^\infty(G_\theta)$ on the space $\Gamma_c^\infty(\mathbf{V}_\alpha)$ can be now explicitly described by the formulas

$$
\begin{aligned}
Ue_k &= \mathrm{e}^{\mathrm{i}(\alpha - 2\pi\theta k)} e_k, \\
Ve_k &= e_{k+1}
\end{aligned}
$$

for any $k \in \mathbb{Z}$.

By the above procedure one obtains a left module of sections $\Gamma_c^\infty(E)$ over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ for an arbitrary representation E of the groupoid G. Any morphism $\varphi\colon E \to F$ of representations of G is in particular a morphism of vector bundles over G_0 and it induces a homomorphism $\Gamma_c^\infty(\varphi)\colon \Gamma_c^\infty(E) \to \Gamma_c^\infty(F)$ of left $\mathcal{C}_c^\infty(G_0)$-modules, given by composition with φ, i.e. $\Gamma_c^\infty(\varphi)(u) = \varphi \circ u$ for any $u \in \Gamma_c^\infty(E)$. Considering that the map φ is fiberwise linear and G-equivariant, we have the equality $\Gamma_c^\infty(\varphi)(au) = a\Gamma_c^\infty(\varphi)(u)$ for any $a \in \mathcal{C}_c^\infty(G)$ and $u \in \Gamma_c^\infty(E)$. It follows that $\Gamma_c^\infty(\varphi)$ is a homomorphism of $\mathcal{C}_c^\infty(G)$-modules, so one can define a covariant functor

$$\Gamma_c^\infty = (\Gamma_c^\infty)_G\colon \mathsf{Rep}(G) \to {}_G\mathsf{Mod}$$

from the category of representations of the groupoid G to the category of left modules over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of the groupoid G.

3.2. Modules of finite type and of constant rank. According to the previous subsection one can associate a left $\mathcal{C}_c^\infty(G)$-module $\Gamma_c^\infty(E)$ with every representation

E of G. However, not every $\mathcal{C}_c^\infty(G)$-module is of this kind and it is not too hard to find counter-examples. In this subsection we define and explain the properties that characterize modules of sections of representations of étale Lie groupoids.

Let M be a Hausdorff manifold and let $\mathcal{C}_c^\infty(M)$ be the algebra of smooth functions with compact support on M. There is a bijective correspondence between the points of the manifold M and nontrivial homomorphisms $\eta\colon \mathcal{C}_c^\infty(M) \to \mathbb{C}$ of $\mathbb{C}$-algebras. With any $x \in M$ one associates the evaluation $\mathrm{ev}_x\colon \mathcal{C}_c^\infty(M) \to \mathbb{C}$ at the point x given by $\mathrm{ev}_x(f) = f(x)$ for $f \in \mathcal{C}_c^\infty(M)$. Suppose now $\eta\colon \mathcal{C}_c^\infty(M) \to \mathbb{C}$ is a nontrivial homomorphism of $\mathbb{C}$-algebras. Its kernel is a maximal ideal of $\mathcal{C}_c^\infty(M)$ of the form $\ker(\eta) = I_x\mathcal{C}_c^\infty(M) = \{f \in \mathcal{C}_c^\infty(M) \,|\, f(x) = 0\}$ for a unique point $x \in M$ (we will use the notation $I_x\mathcal{C}_c^\infty(M)$ for the maximal ideal of functions that vanish at x and $\mathcal{C}_c^\infty(M)(x) = \mathcal{C}_c^\infty(M)/I_x\mathcal{C}_c^\infty(M)$ for the quotient algebra). Both the induced maps $\overline{\eta}, \overline{\mathrm{ev}_x}\colon \mathcal{C}_c^\infty(M)(x) \to \mathbb{C}$ are isomorphisms of $\mathbb{C}$-algebras. Since identity is the only automorphism of the $\mathbb{C}$-algebra $\mathbb{C}$, it follows $\overline{\eta} = \overline{\mathrm{ev}_x}$ and hence $\eta = \mathrm{ev}_x$. Note also that ev_x induces a canonical isomorphism between $\mathcal{C}_c^\infty(M)(x)$ and $\mathbb{C}$. Now choose a left $\mathcal{C}_c^\infty(M)$-module $\mathfrak{M}$. The $\mathcal{C}_c^\infty(M)$-module $I_x\mathfrak{M} = I_x\mathcal{C}_c^\infty(M) \cdot \mathfrak{M}$ is then a submodule of $\mathfrak{M}$ and we denote by

$$\mathfrak{M}(x) = \mathfrak{M}/I_x\mathfrak{M}$$

the quotient $\mathcal{C}_c^\infty(M)(x)$-module and consider it as a complex vector space.

Now let G be an étale Lie groupoid and let $\mathfrak{M}$ be a left $\mathcal{C}_c^\infty(G)$-module. It follows that $\mathfrak{M}$ is a left module over the algebra $\mathcal{C}_c^\infty(G_0)$ as well since $\mathcal{C}_c^\infty(G_0)$ is a subalgebra of $\mathcal{C}_c^\infty(G)$. The $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ is of *finite type* if it is isomorphic, as a $\mathcal{C}_c^\infty(G_0)$-module, to some submodule of the module $\mathcal{C}_c^\infty(G_0)^k$ for some natural number k. The $\mathcal{C}_c^\infty(G_0)$-modules of the form $\mathcal{C}_c^\infty(G_0)^k$ correspond precisely to the modules of sections of trivial vector bundles $G_0 \times \mathbb{C}^k$, so one can roughly think of modules of finite type as corresponding to subfamilies of trivial vector bundles. Suppose now that the $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ is of finite type and choose an injective homomorphism $\Phi\colon \mathfrak{M} \to \mathcal{C}_c^\infty(G_0)^k$ of $\mathcal{C}_c^\infty(G_0)$-modules. For each $x \in G_0$ we obtain an injective complex linear map $\Phi(x)\colon \mathfrak{M}(x) \to \mathcal{C}_c^\infty(G_0)(x)^k \cong \mathbb{C}^k$, which shows that $\mathfrak{M}(x)$ is a finite dimensional complex vector space for each $x \in G_0$. We denote by $\mathrm{rank}_x\mathfrak{M} = \dim_{\mathbb{C}} \mathfrak{M}(x)$ the rank of the module $\mathfrak{M}$ at the point $x \in G_0$. A $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ of finite type is of *constant rank* if the function $x \mapsto \mathrm{rank}_x\mathfrak{M}$ is a constant function on G_0. One can similarly define the notions of modules of locally constant rank and of modules of globally bounded rank.

Suppose now that M is a smooth, Hausdorff and paracompact manifold and let E be a vector bundle over M. The module $\Gamma_c^\infty(E)$ of sections of the bundle E is a basic example of a module of finite type and of constant rank. One can see that as follows. Since M is finite dimensional and paracompact, there exists a vector bundle E' over M such that the bundle $E \oplus E'$ is isomorphic to some trivial vector bundle $M \times \mathbb{C}^k$ over M; vector bundles with this property are said to be of finite type. This property basically follows from the proof of Lemma 5.9 in [19]. As a result we obtain the isomorphism $\Gamma_c^\infty(E) \oplus \Gamma_c^\infty(E') \cong \mathcal{C}_c^\infty(M)^k$, i.e. the module $\Gamma_c^\infty(E)$ is of finite type. Furthermore, there is a natural isomorphism $\Gamma_c^\infty(E)(x) \to E_x$ of complex vector spaces for every $x \in M$, induced by the evaluation at the point x, which shows that the module $\Gamma_c^\infty(E)$ is of constant rank. We state this result as a proposition for future reference.

Proposition 3.2. *Let G be an étale Lie groupoid. The left $\mathcal{C}_c^\infty(G)$-module $\Gamma_c^\infty(E)$ of sections of any representation E of the groupoid G is a module of finite type and of constant rank.*

A crucial ingredient in the proof of the above proposition is the fact that any vector bundle over a paracompact manifold is a subbundle of a trivial bundle. Such vector bundles are referred to as bundles of finite type, a fact which also explains our reason to refer to the corresponding modules as modules of finite type.

Example 3.2. (1) Let M be a Hausdorff manifold and let $\mathfrak{M} \subset \mathcal{C}_c^\infty(M)^k$ be a $\mathcal{C}_c^\infty(M)$-module of finite type and of locally constant rank. We can then view $\mathfrak{M}(x)$ as a vector subspace of $\mathbb{C}^k$ for every $x \in M$ and define $E_x = \mathfrak{M}(x)^\perp$. These vector spaces define a smooth vector bundle E over M such that $\mathfrak{M} \oplus \Gamma_c^\infty(E) \cong \mathcal{C}_c^\infty(M)^k$. In particular, for any $\mathcal{C}_c^\infty(M)$-module $\mathfrak{M}$ of finite type and of locally constant rank there exists a $\mathcal{C}_c^\infty(M)$-module $\mathfrak{M}'$ such that $\mathfrak{M} \oplus \mathfrak{M}' \cong \mathcal{C}_c^\infty(M)^k$.

(2) If a Hausdorff manifold M is compact, it follows that the algebra $\mathcal{C}_c^\infty(M)$ is unital. In this case finitely generated, projective modules over the algebra $\mathcal{C}_c^\infty(M)$ correspond to modules of finite type and of locally constant rank. If the manifold M is connected, any such module is automatically of constant rank.

We show next that modules of finite type and of constant rank form an additive monoidal category for any étale Lie groupoid G. Let $\mathfrak{M}$ and $\mathfrak{N}$ be modules of finite type and of constant rank over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of the groupoid G. It is straightforward to check that the direct sum module $\mathfrak{M} \oplus \mathfrak{N}$ is then again of finite type and of constant rank. Furthermore, the tensor product $\mathfrak{M} \otimes_{\mathcal{C}_c^\infty(G_0)} \mathfrak{N}$ is then a left module over the algebra $\mathcal{C}_c^\infty(G_0)$ which can be made into a left module over $\mathcal{C}_c^\infty(G)$ as follows. Let $a \in \mathcal{C}_c^\infty(G)$ be a smooth function with compact support in a bisection V of G. It can be written as $a = a_0 * a'$ where $a_0 = a \circ (t|_V)^{-1} \in \mathcal{C}_c^\infty(t(V))$ and $a' \in \mathcal{C}_c^\infty(V)$ is any function such that $a \cdot a' = a$ pointwise in V. For any $m \otimes n \in \mathfrak{M} \otimes_{\mathcal{C}_c^\infty(G_0)} \mathfrak{N}$ we define $a(m \otimes n) = a_0(a'm \otimes a'n)$. Any function $a \in \mathcal{C}_c^\infty(G)$

can be decomposed as a sum $a = \sum_{k=1}^{n} a_k * a'_k$ of functions as above and in this case we define

$$a(m \otimes n) = \sum_{k=1}^{n} a_k(a'_k m \otimes a'_k n).$$

A little bit longer calculation shows that the action is well defined so that we obtain a left $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M} \otimes_{\mathcal{C}_c^\infty(G_0)} \mathfrak{N}$ of finite type and of constant rank. The natural isomorphism $(\mathfrak{M} \otimes \mathfrak{M}') \otimes \mathfrak{M}'' \cong \mathfrak{M} \otimes (\mathfrak{M}' \otimes \mathfrak{M}'')$ of left $\mathcal{C}_c^\infty(G_0)$-modules is $\mathcal{C}_c^\infty(G)$-linear for any $\mathcal{C}_c^\infty(G)$-modules $\mathfrak{M}, \mathfrak{M}'$ and $\mathfrak{M}''$ of finite type and of constant rank. We thus obtain the following result.

Proposition 3.3. *Let G be an étale Lie groupoid. Modules of finite type and of constant rank over the Hopf algebroid $\mathcal{C}_c^\infty(G)$, with direct sum and tensor product as described above, form an additive monoidal category $\mathsf{Mod}(G)$.*

Example 3.3. If M is a compact connected manifold, then the category $\mathsf{Mod}(M)$ of modules of finite type and of constant rank over $\mathcal{C}_c^\infty(M)$ coincides with the additive monoidal category of finitely generated, projective modules over $\mathcal{C}_c^\infty(M)$, with operations of direct sum and tensor product of modules.

4. Serre-Swan's theorem for étale Lie groupoids

Modules of sections of representations provide typical examples of modules of finite type and of constant rank over the Hopf algebroid of an étale Lie groupoid. We show in this section that these modules are in fact the only examples of modules of finite type and of constant rank, up to isomorphism. Results of this type were first considered by Serre [36] in the category of algebraic varieties and by Swan in the category of compact Hausdorff topological spaces [37].

Recall that we denote by $\mathsf{Mod}(G)$ the full subcategory of the category of left modules over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of an étale Lie groupoid G, consisting of modules of finite type and of constant rank. Since every module of sections of a representation is such a module by Proposition 3.2, we have a functor

$$(\Gamma_c^\infty)_G \colon \mathsf{Rep}(G) \to \mathsf{Mod}(G)$$

from the category of representations of the groupoid G to the category of modules of finite type and of constant rank over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of G.

Theorem 4.1. *The functor $(\Gamma_c^\infty)_G\colon \mathsf{Rep}(G) \to \mathsf{Mod}(G)$ of smooth sections with compact support is an equivalence of categories for any étale Lie groupoid G.*

Before we begin with the proof, we recall the classical version of the Serre-Swan's theorem in the setting of smooth manifolds and modules over the algebras of smooth functions with compact support.

Theorem 4.2. *The functor $(\Gamma_c^\infty)_M\colon \mathsf{Rep}(M) \to \mathsf{Mod}(M)$ is an equivalence of categories for any paracompact Hausdorff manifold M.*

Proof. The crucial point in the proof of the theorem is the observation that every vector bundle over a paracompact manifold is of finite type, i.e. a subbundle of some trivial bundle. Taking this into account, basically the same proof as in Swan's original paper [37] goes through. □

Now let G be an étale Lie groupoid. We will prove Theorem 4.1 by constructing a quasi-inverse

$$\mathrm{R_{sp}}\colon \mathsf{Mod}(\mathrm{G}) \to \mathsf{Rep}(\mathrm{G})$$

to the functor Γ_c^∞, to show that it is an equivalence of categories. For any $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ of finite type and of constant rank we define a vector bundle $\mathrm{R_{sp}}(\mathfrak{M})$ over G_0 as follows. As a set, the bundle $\mathrm{R_{sp}}(\mathfrak{M})$ is defined as a disjoint union of the spaces $\mathfrak{M}(x)$ for $x \in G_0$,

$$\mathrm{R_{sp}}(\mathfrak{M}) = \coprod_{\mathrm{x}\in \mathrm{G_0}} \mathfrak{M}(\mathrm{x}),$$

together with the natural projection onto the manifold G_0. To define a topology and a smooth structure on the space $\mathrm{R_{sp}}(\mathfrak{M})$, we first choose a vector bundle E over G_0 and an isomorphism $\Phi\colon \Gamma_c^\infty(E) \to \mathfrak{M}$ of left $\mathcal{C}_c^\infty(G_0)$-modules. Such an isomorphism exists due to the classical version of Serre-Swan's Theorem 4.2. The induced map $\Phi(x)\colon E_x \to \mathfrak{M}(x)$ is an isomorphism of complex vector spaces for each x, so we can use the fiberwise linear bijection $\varphi = \coprod \Phi(x)\colon E \to \mathrm{R_{sp}}(\mathfrak{M})$ to define a structure of a smooth vector bundle over G_0 on the space $\mathrm{R_{sp}}(\mathfrak{M})$. The so defined vector bundle structure on the space $\mathrm{R_{sp}}(\mathfrak{M})$ is well defined. Namely, if E' is another vector bundle over G_0 and if $\Phi'\colon \Gamma_c^\infty(E') \to \mathfrak{M}$ is an isomorphism of $\mathcal{C}_c^\infty(G_0)$-modules, we obtain the isomorphism $(\Phi')^{-1} \circ \Phi\colon \Gamma_c^\infty(E) \to \Gamma_c^\infty(E')$ of $\mathcal{C}_c^\infty(G_0)$-modules. Bundles E and E' are therefore isomorphic by Theorem 4.2, so they define the same vector bundle structure on the space $\mathrm{R_{sp}}(\mathfrak{M})$.

We next use the action of $\mathcal{C}_c^\infty(G)$ on the space $\mathfrak{M}$ to define a representation of G on the vector bundle $\mathrm{R_{sp}}(\mathfrak{M})$. Choose any arrow $g \in G(x, y)$ and any vector $v \in \mathfrak{M}(x)$. We can find an element $m \in \mathfrak{M}$ such that $v = m(x)$ and a function

$a \in \mathcal{C}_c^\infty(G)$ with compact support in some bisection such that $a(g) = 1$. Since $\mathfrak{M}$ is a left $\mathcal{C}_c^\infty(G)$-module, the element $am \in \mathfrak{M}$ is well defined and we define

$$g \cdot m(x) = am(y) \in \mathfrak{M}(y).$$

We will denote by

$$\mu_{\mathfrak{M}} \colon G \times_{G_0} \mathrm{R}_{\mathrm{sp}}(\mathfrak{M}) \to \mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$$

the map defined by the above formula.

Proposition 4.1. *The map $\mu_{\mathfrak{M}}$ defines a representation of the étale Lie groupoid G on the vector bundle $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ over G_0.*

Proof. It is straightforward to check that the map $\mu_{\mathfrak{M}}$ is well defined and that it defines a linear action of the groupoid G on the vector bundle $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$.

To see that $\mu_{\mathfrak{M}}$ is a smooth map, we first choose any $(g, v) \in G \times_{G_0} \mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ and a function $a \in \mathcal{C}_c^\infty(G)$ with compact support in a bisection V' such that $a|_V \equiv 1$ for some small neighbourhood $V \subset V'$ of g. There exist elements $m^1, \ldots, m^k \in \mathfrak{M}$ with the property that the vectors $\{m^1(x), \ldots, m^k(x)\}$ form a basis of $\mathfrak{M}(x)$ for all $x \in s(V)$. As a result we obtain smooth functions $\lambda_1, \ldots, \lambda_k \colon \mathrm{R}_{\mathrm{sp}}(\mathfrak{M})|_{\mathrm{s(V)}} \to \mathbb{C}$, implicitly defined by the formula $w = \sum_{i=1}^{k} \lambda_i(w) m^i(x)$ for any $w \in \mathfrak{M}(x)$ where $x \in s(V)$. Locally, on a neighbourhood $V \times_{G_0} \mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ of the point (g, v), we have $\mu_{\mathfrak{M}}(h, w) = \sum_{i=1}^{k} \lambda_i(w) am^i(t(h))$, where am^i are smooth sections of the vector bundle $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$. This concludes the proof of the proposition. $\square$

Representation of the groupoid G on the vector bundle $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ will be referred to as the *spectral representation* of the groupoid G associated with a $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ of finite type and of constant rank.

Now choose left $\mathcal{C}_c^\infty(G)$-modules of finite type and of constant rank $\mathfrak{M}$ and $\mathfrak{N}$ and let $\Phi \colon \mathfrak{M} \to \mathfrak{N}$ be a homomorphism of $\mathcal{C}_c^\infty(G)$-modules. For each $x \in G_0$ we have the induced linear map $\Phi(x) \colon \mathfrak{M}(x) \to \mathfrak{N}(x)$ and these maps together define a fiberwise linear map $\mathrm{R}_{\mathrm{sp}}(\Phi) \colon \mathrm{R}_{\mathrm{sp}}(\mathfrak{M}) \to \mathrm{R}_{\mathrm{sp}}(\mathfrak{N})$, which is a G-equivariant morphism of representations of the groupoid G on $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ and $\mathrm{R}_{\mathrm{sp}}(\mathfrak{N})$, respectively. The functoriality of the assignment $\Phi \mapsto \Phi(x)$ for each $x \in G_0$ extends to the functoriality of the map R_{sp}, so we have the spectral representation functor

$$\mathrm{R}_{\mathrm{sp}} \colon \mathsf{Mod}(\mathrm{G}) \to \mathsf{Rep}(\mathrm{G}).$$

Proof of Theorem 4.1. Theorem 4.1 will be proved by showing that the functor $\mathrm{R}_{\mathrm{sp}} \colon \mathsf{Mod}(\mathrm{G}) \to \mathsf{Rep}(\mathrm{G})$ is a quasi-inverse of the functor $\Gamma_c^\infty \colon \mathsf{Rep}(G) \to \mathsf{Mod}(G)$.

We can naturally identify the $\mathcal{C}_c^\infty(G)$-module $\mathfrak{M}$ with the module $\Gamma_c^\infty(\mathrm{R}_{\mathrm{sp}}(\mathfrak{M}))$, by assigning the section $x \mapsto m(x)$ of the bundle $\mathrm{R}_{\mathrm{sp}}(\mathfrak{M})$ to an element $m \in \mathfrak{M}$. Denote by $\varepsilon_{\mathfrak{M}}\colon \mathfrak{M} \to \Gamma_c^\infty(\mathrm{R}_{\mathrm{sp}}(\mathfrak{M}))$ the corresponding isomorphism of modules and let $\varepsilon\colon \mathrm{Id}_{\mathsf{Mod}}(G) \Rightarrow \Gamma_c^\infty \circ \mathrm{R}_{\mathrm{sp}}$ be the corresponding natural equivalence of functors.

Now let E be a representation of G. We have an isomorphism $\Gamma_c^\infty(E)(x) \to E_x$ of complex vector spaces for every $x \in G_0$, induced by the evaluation at the point x, which induces an isomorphism $\eta_E\colon \mathrm{R}_{\mathrm{sp}}(\Gamma_{\mathrm{c}}^\infty(\mathrm{E})) \to \mathrm{E}$ of representations of G. The natural equivalence of functors $\eta\colon \mathrm{R}_{\mathrm{sp}} \circ \Gamma_{\mathrm{c}}^\infty \Rightarrow \mathrm{Id}_{\mathrm{Rep(G)}}$ together with the equivalence ε shows that the functor Γ_c^∞ is an equivalence of categories. □

Example 4.1. Let G be a finite group and let M be a compact connected manifold. Representations of the translation groupoid $G \ltimes M$ correspond to G-equivariant vector bundles over M and by Theorem 4.1 we can identify G-equivariant bundles over M with modules of finite type and of constant rank over the Hopf algebroid $\mathcal{C}_c^\infty(G \ltimes M)$. On the other hand, by combining the Peter-Weyl theorem for finite groups with the result that any G-equivariant bundle over M is of finite type [35], it follows that G-equivariant bundles over M correspond to finitely generated, projective modules over the convolution algebra $\mathcal{C}_c^\infty(G \ltimes M)$. In particular, modules of finite type and of constant rank over $\mathcal{C}_c^\infty(G \ltimes M)$ are precisely finitely generated projective modules, so we obtain a natural monoidal structure on the category of finitely generated projective modules over the algebra $\mathcal{C}_c^\infty(G \ltimes M)$.

5. Natural equivalence of functors Rep and Mod

Representations of étale Lie groupoids naturally define a contravariant functor Rep from the Morita category of étale Lie groupoids to the category of semirings. In this section we first explain that modules of finite type and of constant rank over their Hopf algebroids similarly define a functor Mod between the same two categories and then show how to interpret Serre-Swan's correspondence as a natural equivalence between these two functors.

We begin with some lemmas that describe the modules of sections of pullback representations and of tensor product representations.

Lemma 5.1. *Let G and H be étale Lie groupoids and let P be a principal H-bundle over G. For any representation E of H there exists a natural isomorphism*

$$\sigma_P(E)\colon \mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)$$

of left modules over the Hopf algebroid $\mathcal{C}_c^\infty(G)$.

Proof. First define a bilinear map $\sigma_P(E)\colon \mathcal{C}_c^\infty(P) \times \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)$ by the formula

$$(\sigma_P(E)(f,u))(x) = \sum_{\pi(p)=x} f(p)(p \otimes u(\varphi(p)))$$

for $f \in \mathcal{C}_c^\infty(P)$ and $u \in \Gamma_c^\infty(E)$. It is not too hard to check that the map $\sigma_P(E)$ is well defined and that it induces a homomorphism

$$\sigma_P(E)\colon \mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)$$

of $\mathcal{C}_c^\infty(G)$-modules, which we claim to be an isomorphism. It suffices to show that $\sigma_P(E)$ is a bijective map.

We first consider the case when the bundle P is trivial, i.e. $P = \langle \psi \rangle$ for some smooth functor $\psi\colon G \to H$. Representations $P \otimes_H E$ and $\psi^* E$ of G are then isomorphic via an isomorphism $f\colon P \otimes_H E \to \psi^* E$, which induces an isomorphism

$$\Gamma_c^\infty(f)\colon \Gamma_c^\infty(P \otimes_H E) \to \Gamma_c^\infty(\psi_0^* E)$$

of modules over $\mathcal{C}_c^\infty(G_0)$. Consider now the representation E of H as a vector bundle over H_0. The map

$$\sigma_{\psi_0}(E)\colon \mathcal{C}_c^\infty(G_0) \otimes_{\mathcal{C}_c^\infty(H_0)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(\psi_0^* E),$$

defined analogously as the map $\sigma_P(E)$, is then an isomorphism of modules over the algebra $\mathcal{C}_c^\infty(G_0)$ [11]. Finally, since $P = G_0 \times_{H_0} H$ is a trivial bundle, we have by [27] the isomorphism $\Omega_{G_0,H}\colon \mathcal{C}_c^\infty(P) \cong \mathcal{C}_c^\infty(G_0) \otimes_{\mathcal{C}_c^\infty(H_0)} \mathcal{C}_c^\infty(H)$, which induces an isomorphism

$$\mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E) \to \mathcal{C}_c^\infty(G_0) \otimes_{\mathcal{C}_c^\infty(H_0)} \Gamma_c^\infty(E)$$

of $\mathcal{C}_c^\infty(G_0)$-modules. We can collect all these isomorphisms into the following commutative diagram of homomorphisms of $\mathcal{C}_c^\infty(G_0)$-modules

$$\begin{tikzcd}
\mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E) \arrow[r, "\sigma_P(E)"] \arrow[d, "\cong"] & \Gamma_c^\infty(P \otimes_H E) \arrow[d, "\Gamma_c^\infty(f)V"] \\
\mathcal{C}_c^\infty(G_0) \otimes_{\mathcal{C}_c^\infty(H_0)} \Gamma_c^\infty(E) \arrow[r, "\sigma_{\psi_0}(E)"] & \Gamma_c^\infty(\psi_0^* E)
\end{tikzcd}$$

Since the remaining maps are bijective, the map $\sigma_P(E)$ is bijective as well.

A principal H-bundle P over G is in general only locally trivial [25]. Let U be an open subset of G_0 such that $P|_U$ is a trivial H-bundle. We then have a natural injective homomorphism $\mathcal{C}_c^\infty(P|_U) \to \mathcal{C}_c^\infty(P)$ of right $\mathcal{C}_c^\infty(H)$-modules which induces an injective homomorphism of abelian groups

$$\mathcal{C}_c^\infty(P|_U) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E) \to \mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \Gamma_c^\infty(E).$$

The map $\sigma_P(E)$ restricts, via this injection and injection $\Gamma_c^\infty(P|_U \otimes_H E) \to \Gamma_c^\infty(P \otimes_H E)$, to the bijection $\sigma_{P|_U}(E)$. One concludes the proof by using a partition of unity subordinated to a trivializing open cover for P over G_0 and the fact that $\sigma_P(E)$ is a homomorphism of $\mathcal{C}_c^\infty(G)$-modules. $\square$

Lemma 5.2. *Let G be an étale Lie groupoid. For any representations E and F of the groupoid G there exists a natural isomorphism*

$$\Omega = \Omega_{E,F} \colon \Gamma_c^\infty(E) \otimes_{\mathcal{C}_c^\infty(G_0)} \Gamma_c^\infty(F) \to \Gamma_c^\infty(E \otimes F)$$

of left modules over the Hopf algebroid $\mathcal{C}_c^\infty(G)$.

Proof. Define a bilinear map $\Omega_{E,F} \colon \Gamma_c^\infty(E) \times \Gamma_c^\infty(F) \to \Gamma_c^\infty(E \otimes F)$ by associating with $(u, v) \in \Gamma_c^\infty(E) \times \Gamma_c^\infty(F)$ the section $u \otimes v \in \Gamma_c^\infty(E \otimes F)$, which is given pointwise by $(u \otimes v)(x) = u(x) \otimes v(x)$ for every $x \in G_0$. From the definitions of actions of the algebra $\mathcal{C}_c^\infty(G_0)$ on modules of sections it follows that the above map is $\mathcal{C}_c^\infty(G_0)$-bilinear and induces a bijective $\mathcal{C}_c^\infty(G_0)$-linear map

$$\Omega_{E,F} \colon \Gamma_c^\infty(E) \otimes_{\mathcal{C}_c^\infty(G_0)} \Gamma_c^\infty(F) \to \Gamma_c^\infty(E \otimes F).$$

It remains to be proved that the map $\Omega_{E,F}$ is $\mathcal{C}_c^\infty(G)$-linear. To this effect let a function $a \in \mathcal{C}_c^\infty(G)$ have support in a bisection V of the groupoid G and let us write it as $a = a_0 * a'$ where $a_0 = a \circ (t|_V)^{-1} \in \mathcal{C}_c^\infty(t(V))$ and $a' \in \mathcal{C}_c^\infty(V)$ is any function such that $a \cdot a' = a$ pointwise in V. Moreover, choose any $u \in \Gamma_c^\infty(E)$ and any $v \in \Gamma_c^\infty(F)$. We then have $\Omega_{E,F}(a(u \otimes v)) = a_0(a'u \otimes a'v)$. It follows that the support of the section $\Omega_{E,F}(a(u \otimes v))$ is contained in the support of the function a_0. Next, for any $y \in G_0$ with $a_0(y) \neq 0$ we have

$$(a_0(a'u \otimes a'v))(y) = a_0(y)(g \cdot u(x) \otimes g \cdot v(x))$$

where $g \in G(x, y)$ is the unique arrow in the bisection V with target y. On the other hand, the definition of the action of the algebra $\mathcal{C}_c^\infty(G)$ on the module $\Gamma_c^\infty(E \otimes F)$ yields

$$(a\Omega_{E,F}(u \otimes v))(y) = a(g)(g \cdot u(x) \otimes g \cdot v(x)).$$

Since $a(g) = a_0(y)$, we have the equality $(\Omega_{E,F}(a(u \otimes v)))(y) = (a\Omega_{E,F}(u \otimes v))(y)$ for every $y \in G_0$, which shows that the map $\Omega_{E,F}$ is $\mathcal{C}_c^\infty(G)$-linear. $\square$

We are now ready to show how to interpret modules of finite type and of constant rank as functors. Let P be a principal H-bundle over G. Tensoring by the $\mathcal{C}_c^\infty(G)$-$\mathcal{C}_c^\infty(H)$-bimodule $\mathcal{C}_c^\infty(P)$ induces a covariant functor $\mathsf{Mod}(P)\colon \mathsf{Mod}(H) \to \mathsf{Mod}(G)$ by Theorem 4.1 and Lemma 5.1. Note also that isomorphic principal bundles induce naturally equivalent functors. Let us now denote by $\mathrm{Mod}(G)$ the set of isomorphism classes of modules of finite type and of constant rank over the Hopf algebroid $\mathcal{C}_c^\infty(G)$ of G. Operations of direct sum and tensor product of $\mathcal{C}_c^\infty(G)$-modules turn the set $\mathrm{Mod}(G)$ into a semiring. For any principal H-bundle P over G we obtain a map $\mathrm{Mod}(P)\colon \mathrm{Mod}(H) \to \mathrm{Mod}(G)$ induced by tensoring by the bimodule $\mathcal{C}_c^\infty(P)$. To see that $\mathrm{Mod}(P)$ is a multiplicative map, we define a natural isomorphism

$$\mathcal{C}_c^\infty(P) \otimes (\mathfrak{M} \otimes_{\mathcal{C}_c^\infty(H_0)} \mathfrak{M}') \to (\mathcal{C}_c^\infty(P) \otimes \mathfrak{M}) \otimes_{\mathcal{C}_c^\infty(G_0)} (\mathcal{C}_c^\infty(P) \otimes \mathfrak{M}')$$

of $\mathcal{C}_c^\infty(G)$-modules, given by $f \otimes (m \otimes n) \mapsto f_0(f' \otimes m) \otimes (f' \otimes n)$, where $f \in \mathcal{C}_c^\infty(P)$ has compact support in a open subset U of P, which maps injectively into G_0 by the bundle projection $\pi\colon P \to G_0$, $f_0 = f \circ (\pi|_U)^{-1} \in \mathcal{C}_c^\infty(\pi(U))$ and $f' \in \mathcal{C}_c^\infty(U)$ is such that $f \cdot f' = f$ pointwise in U. It follows that $\mathrm{Mod}(P)$ is a homomorphism of semirings, depending only on the isomorphism class of the principal bundle P. Furthermore, if P' is a principal K-bundle over H, where K is an étale Lie groupoid, there is a natural homomorphism $\Omega_{P,P'}\colon \mathcal{C}_c^\infty(P) \otimes_{\mathcal{C}_c^\infty(H)} \mathcal{C}_c^\infty(P') \to \mathcal{C}_c^\infty(P \otimes_H P')$ of $\mathcal{C}_c^\infty(G)$-$\mathcal{C}_c^\infty(K)$-bimodules, which is in fact an isomorphism [27]. We thus obtain a contravariant functor

$$\mathrm{Mod}\colon \mathsf{EtGPD} \to \mathsf{Rng}$$

from the Morita category of étale Lie groupoids to the category of semirings.

The functor $(\Gamma_c^\infty)_G\colon \mathsf{Rep}(G) \to \mathsf{Mod}(G)$ is an equivalence of categories and by Lemma 5.2 it induces an isomorphism $(\Gamma_c^\infty)_G\colon \mathrm{Rep}(G) \to \mathrm{Mod}(G)$ of semirings. Now let H be another étale Lie groupoid and let P be a principal H-bundle over G. From Lemma 5.1 it follows that $(\Gamma_c^\infty)_G \circ \mathrm{Rep}(P) = \mathrm{Mod}(P) \circ (\Gamma_c^\infty)_H$, so we can consider the family Γ_c^∞ of isomorphisms of semirings as a natural equivalence between the functors Rep and Mod, and as a result we obtain the following theorem.

Theorem 5.1. *Contravariant functors* Rep *and* Mod *from the Morita category of étale Lie groupoids to the category of semirings are naturally equivalent.*

The Morita category of étale Lie groupoids contains as a full subcategory the category of compact global quotients, whose objects are translation groupoids of finite groups acting on compact manifolds [31]. Restricted to this subcategory, the above functors correspond respectively, to the equivariant topological and the algebraic K-theory. Equivariant maps between global quotients correspond to trivial bundles, so in general we obtain the following improvement of the standard equivalence between K-functors.

Corollary 5.1. *Topological and algebraic K-theories of compact global quotients are naturally isomorphic with respect to generalized maps.*

Acknowledgement. I would like to thank J. Mrčun for many helpful discussions and advice related to the paper. I am also grateful to the referee for suggestions on how to improve the paper.

This work was supported in part by the Slovenian Research Agency (ARRS) project J1-2247.

References

[1] *A. Adem, J. Leida, Y. Ruan*: Orbifolds and Stringy Topology. Cambridge Tracts in Mathematics 171, Cambridge University Press, Cambridge, 2007.
[2] *C. Arias Abad, M. Crainic*: Representations up to homotopy and Bott's spectral sequence for Lie groupoids. Preprint arXiv: 0911.2859 (2009).
[3] *M. F. Atiyah, D. R. Anderson*: K-Theory. With reprints of M. F. Atiyah: Power operations in K-theory. New York-Amsterdam, W. A. Benjamin, 1967.
[4] *R. Bos*: Continuous representations of groupoids. Preprint arXiv: 0612639 (2006).
[5] *Ch. Blohmann, X. Tang, A. Weinstein*: Hopfish structure and modules over irrational rotation algebras. Contemporary Mathematics *462* (2008), 23–40.
[6] *A. Connes*: A survey of foliations and operator algebras. Proc. Symp. Pure Math. *38* (1982), 521–628.
[7] *A. Connes*: Noncommutative Geometry. Academic Press, San Diego, 1994.
[8] *M. Crainic, I. Moerdijk*: A homology theory for étale groupoids. J. Reine Angew. Math. *521* (2000), 25–46.
[9] *M. Crainic, I. Moerdijk*: Foliation groupoids and their cyclic homology. Adv. Math. *157* (2001), 177–197.
[10] *M. Crainic, I. Moerdijk*: Čech-De Rham theory for leaf spaces of foliations. Math. Ann. *328* (2004), 59–85.
[11] *W. Greub, S. Halperin, R. Vanstone*: Connections, Curvature, and Cohomology. Vol. I: De Rham cohomology of manifolds and vector bundles. Pure and Applied Mathematics 47, Volume 1, Academic Press, New York, 1972.
[12] *A. Haefliger*: Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. Comment. Math. Helv. *32* (1958), 248–329.
[13] *A. Haefliger*: Groupoïdes d'holonomie et classifiants. Astérisque *116* (1984), 70–97.
[14] *M. Hilsum, G. Skandalis*: Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov. Ann. Sci. Éc. Norm. Supér. *20* (1987), 325–390.
[15] *J. Kališnik*: Representations of orbifold groupoids. Topology Appl. *155* (2008), 1175–1188.
[16] *J. Kališnik, J. Mrčun*: Equivalence between the Morita categories of étale Lie groupoids and of locally grouplike Hopf algebroids. Indag. Math., New Ser. *19* (2008), 73–96.
[17] *F. Kamber, P. Tondeur*: Foliated Bundles and Characteristic Classes. Lecture Notes in Mathematics 493, Berlin-Heidelberg-New York, Springer-Verlag, 1975.
[18] *K. C. H. Mackenzie*: The General Theory of Lie Groupoids and Lie Algebroids. LMS Lecture Note Series 213, Cambridge University Press, Cambridge, 2005.

[19] *J. W. Milnor, J. D. Stasheff*: Characteristic Classes. Annals of Mathematics Studies 76. Princeton, N.J., Princeton University Press and University of Tokyo Press, 1974.

[20] *S. MacLane*: Categories for the Working Mathematician. 4th corrected printing, Graduate Texts in Mathematics, 5. New York etc., Springer-Verlag, 1988.

[21] *I. Moerdijk*: The classifying topos of a continuous groupoid I. Trans. Am. Math. Soc. *310* (1988), 629–668.

[22] *I. Moerdijk*: Classifying toposes and foliations. Ann. Inst. Fourier *41* (1991), 189–209.

[23] *I. Moerdijk*: Orbifolds as groupoids: an introduction. Contemp. Math. *310* (2002), 205–222.

[24] *I. Moerdijk, J. Mrčun*: Introduction to Foliations and Lie Groupoids. Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003.

[25] *I. Moerdijk, J. Mrčun*: Lie groupoids, sheaves and cohomology. Poisson Geometry, Deformation Quantisation and Group Representations, London Mathematical Society Lecture Note Series 323, Cambridge University Press, Cambridge, 2005, pp. 145–272.

[26] *J. Mrčun*: Stability and invariants of Hilsum-Skandalis maps. PhD Thesis, Utrecht University, 1996.

[27] *J. Mrčun*: Functoriality of the bimodule associated to a Hilsum-Skandalis map. K-Theory *18* (1999), 235–253.

[28] *J. Mrčun*: The Hopf algebroids of functions on étale groupoids and their principal Morita equivalence. J. Pure Appl. Algebra *160* (2001), 249–262.

[29] *J. Mrčun*: On duality between étale groupoids and Hopf algebroids. J. Pure Appl. Algebra *210* (2007), 267–282.

[30] *J. Pradines*: Morphisms between spaces of leaves viewed as fractions. Cah. Topologie Géom. Différ. Catég. *30* (1989), 229–246.

[31] *D. Pronk, L. Scull*: Translation groupoids and orbifold cohomology. Can. J. Math. *62* (2010), 614–645.

[32] *J. Renault*: A Groupoid Approach to C^*-algebras. Lecture Notes in Mathematics 793, Berlin-Heidelberg-New York, Springer-Verlag, 1980.

[33] *M. A. Rieffel*: $C*$-algebras associated with irrational rotations. Pac. J. Math. *93* (1981), 415–429.

[34] *I. Satake*: On a generalization of the notion of manifold. Proc. Natl. Acad. Sci. USA *42* (1956), 359–363.

[35] *G. Segal*: Equivariant K-theory. Publ. Math., Inst. Hates Étud. Sci. *34* (1968), 129–151.

[36] *J.-P. Serre*: Faisceaux algébriques cohérents. Ann. Math. *61* (1955), 197–278.

[37] *R. G. Swan*: Vector bundles and projective modules. Trans. Am. Math. Soc. *105* (1962), 264–277.

[38] *G. Trentinaglia*: On the role of effective representations of Lie groupoids. Adv. Math. *225* (2010), 826–858.

[39] *H. E. Winkelnkemper*: The graph of a foliation. Ann. Global Anal. Geom. *1* (1983), 51–75.

Author's address: Jure Kališnik, Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana; University of Ljubljana, Ljubljana, Slovenia, e-mail: `jure.kalisnik@fmf.uni-lj.si`.