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EXISTENCE OF SOLUTIONS FOR ABSTRACT NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Abstract. In this paper we study the existence of classical solutions for a class of abstract neutral integro-differential equation with unbounded delay. A concrete application to partial neutral integro-differential equations is considered.

Keywords: neutral equations, classical solution, analytic semigroup, unbounded delay

MSC 2010: 34K30, 34K40, 35R10

1. Introduction

This paper is devoted to the study of the existence of classical solutions for a class of neutral integro-differential equations with unbounded delay of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)+\int_{-\infty}^{t}B(t,\tau)u(\tau)\,\mathrm{d}\tau\right]=Au(t)+\int_{-\infty}^{t}C(t,\tau)u(\tau)\,\mathrm{d}\tau,\quad t\in[0,a], \tag{1.1}$$

$$u_0=\varphi, \tag{1.2}$$

where $A\colon D(A)\subset X\rightarrow X$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ defined on a Banach space $(X,\|\cdot\|)$, the history $x_t\colon(-\infty,0]\rightarrow X$, $x_t(\theta):=x(t+\theta)$, belongs to an abstract phase space $\mathscr{B}$ defined axiomatically and $B(t,\tau)$, $C(t,\tau)$, $t\geqslant\tau$, are suitable bounded linear operators.

In this work we continue our recent developments in Hernández & O'Regan [16], [17], [18] and Hernández & Balachandran [19] on the existence of solutions to abstract neutral differential systems. As in the cited papers, our purpose is to establish the existence of solutions without employing a strong technical restriction which is usually used to treat neutral systems. To clarify the above, we next make some bibliographical comments related to the abstract neutral system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + g(t, x_t)] = Ax(t) + f(t, x_t), \quad t \in [0, a], \tag{1.3}$$

$$x_0 = \varphi \in \mathscr{C}, \tag{1.4}$$

where $\mathscr{C}$ (the phase space) is a Banach space formed by functions defined from a connected interval $J \subset (-\infty, 0]$ into X and $f, g\colon [0, a] \times \mathscr{C} \to X$ are given functions.

In Datko [12] and Adimy & Ezzinbi [1] some linear neutral systems similar to (1.3)–(1.4) are studied under the strong assumption that the range of $g(\cdot)$ is contained in $D(A)$. If A is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ (the case studied by Datko), this assumption arises from the treatment of the associated integral equation

$$\begin{aligned} u(t) = {} & T(t)[\varphi(0) + g(0, \varphi)] - g(t, u_t) \\ & - \int_0^t AT(t-s) g(s, u_s) \,\mathrm{d}s + \int_0^t T(t-s) f(s, u_s) \,\mathrm{d}s, \end{aligned}$$

since, except in trivial cases, the operator function $AT(\cdot)$ is not integrable in the operator topology on $[0, b]$ for $b > 0$. The same reason explains the use of a similar assumption in Adimy & Ezzinbi [1] where the case in which A is a Hille-Yosida type operator is studied.

In the papers [20], [21], [22] the system (1.3)–(1.4) is studied assuming that the semigroup $(T(t))_{t\geqslant 0}$ is analytic and g has values in the domain of a fractional power of A. The same assumption was later used in many papers, see for example [2], [5], [9], [26], [27], [29].

We note that the restrictions cited above are particular cases of a more general assumption which we call condition $(\mathbf{H_g})$. In the condition $(\mathbf{H_g})$ below, $\mathscr{L}(Y, X)$ denotes the space of bounded linear operators from Y into X endowed with the norm of operators.

Condition $(\mathbf{H_g})$. There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously embedded into X such that $g \in C([0, a] \times \mathscr{B}, Y)$ and $AT(\cdot) \in L^1([0, a], \mathscr{L}(Y, X))$.

The condition $(\mathbf{H_g})$ is verified in several situations, but it is a severe restriction. To understand this fact, it is necessary to remark that the abstract formulation of (1.3)–(1.4) is motivated by the interest in the applications to partial neutral differential equations, so that it is natural to think that A represents a differential operator. In the applications in Datko [12], for example, X is the space formed by all uniformly continuous functions from $[0, \infty)$ into $\mathbb{R}$ endowed with the supremum norm, $D(A) = \{x \in X\colon x' \in X\}$, $A\colon D(A) \subset X \to X$ is given by $Ax = x'$ and the assumption $(\mathbf{H_g})$ is verified with $Y = [D(A)]$, where $[D(A)]$ denotes the domain of A endowed with the graph norm. In this case, the function $g(\cdot)$ has an unusual regularizing effect since $g(t,u)$ is of class C^1 for all $(t,u) \in [0,\infty)\times X$. In the papers [20], [22], [21] a similar situation occurs. In these works, $X = L^2([0,\pi])$, $D(A) = \{x \in X\colon\ x(0) = x(\pi) = 0,\ x', x'' \in X\}$, $A\colon\ D(A) \subset X \to X$ is given by $Ax = x''$, $Y = [D(-A)^{1/2}] = W_0^1([0,1])$ and $g(\cdot)$ is a continuous function from $[0,a] \times X$ into $W_0^1([0,1])$.

It is convenient to note that in the papers [4], [3], [6], [7], [13], among several other works, an alternative assumption (which is really a particular case of condition $(\mathbf{H_g})$) has been used to treat neutral systems. In these works it is assumed that the semigroup $(T(t))_{t\geqslant 0}$ is compact and the set of operators $\{AT(t)\colon\ t \in (0,b]\}$ is bounded. However, as was pointed out in [21], these conditions are valid if and only if A is bounded and $\dim X < \infty$, which restricts the applications to ordinary differential equations. Moreover, if the compactness assumption is removed, it follows that A is bounded, which remains a strong restriction.

The purpose of this paper is to study the existence of α-Hölder "classical" solutions for (1.1)–(1.2) without using condition $(\mathbf{H_g})$, and the novelty compared to [16], [17], [18], [19] is that in the current work we consider a system with unbounded delay. We observe that our results are based on some types of optimal regularity results for abstract systems in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) + \xi_1(t)) = A(t)x(t) + \xi_2(t), \quad t \in [0,a].$$

We now consider a motivation for the study of abstract neutral systems as (1.1)–(1.2). This type of systems arises, for example, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature u and on its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [14], [28], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see [8], [10], [11], [25], has been frequently used to describe this phenomenon:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)\,\mathrm{d}s\Big] = c\Delta u(t,x) + \int_{-\infty}^{t} k_2(t-s)\Delta u(s,x)\,\mathrm{d}s,$$
$$u(t,x) = 0, \quad x \in \partial\Omega.$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x) \in [0,\infty)\times\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i\colon\ \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are the internal energy and the heat flux relaxation, respectively.

By assuming the solution $u(\cdot)$ to be known on $(-\infty,0]$ and $k_2\equiv 0$, we can transform this system into the abstract form (1.1)–(1.2).

Next, we introduce some notation and technicalities needed to establish our results. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this paper, we denote by $\mathscr{L}(Z,W)$ the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathscr{L}(Z,W)}$, and we write $\mathscr{L}(Z)$ and $\|\cdot\|_{\mathscr{L}(Z)}$ when $Z=W$. In addition, we write $Z\hookrightarrow W$ when Z is continuously included in W. As usual, $C([0,b],Z)$ is the space of continuous functions from $[0,b]$ into Z with the sup-norm denoted by $\|\cdot\|_{C([0,b],Z)}$, and $C^{\gamma}([0,b],Z)$, $\gamma\in(0,1)$, is the space formed by all γ-Hölder Z-valued functions from $[0,b]$ into Z provided with the norm $\|\xi\|_{C^{\gamma}([0,b],Z)}=\|\xi\|_{C([0,b],Z)}+|[\xi]|_{C^{\gamma}([0,b],Z)}$ where $[|\xi|]_{C^{\gamma}([0,b],Z)}=\sup\limits_{t,s\in[0,b],\,t\neq s}\|\xi(s)-\xi(t)\|_Z/|t-s|^{\gamma}$. The notation $C^{1+\gamma}([0,b],Z)$ stands for the space formed by all C^1 functions ξ such that $\xi'\in C^{\gamma}([0,b],Z)$, endowed with the norm $\|\xi\|_{C^{1+\gamma}([0,b],Z)}=\|\xi\|_{C([0,b],Z)}+\|\xi'\|_{C^{\gamma}([0,b],Z)}$. In addition, $B([0,b],Z)$ is the space of bounded measurable functions from $[0,b]$ into Z with the sup norm denoted by $\|\cdot\|_{B([0,b],Z)}$.

In this paper, $(X,\|\cdot\|)$ is a Banach space and $A\colon D(A)\subset X\to X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geqslant 0}$ on X. To simplify our developments, we assume $0\in\varrho(A)$ and we use the notation $(\mathscr{D},\|\cdot\|_{\mathscr{D}})$ for the space $D(A)$ endowed with the norm $\|x\|_{\mathscr{D}}=\|Ax\|$. In addition, for $\eta\in(0,1)$ we denote by $(X,\mathscr{D})_{\eta,\infty}$ the space

$$(X,\mathscr{D})_{\eta,\infty}=\left\{x\in X\colon\ [x]_{\eta,\infty}=\int_0^1 t^{1-\eta}\|AT(t)x\|\,\mathrm{d}t<\infty\right\}$$

provided with the norm $\|x\|_{\eta,\infty}=[x]_{\eta,\infty}+\|x\|$. In the remainder of this paper, we will assume that C_i, $i=0,1,2$, and $C^1_{\alpha,\infty}$ are positive constants such that $\|s^iA^iT(s)\|<C_i$, $i=0,1,2$, and $\|s^{1-\alpha}AT(s)\|_{\mathscr{L}((X,\mathscr{D})_{\alpha,\infty},X)}<C^1_{\alpha,\infty}$ for all $s>0$.

For additional details on analytical semigroups we refer the reader to Lunardi [24].

Throughout this work, we use an axiomatic definition for the phase space $\mathscr{B}$ which is similar to the one in [15], [23]. Specifically, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty,0]$ into $\mathscr{D}$ endowed with a semi-norm $\|\cdot\|_{\mathscr{B}}$ and satisfying the following axioms:

(A) If $x\colon(-\infty,\sigma+b)\to\mathscr{D}$, $b>0$, $\sigma\in\mathbb{R}$, is continuous on $[\sigma,\sigma+b)$ and $x_\sigma\in\mathscr{B}$, then for every $t\in[\sigma,\sigma+b)$ the following conditions hold:
  (i) x_t is in $\mathscr{B}$.
  (ii) $\|x(t)\|\leqslant H\|x_t\|_{\mathscr{B}}$.
  (iii) $\|x_t\|_{\mathscr{B}}\leqslant K(t-\sigma)\sup\{\|x(s)\|_{\mathscr{D}}\colon\ \sigma\leqslant s\leqslant t\}+M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$, where $H>0$ is a constant; $K,M\colon[0,\infty)\to[1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), the function $t \to x_t$ belongs to $C([\sigma, \sigma+b), \mathscr{B})$.
(B) The space $\mathscr{B}$ is complete.

Example 1.1. The phase space $C_r \times L^p(\varrho, \mathscr{D})$. Let $r \geqslant 0$, $1 \leqslant p < \infty$ and let $\varrho\colon (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [23]. Briefly, this means that ϱ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\varrho(\xi+\theta) \leqslant \gamma(\xi)\varrho(\theta)$ for all $\xi \leqslant 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\varrho, \mathscr{D})$ consists of all classes of functions $\varphi\colon (-\infty, 0] \to \mathscr{D}$ such that $\varphi \in C([-r, 0]; \mathscr{D})$, φ is Lebesgue-measurable and $\varrho^{1/p}\varphi \in L^p((-\infty, -r]; \mathscr{D})$. The seminorm in $C_r \times L^p(\varrho, \mathscr{D})$ is given by $\|\varphi\|_{\mathscr{B}} = \|\varphi\|_{C([-r,0];\mathscr{D})} + \|\varrho^{1/p}\varphi\|_{L^p((-\infty,-r];\mathscr{D})}$.

The space $\mathscr{B} = C_r \times L^p(\varrho, \mathscr{D})$ satisfies axioms (A), (A1) and (B). Moreover, for $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \bigl(\int_{-t}^0 \varrho(\theta)\,\mathrm{d}\theta\bigr)^{1/2}$ for $t \geqslant 0$. See [23, Theorem 1.3.8] for details.

The paper has two main sections. In the next section, we study the existence of α-Hölder classical solutions for the abstract neutral system (1.1)–(1.2). In the last section, a concrete application to partial neutral integro-differential equations is considered.

2. Existence of solutions

In this section we discuss the existence of α-Hölder classical solutions for the abstract neutral integro-differential system (1.1)–(1.2). To simplify our next developments, it is convenient to introduce some additional notation. In the remainder of this section, $\alpha \in (0,1)$, $\varphi \in \mathscr{B}$ and $y\colon \mathbb{R} \to X$ is the function defined by $y(s) = \varphi(s)$ for $s \leqslant 0$ and $y(s) = T(s)\varphi(0)$ for $s \geqslant 0$. In addition, $G, F\colon [0,a] \to \mathscr{L}(\mathscr{B}, X)$ are the functions given by

$$G(t)\psi = \int_0^\infty B(t, t-s)\psi(-s)\,\mathrm{d}s \quad \text{and} \quad F(t)\psi = \int_0^\infty C(t, t-s)\psi(-s)\,\mathrm{d}s.$$

Let $b \in (0, a]$ and let $w\colon (-\infty, b] \to \mathscr{D}$ be a function such that $w_0 \in \mathscr{B}$ and $w_{|[0,b]} \in C([0,b], \mathscr{D})$. Next, we use the notation G_w, F_w and P_w for the functions $G_w, F_w\colon [0,b] \to X$ and $P_w\colon [0,b] \to \mathscr{B}$ defined by $G_w(t) = G(t)w_t$, $F_w(t) = F(t)w_t$ and $P_w(t) = w_t$. In addition, $\mathscr{C}^\alpha_{\mathscr{B}}(b)$ denotes the space

(2.5)
$$\mathscr{C}^\alpha_{\mathscr{B}}(b) = \{u\colon (-\infty, b] \to \mathscr{D}\colon\ u_0 \in \mathscr{B},\ u_{|[0,b]} \in C([0,b], \mathscr{D}),\ P_u \in C^\alpha([0,b], \mathscr{B})\},$$

endowed with the norm $\|u\|_{\mathscr{C}^\alpha_{\mathscr{B}}(b)} = \|P_u\|_{C^\alpha([0,b],\mathscr{B})}$.

We note that using the above notation, the integro-differential system (1.1)–(1.2) can be represented in the abstract form as

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+G(t)x_t]=Ax(t)+F(t)x_t,\quad t\in[0,a], \tag{2.6}$$

$$x_0=\varphi\in\mathscr{B}. \tag{2.7}$$

Definition 2.1. A function $u\colon(-\infty,b]\to\mathscr{D}$, $0<b\leqslant a$, is called a classical solution of the neutral system (1.1)–(1.2) on $[0,b]$, if $u_0=\varphi$, $u_{|[0,b]}\in C([0,b],\mathscr{D})$, the function $t\to\bigl[u(t)+\int_{-\infty}^t B(t,\tau)u(\tau)\,\mathrm{d}\tau\bigr]$ belongs to $C^1([0,b],X)$ and the equation (1.1) is satisfied on $[0,b]$.

To prove the main results of this section, Theorem 2.1, we need some preliminary lemmas. The next results, Lemma 2.1 and Lemma 2.2, are proved in [17]. However, we include the proof of them for completeness.

Lemma 2.1. *Assume that $\xi_1\in C^\alpha([0,b],\mathscr{D})$, $\xi_2\in C^\alpha([0,b],X)$, $x\in\mathscr{D}$ and let $u\colon[0,b]\to X$ be the function defined by*

$$u(t)=T(t)(x+\xi_1(0))-\xi_1(t)-\int_0^t AT(t-s)\xi_1(s)\,\mathrm{d}s+\int_0^t T(t-s)\xi_2(s)\,\mathrm{d}s.$$

If $Ax+\xi_2(0)\in(X,\mathscr{D})_{\alpha,\infty}$, then $u\in C^\alpha([0,b],\mathscr{D})$, $\frac{\mathrm{d}}{\mathrm{d}t}(u+\xi_1)\in B([0,b],(X,\mathscr{D})_{\alpha,\infty})$, $u+\xi_1\in C^\alpha([0,b],\mathscr{D})\cap C^{1+\alpha}([0,b],X)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t)+\xi_1(t)]=Au(t)+\xi_2(t),\quad\forall\,t\in[0,b].$$

Moreover,

$$[|u|]_{C^\alpha([0,b],\mathscr{D})}\leqslant\Lambda_1([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}+[|\xi_2|]_{C^\alpha([0,b],X)})+\frac{C^1_{\alpha,\infty}}{\alpha}\|Ax+\xi_2(0)\|_{\alpha,\infty},$$

$$\|u\|_{C^\alpha([0,b],\mathscr{D})}\leqslant C_0\|Ax\|+([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}[|\xi_2|]_{C^\alpha([0,b],X)})\Bigl(C_0+\frac{C_1}{\alpha}\Bigr)b^\alpha+2C_0\|\xi_2\|,$$

where $\Lambda_1=2C_1/\alpha+3C_0+2+C_2/\alpha(1-\alpha)$.

Proof. Let $v\colon[0,b]\to X$ be the function defined by

$$v(t)=T(t)(x+\xi_1(0))+\int_0^t T(t-s)(-A\xi_1(s)+\xi_2(s))\,\mathrm{d}s.$$

From [24, Theorem 4.3.1] we have that $v\in C^\alpha([0,b],\mathscr{D})\cap C^{1+\alpha}([0,b],X)$ and $v(\cdot)$ is the unique strict solution of

$$x'(t)=Ax(t)-A\xi_1(t)+\xi_2(t),\quad\forall\,t\in[0,a],$$
$$x(0)=x+\xi_1(0).$$

Since $v = u + \xi_1$, we obtain $u \in C^\alpha([0,b],\mathscr{D})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t)+\xi_1(t)] = A(u(t)+\xi_1(t)) - A\xi_1(t) + \xi_2(t) = Au(t) + \xi_2(t), \quad \forall\, t \in [0,b].$$

On the other hand, from the proof of [24, Theorem 4.3.1] we obtain that

$$[|v|]_{C^\alpha([0,b],\mathscr{D})} \leqslant d([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})} + [|\xi_2|]_{C^\alpha([0,b],X)}) + \frac{C^1_{\alpha,\infty}}{\alpha}\|Ax+\xi_2(0)\|_{\alpha,\infty},$$

where $d = 2C_1/\alpha + 3C_0 + 1 + C_2/\alpha(1-\alpha)$. Now, from the definition of $v(\cdot)$ we get

$$[|u|]_{C^\alpha([0,b],\mathscr{D})} \leqslant \Lambda_1([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})} + [|\xi_2|]_{C^\alpha([0,b],X)}) + \frac{C^1_{\alpha,\infty}}{\alpha}\|Ax+\xi_2(0)\|_{\alpha,\infty}.$$

Moreover, by re-writing $u(\cdot)$ in the form

$$\begin{aligned}(2.8)\qquad u(t) = T(t)x + T(t)(\xi_1(0)-\xi_1(t)) - \int_0^t T(t-s)[A\xi_1(s) - A\xi_1(t)]\,\mathrm{d}s \\ + \int_0^t T(t-s)(\xi_2(s)-\xi_2(t))\,\mathrm{d}s + \int_0^t T(t-s)\xi_2(t)\,\mathrm{d}s,\end{aligned}$$

we have that

$$\begin{aligned}\|Au(t)\| &\leqslant \|T(t)Ax\| + C_0\|A\xi_1(0) - A\xi_1(t)\| \\ &\quad + \int_0^t \|AT(t-s)[A\xi_1(s) - A\xi_1(t)]\|\,\mathrm{d}s \\ &\quad + \int_0^t \|AT(t-s)(\xi_2(s)-\xi_2(t))\|\,\mathrm{d}s + \|(T(t)-I)\xi_2(t)\| \\ &\leqslant C_0\|Ax\| + C_0[|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}t^\alpha + [|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}\int_0^t \frac{C_1}{(t-s)^{1-\alpha}}\,\mathrm{d}s \\ &\quad + [|\xi_2|]_{C^\alpha([0,b],X)}\int_0^t \frac{C_1}{(t-s)^{1-\alpha}}\,\mathrm{d}s + 2C_0\|\xi_2\|_{C([0,b],X)} \\ &\leqslant C_0\|Ax\| + ([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})} + [|\xi_2|]_{C^\alpha([0,b],X)})\Big(C_0 + \frac{C_1}{\alpha}\Big)b^\alpha \\ &\quad + 2C_0\|\xi_2\|_{C([0,b],X)},\end{aligned}$$

which completes the proof. □

Lemma 2.2. *If the conditions of Lemma* 2.1 *are satisfied and* $\xi_2(0) \in (X,\mathscr{D})_{\alpha,\infty}$, *then*

$$\|u\|_{C([0,b],\mathscr{D})} \leqslant C_0\|Ax\| + \Lambda_2([|\xi_1|]_{C^\alpha(\mathscr{D})} + [|\xi_2|]_{C^\alpha(X)})b^\alpha + \frac{C^1_{\alpha,\infty}}{\alpha}\|\xi_2(0)\|_{\alpha,\infty}b^\alpha,$$

where $\Lambda_2 = C_0 + C_1/\alpha + 1$.

P r o o f. By re-writing $u(t)$ as in (2.8), we obtain

$$Au(t) = T(t)Ax + T(t)(A\xi_1(0) - A\xi_1(t)) - \int_0^t AT(t-s)[A\xi_1(s) - A\xi_1(t)]\,\mathrm{d}s + \int_0^t AT(t-s)(\xi_2(s) - \xi_2(t))\,\mathrm{d}s + T(t)\xi_2(t) - \xi_2(t).$$

Consequently,

$$\begin{aligned}\|Au(t)\| &\leqslant C_0\|Ax\| + C_0[|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}b^\alpha \\ &\quad + \int_0^t \|AT(t-s)[A\xi_1(s) - A\xi_1(t)]\|\,\mathrm{d}s \\ &\quad + \int_0^t \|AT(t-s)(\xi_2(s) - \xi_2(t))\|\,\mathrm{d}s + \|\xi_2(0) - \xi_2(t)\| \\ &\quad + \|T(t)\xi_2(0) - \xi_2(0)\| + \|T(t)(\xi_2(t) - \xi_2(0))\| \\ &\leqslant C_0\|Ax\| + C_0[|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}b^\alpha + \frac{C_1}{\alpha}[|\xi_1|]_{C^\alpha([0,b],\mathscr{D})}b^\alpha \\ &\quad + \frac{C_1}{\alpha}[|\xi_2|]_{C^\alpha([0,b],X)}b^\alpha + [|\xi_2|]_{C^\alpha([0,b],X)}b^\alpha \\ &\quad + \left\|\int_0^t AT(s)\xi_2(0)\,\mathrm{d}s\right\| + C_0[|\xi_2|]_{C^\alpha([0,b],X)}b^\alpha,\end{aligned}$$

and hence,

$$\|u\|_{C([0,b],\mathscr{D})} \leqslant C_0\|Ax\| + ([|\xi_1|]_{C^\alpha([0,b],\mathscr{D})} + [|\xi_2|]_{C^\alpha([0,b],X)})\Big(C_0 + \frac{C_1}{\alpha} + 1\Big)b^\alpha + \frac{C^1_{\alpha,\infty}}{\alpha}\|\xi_2(0)\|_{\alpha,\infty}b^\alpha.$$

□

Next, we establish some properties for the functions $F(\cdot)$ and $G(\cdot)$.

Lemma 2.3. *Assume there are a measurable function $L_B\colon [0,\infty) \to \mathbb{R}^+$ and $\Lambda_B > 0$ such that*

$$\|B(t,t-\tau) - B(s,s-\tau)\|_{\mathscr{L}_{\mathscr{B}}(\mathscr{D})} \leqslant L_B(\tau)|t-s|^\alpha, \quad t,s\in[0,a],\ \tau>0,$$

$$\sup_{t\in[0,a]}\int_0^\infty (L_B(\tau) + \|B(t,t-\tau)\|_{\mathscr{L}(\mathscr{B},\mathscr{D})})\|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau \leqslant \Lambda_B\|\psi\|_{\mathscr{B}}, \quad \psi\in\mathscr{B}.$$

Then $G\in C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))$ and $\|G\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant 2\Lambda_B$.

P r o o f. From the assumptions, for $\psi \in \mathscr{B}$ we see that

$$\|G(t)\psi\|_{\mathscr{D}} \leqslant \int_0^\infty \|B(t,t-s)\|_{\mathscr{L}(\mathscr{B})}\|\psi(-s)\|_{\mathscr{D}}\,\mathrm{d}s \leqslant \Lambda_B\|\psi\|_{\mathscr{B}},$$

which implies that $\|G\|_{C([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant \Lambda_B$. Also, for $0 \leqslant s < t \leqslant a$ and $\psi \in \mathscr{B}$ we get

$$\begin{aligned}\|G(t)\psi - G(s)\psi\|_{\mathscr{D}} &\leqslant \int_0^\infty \|B(t,t-\tau)-B(s,s-\tau)\|_{\mathscr{L}(\mathscr{D})}\|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau\\ &\leqslant (t-s)^\alpha \int_0^\infty L_B(\tau)\|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau\\ &\leqslant (t-s)^\alpha\Lambda_B\|\psi\|_{\mathscr{B}},\end{aligned}$$

so that $[|G|]_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant \Lambda_B$. From the above it follows that $G \in C^\alpha([0,a], \mathscr{L}(\mathscr{B},\mathscr{D}))$ and $\|G\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant \Lambda_B$. The proof is complete. □

The proof of the next lemma is similar to the proof of Lemma 2.3, so we omit it.

Lemma 2.4. *Assume there are a measurable function $L_C\colon [0,\infty) \to \mathbb{R}^+$ and $\Lambda_C > 0$ such that*

$$\begin{gathered}\|C(t,t-\tau)-C(s,s-\tau)\|_{\mathscr{L}(\mathscr{D},X)} \leqslant L_C(\tau)|t-s|^\alpha, \quad t,s\in[0,a],\ \tau>0,\\ \sup_{t\in[0,a]}\int_0^\infty (L_C(\tau)+\|C(t,t-\tau)\|_{\mathscr{L}(\mathscr{D},X)})\|\psi(-\tau)\|\,\mathrm{d}\tau \leqslant \Lambda_C\|\psi\|_{\mathscr{B}}, \quad \psi\in\mathscr{B}.\end{gathered}$$

Then $F \in C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))$ and $\|F\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))} \leqslant 2\Lambda_C$.

In the application considered in the last section, it is possible to see that the assumptions in the above lemmas are verified in many situations and are not restrictive. Now, we consider some variants of the above lemmas.

Lemma 2.5. *Assume $\mathscr{B} \hookrightarrow L^1((-\infty,0],\mathscr{D}) \cap L^p((-\infty,0],\mathscr{D})$ for some $p>1$, and there are constants Λ_B^i, $i=1,2,3$, such that*

$$\begin{gathered}\|B(t,\tau)-B(s,\tau)\|_{\mathscr{D}} \leqslant \Lambda_B^1|t-s|^\alpha, \quad t,s\in[0,a],\ \tau\leqslant\min\{t,s\},\\ \|B(t,s)-B(t,\tau)\|_{\mathscr{D}} \leqslant \Lambda_B^2|s-\tau|^\alpha, \quad t\in[0,a],\ s,\tau\in(-\infty,t],\\ \sup_{t\in[0,a]}\biggl(\int_0^\infty \|B(t,t-\tau)\|_{\mathscr{D}}^{p'}\,\mathrm{d}\tau\biggr)^{1/p'} \leqslant \Lambda_B^3, \quad \text{where } p'=\frac{p}{p-1}.\end{gathered}$$

Then $G \in C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))$ and

$$\|G\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant (\Lambda_B^1+\Lambda_B^2)\|i\|_{\mathscr{L}(\mathscr{B},L^1)} + \Lambda_B^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)},$$

where i represents the inclusion map from $\mathscr{B}$ into the space $L^j((-\infty,0],\mathscr{D})$, $j=1,p$, and

$$\|i\|_{\mathscr{L}(\mathscr{B},L^j)} = \|i\|_{\mathscr{L}(\mathscr{B},L^j(-\infty,0],\mathscr{D})}. \tag{2.9}$$

For $t\in[0,a]$ and $\psi\in\mathscr{B}$ we have that

$$\begin{aligned}\|G(t)\psi\|_{\mathscr{D}} &\leqslant \int_0^\infty \|B(t,t-\tau)\|_{\mathscr{L}(\mathscr{D})}\|\psi(-\tau)\|_{\mathscr{D}}\\ &\leqslant \Lambda_B^3\|\psi\|_{L^p(-\infty,0],\mathscr{D})}\\ &\leqslant \Lambda_B^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)}\|\psi\|_{\mathscr{B}},\end{aligned}$$

and hence $\|G(t)\|_{\mathscr{L}(\mathscr{B},\mathscr{D})}\leqslant \Lambda_B^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)}$ for all $t\in[0,a]$. Also, for $t>s$, $t,s\in[0,a]$ and $\psi\in\mathscr{B}$ we see that

$$\begin{aligned}\|G(t)\psi-G(s)\psi\|_{\mathscr{D}} &\leqslant \int_0^\infty \|B(t,t-\tau)-B(s,t-\tau)\|_{\mathscr{L}(\mathscr{D})}\|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau\\ &\quad+\int_0^\infty \|B(s,t-\tau)-B(s,s-\tau)\|_{\mathscr{L}(\mathscr{D})}\|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau\\ &\leqslant (t-s)^\alpha(\Lambda_1+\Lambda_2)\int_0^\infty \|\psi(-\tau)\|_{\mathscr{D}}\,\mathrm{d}\tau,\end{aligned}$$

which shows that

$$[|G|]_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))}\leqslant(\Lambda_B^1+\Lambda_B^2)\|i\|_{\mathscr{L}(\mathscr{B},L^1)},\quad G\in C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))$$

and (2.9) is satisfied. This completes the proof. □

The proof of Lemma 2.6 follows reasoning similar to that in the proof of Lemma 2.5.

Lemma 2.6. *Assume $\mathscr{B}\hookrightarrow L^1((-\infty,0],X)\cap L^p((-\infty,0],X)$ for some $p>1$, and there are constants Λ_C^i, $i=1,2,3$, such that*

$$\begin{aligned}\|C(t,\tau)-C(s,\tau)\|_{\mathscr{L}(\mathscr{D},X)}&\leqslant\Lambda_C^1|t-s|^\alpha,\quad t,s\in[0,a],\ \tau\leqslant\min\{t,s\},\\ \|C(t,s)-C(t,\tau)\|_{\mathscr{L}(\mathscr{D},X)}&\leqslant\Lambda_C^2|s-\tau|^\alpha,\quad t\in[0,a],\ s,\tau\in(-\infty,t],\\ \sup_{t\in[0,a]}\left(\int_0^\infty\|C(t,t-\tau)\|^{p'}\,\mathrm{d}\tau\right)^{1/p'}&\leqslant\Lambda_C^3,\quad \textit{where } p'=\frac{p}{p-1}.\end{aligned}$$

Then $F\in C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))$ and

$$\|F\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))}\leqslant+(\Lambda_C^1+\Lambda_C^2)\|i\|_{\mathscr{L}(\mathscr{B},L^1)}\Lambda_C^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)},$$

where i represents the inclusion map from $\mathscr{B}$ into the space $L^j((-\infty,0],X)$, $j=1,p$, and $\|i\|_{\mathscr{L}(\mathscr{B},L^j)}=\|i\|_{\mathscr{L}(\mathscr{B},L^j(-\infty,0],X)}$.

The proof of Lemma 2.7 below is a straightforward estimate argument.

Lemma 2.7. *Let $b \in (0,a]$, $u, v \in \mathscr{C}^{\alpha}_{\mathscr{B}}(b)$ with $u_0 = v_0$ and assume $F \in C^{\alpha}(\mathscr{L}(\mathscr{B}, X))$ and $G \in C^{\alpha}(\mathscr{L}(\mathscr{B}, \mathscr{D}))$. Then $G_u \in C^{\alpha}([0,b], \mathscr{D})$, $F_u \in C^{\alpha}([0,b], X)$ and*

$$\begin{aligned}
\|G_u\|_{C^{\alpha}([0,b],\mathscr{D})} &\leqslant \|G\|_{C^{\alpha}([0,b],\mathscr{L}(\mathscr{B},\mathscr{D}))}\|u\|_{\mathscr{C}^{\alpha}_{\mathscr{B}}(b)},\\
\|G_{(u-v)}\|_{C^{\alpha}([0,b],\mathscr{D})} &\leqslant \|G\|_{C^{\alpha}([0,b],\mathscr{L}(\mathscr{B},\mathscr{D}))}(b^{\alpha}+1)\|u-v\|_{\mathscr{C}^{\alpha}_{\mathscr{B}}(b)},\\
\|F_u\|_{C^{\alpha}([0,b],X)} &\leqslant \|F\|_{C^{\alpha}([0,b],\mathscr{L}(\mathscr{B},X))}\|u\|_{\mathscr{C}^{\alpha}_{\mathscr{B}}(b)},\\
\|F_{(u-v)}\|_{C^{\alpha}([0,b],X)} &\leqslant \|F\|_{C^{\alpha}([0,b],\mathscr{L}(\mathscr{B},X))}(b^{\alpha}+1)\|u-v\|_{\mathscr{C}^{\alpha}_{\mathscr{B}}(b)}.
\end{aligned}$$

In the next theorem, the main result of this work, we use all the notation introduced in this section. In addition, for $b \in (0,a]$, we use the notation $M^b = \sup\limits_{s\in[0,b]} M(s)$ and $K^b = \sup\limits_{s\in[0,b]} K(s)$.

Theorem 2.1. *Assume the conditions in Lemmas 2.3 and 2.4 are satisfied. Suppose, in addition, $y \in \mathscr{C}^{\alpha}_{\mathscr{B}}(a)$, $F(0)\varphi \in (X,\mathscr{D})_{\alpha,\infty}$, $A\varphi(0) \in (X,\mathscr{D})_{\alpha,\infty}$ and there is $\delta \in (0,a]$ such that $2(\Lambda_B + \Lambda_C)K^{\delta}[M^{\delta}\Lambda_2 + \Lambda_1] < 1$. Then there exists a unique solution $u(\cdot)$ of (1.1)–(1.2) in $C^{\alpha}([0,b],\mathscr{D})$ for some $0 < b \leqslant a$.*

Proof. Let $0 < b \leqslant a$ be such that $2(1+b^{\alpha})^2(\Lambda_B + \Lambda_C)K^b[M^b\Lambda_2 + \Lambda_1] < 1$. On the space

$$\mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi, b) = \{u \in \mathscr{C}^{\alpha}_{\mathscr{B}}(b)\colon\ u_0 = \varphi\},$$

endowed with the metric $d(u,v) = \|u-v\|_{\mathscr{C}^{\alpha}_{\mathscr{B}}(b)}$, we define the map $\Gamma\colon \mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi,b) \to \mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi,b)$ by $(\Gamma u)_0 = \varphi$ and

$$\begin{aligned}
\Gamma u(t) = {}& T(t)(\varphi(0) + G(0)\varphi) - G_u(t)\\
&- \int_0^t AT(t-s)G_u(s)\,\mathrm{d}s + \int_0^t T(t-s)F_u(s)\,\mathrm{d}s, \quad t \in [0,b].
\end{aligned}$$

From Lemmas 2.1 and 2.7, it is easy to see that $\Gamma u_{|_{[0,b]}} \in C^{\alpha}([0,b],\mathscr{D})$. In order to prove that Γ is a contraction on $\mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi,b)$, next we show that Γ has values in $\mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi,b)$. For $u \in \mathscr{C}^{\alpha}_{\mathscr{B}}(\varphi,b)$, $t \in [0,b)$ and $h > 0$ such that $t+h \in [0,b]$, we get

$$\begin{aligned}
(2.10)\quad & \|P_{\Gamma u}(t+h) - P_{\Gamma u}(t)\|_{\mathscr{B}}\\
&= \|(\Gamma u)_{(t+h)} - (\Gamma u)_t\|_{\mathscr{B}}\\
&\leqslant M^b\|(\Gamma u)_h - \varphi\|_{\mathscr{B}} + K^b \sup_{s\in[0,b]}\|\Gamma u(s+h) - \Gamma u(s)\|_{\mathscr{D}}\\
&\leqslant M^b\|y_h - \varphi\|_{\mathscr{B}} + M^b\|(\Gamma u - y)_h\|_{\mathscr{B}} + K^b[|\Gamma u|]_{C^{\alpha}([0,b],\mathscr{D})}h^{\alpha}\\
&\leqslant M^b[|P_y|]_{C^{\alpha}([0,b],\mathscr{B})}h^{\alpha} + M^b\|(\Gamma u - y)_h\|_{\mathscr{B}}\\
&\quad + K^b[|\Gamma u_{|_{[0,b]}}|]_{C^{\alpha}([0,b],\mathscr{D})}h^{\alpha}.
\end{aligned}$$

To estimate $\|(\Gamma u - y)_h\|_{\mathscr{B}}$, we use Lemma 2.2 with $x = 0$. By noting that $F(0)\varphi \in (X, \mathscr{D})_{\alpha,\infty}$ we conclude that

$$\begin{aligned}\|(\Gamma u - y)_h\|_{\mathscr{B}} &\leqslant K^b \sup_{s\in[0,h]} \|(\Gamma u - y)(s)\|_{\mathscr{D}} \\ &\leqslant K^b \Lambda_2([|G_u|]_{C^\alpha([0,b],\mathscr{D})} + [|F_u|]_{C^\alpha([0,b],X)})h^\alpha \\ &\quad + \frac{K^b C^1_{\alpha,\infty}}{\alpha} \|F(0)\varphi\|_{\alpha,\infty} h^\alpha.\end{aligned}$$

Using this inequality in (2.10), we find that

$$\begin{aligned}[|P_{\Gamma u}|]_{C^\alpha([0,b],\mathscr{B})} &\leqslant M^b[|P_y|]_{C^\alpha([0,b],\mathscr{B})} + M^b K^b \Lambda_2([|G_u|]_{C^\alpha([0,b],\mathscr{D})} \\ &\quad + [|F_u|]_{C^\alpha([0,b],X)}) + M^b K^b \frac{C^1_{\alpha,\infty}}{\alpha} \|F(0)\varphi\|_{\alpha,\infty} \\ &\quad + K^b [|\Gamma u_{|[0,b]}|]_{C^\alpha([0,b],\mathscr{D})},\end{aligned} \tag{2.11}$$

which proves that $\Gamma u \in \mathscr{C}^\alpha_{\mathscr{B}}(\varphi, b)$. Thus, Γ has values in $\mathscr{C}^\alpha_{\mathscr{B}}(\varphi, b)$.

Let $u, v \in \mathscr{C}^\alpha_{\mathscr{B}}(\varphi, b)$. From the inequality (2.11) and Lemmas 2.7, 2.1, 2.3 and 2.4 we infer that

$$\begin{aligned}[|P_{\Gamma u} - P_{\Gamma v}|]_{\mathscr{C}^\alpha([0,b],\mathscr{B})} &\leqslant M^b K^b \Lambda_2([|G_{u-v}|]_{C^\alpha([0,b],\mathscr{D})} + [|F_{u-v}|]_{C^\alpha([0,b],X)}) \\ &\quad + K^b[|\Gamma(u-v)|]_{C^\alpha([0,b],\mathscr{D})} \\ &\leqslant 2M^b K^b \Lambda_2 (b^\alpha + 1)(\Lambda_B + \Lambda_C)\|u - v\|_{\mathscr{C}^\alpha_{\mathscr{B}}(b)} \\ &\quad + K^b \Lambda_1([|G_{u-v}|]_{C^\alpha([0,b],\mathscr{D})} + [|F_{u-v}|]_{C^\alpha([0,b],X)}) \\ &\leqslant 2M^b K^b \Lambda_2 (b^\alpha + 1)(\Lambda_B + \Lambda_C)\|u - v\|_{\mathscr{C}^\alpha_{\mathscr{B}}(b)} \\ &\quad + 2K^b \Lambda_1 (b^\alpha + 1)(\Lambda_B + \Lambda_C)\|u - v\|_{\mathscr{C}^\alpha_{\mathscr{B}}(b)},\end{aligned}$$

which implies that

$$[|P_{\Gamma u} - P_{\Gamma v}|]_{C^\alpha([0,b],\mathscr{B})} \leqslant 2(b^\alpha + 1)(\Lambda_B + \Lambda_C)K^b[M^b\Lambda_2 + \Lambda_1]\|u - v\|_{\mathscr{C}^\alpha_{\mathscr{B}}(b)}, \tag{2.12}$$

and

$$\|P_{\Gamma u} - P_{\Gamma v}\|_{C([0,b],\mathscr{B})} \leqslant 2(b^\alpha + 1)(\Lambda_B + \Lambda_C)K^b[M^b\Lambda_2 + \Lambda_1]b^\alpha[|u - v|]_{\mathscr{C}^\alpha_{\mathscr{B}}(b)}, \tag{2.13}$$

since $P_{\Gamma u}(0) = P_{\Gamma v}(0)$. Now, from (2.12) and (2.13) it follows that

$$d(\Gamma u, \Gamma v) \leqslant 2(b^\alpha + 1)^2(\Lambda_B + \Lambda_C)K^b[M^b\Lambda_2 + \Lambda_1]d(u, v),$$

which shows that $\Gamma(\cdot)$ is a contraction on $\mathscr{C}^\alpha_{\mathscr{B}}(\varphi, b)$ and there exists a unique fixed point $u \in \mathscr{C}^\alpha_{\mathscr{B}}(\varphi, b)$ of Γ. Finally, Lemma 2.1 implies that $u(\cdot)$ is a classical solution of (1.1)–(1.2) in $C^\alpha([0,b],\mathscr{D})$. This completes the proof. □

The proof of the next theorem follows reasoning similar to that in the proof of Theorem 2.1. In this result we use the notation introduced in Lemmas 2.5 and 2.6.

Theorem 2.2. *Assume the conditions in Lemmas* 2.5 *and* 2.6 *are verified. Suppose* $y \in \mathscr{C}^{\alpha}_{\mathscr{B}}(a)$, $F(0)\varphi \in (X, \mathscr{D})_{\alpha,\infty}$, $A\varphi(0) \in (X, \mathscr{D})_{\alpha,\infty}$ *and there is* $\delta \in (0, a]$ *such that* $(\widetilde{\Lambda_B} + \widetilde{\Lambda_C})K^{\delta}[M^{\delta}\Lambda_2 + \Lambda_1] < 1$, *where* $\widetilde{\Lambda_B} = (\Lambda_B^1 + \Lambda_B^2)\|i\|_{\mathscr{L}(\mathscr{B},L^1)} + \Lambda_B^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)}$ *and* $\widetilde{\Lambda_C} = (\Lambda_C^1 + \Lambda_C^2)\|i\|_{\mathscr{L}(\mathscr{B},L^1)} + \Lambda_C^3\|i\|_{\mathscr{L}(\mathscr{B},L^p)}$. *Then there exists a unique classical solution* $u(\cdot)$ *of* (1.1)–(1.2) *in* $C^{\alpha}([0, b], \mathscr{D})$ *for some* $0 < b \leqslant a$.

3. Application

In this section we consider an application of our abstract results. Specifically, we study the existence of a classical solution for the partial neutral integro-differential system

$$\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-\infty}^{t} a_1(t-s)u(s,\xi)\,\mathrm{d}s\right] = \frac{\partial^2}{\partial\xi^2}u(t,\xi) + \int_{-\infty}^{t} a_2(t-s)u(s,\xi)\,\mathrm{d}s, \tag{3.14}$$

$$u(t,0) = u(t,\pi) = 0, \tag{3.15}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta \leqslant 0,\ \xi \in [0,\pi], \tag{3.16}$$

for $(t,\xi) \in [0,a] \times [0,\pi]$, where $a_i \in C([0,\infty),\mathbb{R})$.

We note that this system appears in the theory of heat conduction in fading memory materials, see our bibliographical comments in Section 1.

To study the integro-differential system (3.14)–(3.16), we consider the Banach space $X = L^2([0,\pi])$ and the operator $A\colon D(A) \subset X \to X$ given by $Ax = x''$ with the domain $D(A) = \{x \in X\colon x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geqslant 0}$ on X. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and normalized eigenfunctions given by $z_n(\xi) := (2/\pi)^{1/2}\sin(n\xi)$. The set of functions $\{z_n\colon n \in \mathbb{N}\}$ is an orthonormal basis for X, $T(t)x = \sum\limits_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\langle x, z_n\rangle z_n$ for $x \in X$ and $Ax = -\sum\limits_{n=1}^{\infty} n^2\langle x, z_n\rangle z_n$ for $x \in D(A)$. From the above, it is easy to see that $\|T(t)\| \leqslant \mathrm{e}^{-t}$, $\|AT(t)\| \leqslant \mathrm{e}^{-t}t^{-1}$ and $\|A^2T(t)\| \leqslant 4\mathrm{e}^{-t}t^{-2}$ for every $t > 0$.

Next, $[D((-A)^{\beta})]$ denotes the domain of the fractional power $(-A)^{\beta}$, $\beta \in (0,1)$, of A endowed with the graph norm $\|x\|_{X_\beta} = \|(-A)^{-\beta}x\|$, $\mathscr{D}$ is the domain of A with the graph norm and $(X,\mathscr{D})_{\beta,\infty}$ is the space introduced in Section 1. We know from Lunardi [24] that $[(-A)^{\beta}] \hookrightarrow (X,\mathscr{D})_{\beta,\infty}$ for all $\beta \in (0,1)$.

As a phase space we select $\mathscr{B} = C_0 \times L^p(\varrho, \mathscr{D})$. Next, K^δ and M^δ are the constants given by $K^\delta = 1 + \big(\int_{-\delta}^0 \varrho(\theta)\,\mathrm{d}\theta\big)^{1/2}$ and $M^\delta = \sup\limits_{s\in[0,\delta]} \sqrt{\gamma(-s)}$, see Example 1.1.

The proof of Proposition 3.1 below follows directly from Theorem 2.1. In this result, we say that a function $u(\cdot)$ is a classical solution of (3.14)–(3.16) if $u(\cdot)$ is a classical solution of the associated abstract system (1.1)–(1.2). In addition, $y(\cdot)$ and $\mathscr{C}^\alpha_{\mathscr{B}}(a)$ are as in Section 2.

Proposition 3.1. *Assume there are* $\alpha \in (0,1)$ *and* $\delta \in [0,a]$ *such that* $y \in \mathscr{C}^\alpha_{\mathscr{B}}(a)$, $\int_0^\infty a_2(s)\varphi(-s) \in X_\alpha$, $A\varphi(0) \in X_\alpha$ *and*

$$K^\delta\Big(M^\delta\Big(2+\frac{1}{\alpha}\Big)+\Big(5+\frac{2}{\alpha}+\frac{4}{\alpha(1-\alpha)}\Big)\Big)\sum_{i=1}^{2}\Big(\int_{-\infty}^{0}\frac{|a_i(-s)|^2}{\varrho(s)}\,\mathrm{d}s\Big)^{1/2} < 1.$$

Then there exists a classical solution $u(\cdot)$ *of* (3.14)–(3.16) *in* $C^\alpha([0,b],\mathscr{D})$ *for some* $0 \leqslant b \leqslant a$.

P r o o f. Let $B(t,s)\colon \mathscr{B} \to \mathscr{D}$ and $C(t,s)\colon \mathscr{B} \to X$ be defined by $B(t,s)\psi = a_1(t-s)\psi$ and $C(t,s)\psi = a_2(t-s)\psi$, and let $G, F\colon [0,a] \to \mathscr{L}(\mathscr{B}, X)$ be the operators given by

$$G(t)\psi = \int_0^\infty B(t,t-s)\psi(-s)\,\mathrm{d}s \quad \text{and} \quad F(t)\psi = \int_0^\infty C(t,t-s)\psi(-s)\,\mathrm{d}s.$$

It is easy to see that the assumptions in Lemmas 2.3 and 2.4 are verified with $L_B = L_C = 0$, $\Lambda_B = (\int_0^\infty (|a_1(-s)|^2/\varrho(s))\,\mathrm{d}s)^{1/2}$ and $\Lambda_C = (\int_0^\infty (|a_2(-s)|^2/\varrho(s))\,\mathrm{d}s)^{1/2}$. Now, from Lemmas 2.3 and 2.4, we obtain that $G \in C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))$, $F \in C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))$, $\|G\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},\mathscr{D}))} \leqslant 2\Lambda_B$ and $\|F\|_{C^\alpha([0,a],\mathscr{L}(\mathscr{B},X))} \leqslant 2\Lambda_C$. Moreover, in this case, the constants Λ_1 and Λ_2 in Lemmas 2.1 and 2.2 are given by $\Lambda_1 = 5 + 2/\alpha + 4/(\alpha(1-\alpha))$ and $\Lambda_2 = 2 + 1/\alpha$.

Finally, an application of Theorem 2.1 guarantees the existence of a unique classical solution of (3.14)–(3.16) in $C^\alpha([0,b],\mathscr{D})$. The proof is now complete. □

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