Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 3, 707–720

Persistent URL: `http://dml.cz/dmlcz/141632`

Czechoslovak Mathematical Journal, 61 (136) (2011), 707–720

SOME REMARKS ON THE AKIVIS ALGEBRAS AND THE PRE-LIE ALGEBRAS

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(Received April 24, 2010)

Abstract. In this paper, by using the Composition-Diamond lemma for non-associative algebras invented by A. I. Shirshov in 1962, we give Gröbner-Shirshov bases for free Pre-Lie algebras and the universal enveloping non-associative algebra of an Akivis algebra, respectively. As applications, we show I. P. Shestakov's result that any Akivis algebra is linear and D. Segal's result that the set of all good words in X^{**} forms a linear basis of the free Pre-Lie algebra $\mathrm{PLie}(X)$ generated by the set X. For completeness, we give the details of the proof of Shirshov's Composition-Diamond lemma for non-associative algebras.

Keywords: non-associative algebra, Akivis algebra, universal enveloping algebra, Pre-Lie algebra, Gröbner-Shirshov basis

MSC 2010: 17A01, 16S15, 13P10

1. Introduction

A. G. Kurosh [11] initiated the study of free non-associative algebras over a field proving that any subalgebra of a free non-associative algebra is free. His student, A. I. Zhukov, proved in [22] that the word problem is algorithmically decidable in the class of non-associative algebras. Namely, he proved that the word problem is decidable for any finitely presented non-associative algebra. A. I. Shirshov, also a student of Kurosh, proved in [16], [20], 1953, that any subalgebra of a free Lie algebra is free. This theorem is now known as the Shirshov-Witt theorem (see, for example, [12]) for it was proved also by E. Witt [21]. Some time later, Shirshov [17], [20] gave a direct construction of a free (anti-) commutative algebra and proved that any subalgebra of such an algebra is again free (anti-) commutative algebra. Almost ten years later,

Supported by the NNSF of China (Nos. 10771077; 10911120389).

Shirshov came back to, we may say, the Kurosh programme, and published two papers [18] and [19]. In the former, he gave a conceptual proof that the word problem is decidable in the class of (anti-) commutative non-associative algebras. Namely, he created the theory that is now known as Gröbner-Shirshov bases theory for (anti-) commutative non-associative algebras. In the latter, he did the same for Lie algebras (explicitly) and associative algebras (implicitly). Their main applications were the decidability of the word problem for any one-relator Lie algebra, the Freiheitsatz (the Freeness theorem) for Lie algebras, and the algorithm for decidability of the word problem for any finitely presented homogeneous Lie algebra. The same algorithm is valid for any finitely presented homogeneous associative algebra as well. Shirshov's main technical discovery in [19], [20] was the notion of the composition of two Lie polynomials and implicitly two associative polynomials. Based on it, he gave the algorithm to construct a Gröbner-Shirshov basis for any ideal of a free Lie algebra. The same algorithm is valid in the associative case. This algorithm is in general infinite as well as, for example, the Knuth-Bendix algorithm [10]. Shirshov proved that if a Gröbner-Shirshov basis of an ideal is recursive, then the word problem for the quotient algebra is decidable. It follows from Shirshov's Composition-Diamond lemma that it is valid for free non-associative, free (anti-) commutative, free Lie and free associative algebras (see [18], [19], [20]). Explicitly the associative case was treated in the papers by L. A. Bokut [3] and G. Bergman [2].

Independently, B. Buchberger in his thesis (1965) (see [7]) created the Gröbner bases theory for the classical case of commutative associative algebras. Also, H. Hironaka in his famous paper [9] did the same for (formal or convergent) infinite series rather than polynomials. He called his bases the standard bases. This term has been used until now as a synonym of Gröbner (in commutative case) or Gröbner-Shirshov (in non-associative and non-commutative cases) bases.

There are a lot of sources of the history of the Gröbner and Gröbner-Shirshov bases theory (see, for example, [8], [4], [5], [6]).

In the present paper we are dealing with the Composition-Diamond lemma for a free non-associative algebra, calling it the non-associative Composition-Diamond lemma. Shirshov mentioned in [18], [20] that all his results are valid for the case of free non-associative algebras rather than free (anti-) commutative algebras. For completeness, we prove this lemma in Section 2 in this paper. Then we apply this lemma to the universal enveloping non-associative algebra of an Akivis algebra and the Pre-Lie algebra to obtain the Gröbner-Shirshov bases for such algebras, respectively. In particular, as applications, we show I. P. Shestakov's result that any Akivis algebra is linear (see [14]) and D. Segal's result that the set of all good words in X^{**} forms a linear basis of the free Pre-Lie algebra $\mathrm{PLie}(X)$ generated by the set X (see [13]).

An Akivis algebra is a vector space V over a field k endowed with a skew-symmetric bilinear product $[x,y]$ and a trilinear product (x,y,z) that satisfy the identity $[[x,y],z]+[[y,z],x]+[[z,x],y]=(x,y,z)+(z,x,y)+(y,z,x)-(x,z,y)-(y,x,z)-(z,y,x)$. These algebras were introduced in 1976 by M. A. Akivis [1] as tangent algebras of local analytic loops. For any (non-associative) algebra B one may obtain an Akivis algebra $\mathrm{Ak}(B)$ by considering in B the usual commutator $[x,y]=xy-yx$ and associator $(x,y,z)=(xy)z-x(yz)$. Let $\{e_i\}_I$ be a linear basis of an Akivis algebra A. Then the nonassociative algebra $U(A)=M(\{e_i\}_I \mid e_ie_j-e_je_i=[e_i,e_j],\ (e_ie_j)e_k-e_i(e_je_k)=(e_i,e_j,e_k),\ i,j,k\in I)$ given by the generators and relations is the universal enveloping non-associative algebra of A, where $[e_i,e_j]=\sum\limits_m \alpha_{ij}^m e_m$, $(e_i,e_j,e_k)=\sum\limits_n \beta_{ijk}^n e_n$ and each $\alpha_{ij}^m,\beta_{ijk}^n\in k$. The linearity of A means that A is a subspace of $U(A)$ (see [14]). Let us remark also that any subalgebra of a free Akivis algebra is again free (see [15]).

A Pre-Lie algebra A over a field k is a non-associative algebra with identity:

$$(x,y,z)=(x,z,y),\quad x,y,z\in A.$$

2. Composition-Diamond lemma for non-associative algebras

Let $X=\{x_i\colon i\in I\}$ be a set, X^* the set of all associative words u in X, and X^{**} the set of all non-associative words (u) in X. Let k be a field and $M(X)$ a k-linear space spanned by X^{**}. We define the product of non-associative words in the following way:

$$(u)(v)=((u)(v)).$$

Then $M(X)$ is a free non-associative algebra generated by X.

Let I be a well-ordered set. We order X^{**} by the induction on the length $|((u)(v))|$ of the words (u) and (v) in X^{**}:

(i) If $|((u)(v))|=2$, then $(u)=x_i>(v)=x_j$ if and only if $i>j$.
(ii) If $|((u)(v))|>2$, then $(u)>(v)$ if and only if one of the following cases holds:
  (a) $|(u)|>|(v)|$.
  (b) If $|(u)|=|(v)|$ and $(u)=((u_1)(u_2))$, $(v)=((v_1)(v_2))$, then $(u_1)>(v_1)$ or $((u_1)=(v_1)$ and $(u_2)>(v_2))$.

It is easy to check that $>$ is a monomial ordering on X^{**} in the following sense:

(a) $>$ is a well ordering.
(b) $(u)>(v)\Longrightarrow(u)(w)>(v)(w)$ and $(w)(u)>(w)(v)$ for any $(w)\in X^{**}$.

Such an ordering is called deg-lex (degree-lexicographical) ordering and we use this ordering throughout this paper.

Given a polynomial $f \in M(X)$, it has the leading word $(\overline{f}) \in X^{**}$ according to the deg-lex ordering on X^{**} such that

$$f = \alpha(\overline{f}) + \sum \alpha_i(u_i),$$

where $(\overline{f}) > (u_i)$, $\alpha, \alpha_i \in k$, $(u_i) \in X^{**}$. We call $(\overline{f})$ the leading term of f. f is called monic if $\alpha = 1$.

Let $S \subset M(X)$ be a set of monic polynomials, $s \in S$ and $(u) \in X^{**}$. We define an S-word $(u)_s$ by induction:

(i) $(s)_s = s$ is an S-word of S-length 1.

(ii) If $(u)_s$ is an S-word of S-length k and (v) is a non-associative word of length l, then

$$(u)_s(v) \quad \text{and} \quad (v)(u)_s$$

are S-words of length $k + l$.

Note that for any S-word $(u)_s = (asb)$, where $a, b \in X^*$, we have $\overline{(asb)} = (a\bar{s}b)$.

Let f, g be monic polynomials in $M(X)$. Suppose that there exist $a, b \in X^*$ such that $(\overline{f}) = (a(\overline{g})b)$. Then we define the composition of inclusion

$$(f, g)_{(\overline{f})} = f - (agb).$$

It is clear that

$$(f, g)_{(\overline{f})} \in \mathrm{Id}(f, g) \quad \text{and} \quad \overline{(f, g)_{(\overline{f})}} < (\overline{f})$$

where $\mathrm{Id}(f, g)$ is the ideal of $M(X)$ generated by f, g.

The composition $(f, g)_{(\overline{f})}$ is trivial modulo $(S, (\overline{f}))$, if

$$(f, g)_{(\overline{f})} = \sum_i \alpha_i(a_i s_i b_i)$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, $(a_i s_i b_i)$ is an S-word and $(a_i(\bar{s_i})b_i) < (\overline{f})$. If this is the case, then we write $(f, g)_{(\overline{f})} \equiv 0 \bmod(S, (\overline{f}))$. In general, for $p, q \in M(X)$ and $(w) \in X^{**}$, we write

$$p \equiv q \bmod(S, (w))$$

which means that $p - q = \sum \alpha_i(a_i s_i b_i)$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, $(a_i s_i b_i)$ is an S-word and $(a_i(\bar{s_i})b_i) < (w)$.

Definition 2.1 ([18], [20]). Let $S \subset M(X)$ be a nonempty set of monic polynomials and let the ordering $>$ be defined as before. Then S is called a Gröbner-Shirshov basis in $M(X)$ if any composition $(f, g)_{(\overline{f})}$ with $f, g \in S$ is trivial modulo $(S, (\overline{f}))$, i.e., $(f, g)_{(\overline{f})} \equiv 0 \bmod(S, (\overline{f}))$.

Lemma 2.2. *Let $(a_1s_1b_1)$, $(a_2s_2b_2)$ be S-words. If S is a Gröbner-Shirshov basis in $M(X)$ and $(w) = (a_1(\overline{s_1})b_1) = (a_2(\overline{s_2})b_2)$, then*

$$(a_1s_1b_1) \equiv (a_2s_2b_2) \bmod(S,(w)).$$

Proof. We have $a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2$ as associative words in the alphabet X. There are two cases to consider.

Case 1. Suppose that subwords $\bar{s}_1$ and $\bar{s}_2$ of w are disjoint, say, $|a_2| \geqslant |a_1| + |\bar{s}_1|$. Then we can assume that

$$a_2 = a_1\bar{s}_1c \quad \text{and} \quad b_1 = c\bar{s}_2b_2$$

for some $c \in X^*$, and so, $w = (a_1(\bar{s}_1)c(\bar{s}_2)b_2)$. Now,

$$\begin{aligned}(a_1s_1b_1) - (a_2s_2b_2) &= (a_1s_1c(\bar{s}_2)b_2) - (a_1(\bar{s}_1)cs_2b_2)\\ &= (a_1s_1c((\bar{s}_2) - s_2)b_2) + (a_1(s_1 - (\bar{s}_1))cs_2b_2).\end{aligned}$$

Since $(\overline{(\overline{s_2}) - s_2}) < (\bar{s}_2)$ and $(\overline{s_1 - (\overline{s_1})}) < (\bar{s}_1)$, we conclude that

$$(a_1s_1b_1) - (a_2s_2b_2) = \sum_i \alpha_i(u_is_1v_i) + \sum_j \beta_j(u_js_2v_j)$$

for some $\alpha_i, \beta_j \in k$, and S-words $(u_is_1v_i)$ and $(u_js_2v_j)$ such that

$$(u_i(\bar{s}_1)v_i), (u_j(\bar{s}_2)v_j) < (w).$$

Thus,

$$(a_1s_1b_1) \equiv (a_2s_2b_2) \bmod(S,(w)).$$

Case 2. Suppose that the subword $\bar{s}_1$ of w contains $\bar{s}_2$ as a subword. We assume that

$$(\bar{s}_1) = (a(\bar{s}_2)b), \quad a_2 = a_1a \quad \text{and} \quad b_2 = bb_1, \quad \text{that is,} \quad (w) = (a_1a(\bar{s}_2)bb_1)$$

for some S-word (as_2b). We have

$$\begin{aligned}(a_1s_1b_1) - (a_2s_2b_2) &= (a_1s_1b_1) - (a_1(as_2b)b_1)\\ &= (a_1(s_1 - (as_2b))b_1)\\ &= (a_1(s_1, s_2)_{(\overline{s_1})}b_1).\end{aligned}$$

Since S is a Gröbner-Shirshov basis, $(s_1, s_2)_{(\overline{s_1})} = \sum\limits_i \alpha_i(c_i s_i d_i)$ for some $\alpha_i \in k$, and S-words $(c_i s_i d_i)$ with each $(c_i(\bar{s}_i)d_i) < (\bar{s_1})$. Then

$$\begin{aligned}(a_1 s_1 b_1) - (a_2 s_2 b_2) &= (a_1 (s_1, s_2)_{(\overline{s_1})} b_1) \\ &= \sum_i \alpha_i (a_1 (c_i s_i d_i) b_1) = \sum_j \beta_j (a_j s_j b_j)\end{aligned}$$

for some $\beta_j \in k$, and S-words $(a_j s_j b_j)$ with each $(a_j(\bar{s}_j)b_j) < (w) = (a_1(\bar{s_1})b_1)$. Thus,

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \bmod (S, (w)).$$

□

Lemma 2.3. *Let $S \subset M(X)$ be a subset of monic polynomials and let $\mathrm{Irr}(S) = \{(u) \in X^{**}\colon\ (u) \neq (a(\bar{s})b),\ a, b \in X^*,\ s \in S$ and (asb) is an S-word$\}$. Then for any $f \in M(X)$,*

$$f = \sum_{(u_i) \leqslant (\overline{f})} \alpha_i(u_i) + \sum_{(a_j(\overline{s_j})b_j) \leqslant (\overline{f})} \beta_j (a_j s_j b_j)$$

where $\alpha_i, \beta_j \in k$, $(u_i) \in \mathrm{Irr}(S)$ and $(a_j s_j b_j)$ is an S-word.

Proof. Let $f = \sum\limits_i \alpha_i (u_i) \in M(X)$, where $0 \neq \alpha_i \in k$ and $(u_1) > (u_2) > \ldots$. If $(u_1) \in \mathrm{Irr}(S)$, then let $f_1 = f - \alpha_1(u_1)$. If $(u_1) \notin \mathrm{Irr}(S)$, then there exist $s \in S$ and $a_1, b_1 \in X^*$ such that $(\overline{f}) = (u_1) = (a_1(\bar{s_1})b_1)$. Let $f_1 = f - \alpha_1(a_1 s_1 b_1)$. In both cases, we have $(\bar{f_1}) < (\overline{f})$. Then the result follows by the induction on $(\overline{f})$. □

The proof of the next theorem is analogous to the one in Shirshov [18]. For convenience, we give the details.

Theorem 2.4 (A. I. Shirshov [18], [20], Composition-Diamond lemma for non-associative algebras)**.** *Let $S \subset M(X)$ be a nonempty set of monic polynomials, $\mathrm{Id}(S)$ the ideal of $M(X)$ generated by S and let the ordering $>$ on X^{**} be defined as before. Then the following statements are equivalent.*

(i) *S is a Gröbner-Shirshov basis in $M(X)$.*

(ii) *$f \in \mathrm{Id}(S) \Rightarrow (\overline{f}) = (a(\bar{s})b)$ for some $s \in S$ and $a, b \in X^*$, where (asb) is an S-word.*

(iii) *$\mathrm{Irr}(S) = \{(u) \in X^{**}\colon\ (u) \neq (a(\bar{s})b),\ a, b \in X^*,\ s \in S$ and (asb) is an S-word$\}$ is a linear basis of the algebra $M(X|S)$.*

P r o o f. (i) $\Rightarrow$ (ii). Let S be a Gröbner-Shirshov basis and $0 \neq f \in \mathrm{Id}(S)$. Then we have

$$f = \sum_{i=1}^{n} \alpha_i(a_i s_i b_i)$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $(a_i s_i b_i)$ is an S-word. Let

$$(w_i) = (a_i(\overline{s_i})b_i), \quad (w_1) = (w_2) = \ldots = (w_l) > (w_{l+1}) \geqslant \ldots$$

We will use the induction on l and (w_1) to prove that $(\overline{f}) = (a(\overline{s})b)$ for some $s \in S$ and $a, b \in X^*$.

If $l = 1$, then $(\overline{f}) = \overline{(a_1 s_1 b_1)} = (a_1(\overline{s_1})b_1)$ and hence the result holds. Assume that $l \geqslant 2$. Then, by Lemma 2.2, we have

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \bmod(S, (w_1)).$$

Thus, if $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$, then the result holds. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use the induction on (w_1). Now, the result follows.

(ii) $\Rightarrow$ (iii). Suppose that $\sum\limits_i \alpha_i(u_i) = 0$ in $M(X|S)$, where $\alpha_i \in k$, $(u_i) \in \mathrm{Irr}(S)$. It means that $\sum\limits_i \alpha_i(u_i) \in \mathrm{Id}(S)$. Then all α_i must be equal to zero. Otherwise, $\overline{\sum\limits_i \alpha_i(u_i)} = (u_j) \in \mathrm{Irr}(S)$ for some j, which contradicts (ii).

Now, by Lemma 2.3, (iii) follows.

(iii) $\Rightarrow$ (i). For any $f, g \in S$, by Lemma 2.3 and (iii), we have $(f,g)_{(\overline{f})} \equiv 0 \bmod(S, (\overline{f}))$. Therefore, S is a Gröbner-Shirshov basis. □

3. Gröbner-Shirshov basis for the universal enveloping algebra of an Akivis algebra

In this section, we obtain a Gröbner-Shirshov basis for the universal enveloping non-associative algebra of an Akivis algebra.

Theorem 3.1. *Let $(A, +, [-,-], (-,-,-))$ be an Akivis algebra over a field k with a well-ordered k-basis $\{e_i : i \in I\}$. Let*

$$[e_i, e_j] = \sum_m \alpha_{ij}^m e_m, \quad (e_i, e_j, e_k) = \sum_n \beta_{ijk}^n e_n,$$

where $\alpha_{ij}^m, \beta_{ijk}^n \in k$. We denote $\sum\limits_m \alpha_{ij}^m e_m$ and $\sum\limits_n \beta_{ijk}^n e_n$ by $\{e_i e_j\}$ and $\{e_i e_j e_k\}$, respectively. Let

$$U(A) = M(\{e_i\}_I \mid e_i e_j - e_j e_i = \{e_i e_j\},\ (e_i e_j)e_k - e_i(e_j e_k) = \{e_i e_j e_k\},\ i, j, k \in I)$$

be the universal enveloping non-associative algebra of A*. Let*

$$\begin{aligned} S = \{f_{ij} &= e_ie_j - e_je_i - \{e_ie_j\} \ (i > j),\\ g_{ijk} &= (e_ie_j)e_k - e_i(e_je_k) - \{e_ie_je_k\} \ (i, j, k \in I),\\ h_{ijk} &= e_i(e_je_k) - e_j(e_ie_k) - \{e_ie_j\}e_k - \{e_je_ie_k\} + \{e_ie_je_k\} \ (i > j,\ k \geqslant j)\}. \end{aligned}$$

Then

(i) S *is a Gröbner-Shirshov basis in* $M(\{e_i\}_I)$*.*

(ii) $\mathrm{Irr}(S) = \{u\colon u \in \{e_i\colon i \in I\}^{**}$ *and* u *does not contain one of the words* e_ie_j $(i > j)$, $(e_ie_j)e_k$ $(i, j, k \in I)$, $e_i(e_je_k)$ $(i > j,\ k \geqslant j)$ *as a subword*$\}$ *is a linear basis of the universal enveloping non-associative algebra* $U(A)$ *of* A*.*

(iii) A *can be embedded into its universal enveloping non-associative algebra* $U(A)$*.*

Proof. (i) It is easy to check that

$$\overline{f_{ij}} = e_ie_j \ (i > j), \quad \overline{g_{ijk}} = (e_ie_j)e_k \ (i, j, k \in I), \quad \overline{h_{ijk}} = e_i(e_je_k) \ (i > j,\ k \geqslant j).$$

So, we have only two kinds of compositions to consider:

$$(g_{ijk}, f_{ij})_{(e_ie_j)e_k} \ (i > j,\ j \leqslant k) \quad \text{and} \quad (g_{ijk}, f_{ij})_{(e_ie_j)e_k} \ (i > j > k).$$

For $(g_{ijk}, f_{ij})_{(e_ie_j)e_k}$ $(i > j, j \leqslant k)$, we have, $\mathrm{mod}(S, (e_ie_j)e_k)$,

$$\begin{aligned} (g_{ijk}, f_{ij})_{(e_ie_j)e_k} &= (e_je_i)e_k - e_i(e_je_k) + \{e_ie_j\}e_k - \{e_ie_je_k\}\\ &\equiv - e_i(e_je_k) + e_j(e_ie_k) + \{e_ie_j\}e_k + \{e_je_ie_k\} - \{e_ie_je_k\}\\ &\equiv 0. \end{aligned}$$

For $(g_{ijk}, f_{ij})_{(e_ie_j)e_k}$ $(i > j > k)$, by noting that, in A,

$$\begin{aligned} &[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j]\\ &\quad = (e_i, e_j, e_k) + (e_k, e_i, e_j) + (e_j, e_k, e_i) - (e_i, e_k, e_j) - (e_j, e_i, e_k) - (e_k, e_j, e_i), \end{aligned}$$

we have, $\mathrm{mod}(S,(e_ie_j)e_k)$,

$$
\begin{aligned}
&(g_{ijk}, f_{ij})_{(e_ie_j)e_k}\\
&\quad= (e_je_i)e_k - e_i(e_je_k) + \{e_ie_j\}e_k - \{e_ie_je_k\}\\
&\quad\equiv\ - e_i(e_je_k) + e_j(e_ie_k) + \{e_ie_j\}e_k + \{e_je_ie_k\} - \{e_ie_je_k\}\\
&\quad\equiv\ - e_i(e_ke_j) - e_i\{e_je_k\} + e_j(e_ie_k) + \{e_ie_j\}e_k + \{e_je_ie_k\} - \{e_ie_je_k\}\\
&\quad\equiv e_j(e_ie_k) - e_k(e_ie_j) - \{e_ie_k\}e_j + \{e_ie_j\}e_k - e_i\{e_je_k\}\\
&\qquad - \{e_ke_ie_j\} + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_ie_je_k\}\\
&\quad\equiv e_j(e_ke_i) + e_j\{e_ie_k\} - e_k(e_je_i) - e_k\{e_ie_j\} - \{e_ie_k\}e_j + \{e_ie_j\}e_k\\
&\qquad - e_i\{e_je_k\} - \{e_ke_ie_j\} + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_ie_je_k\}\\
&\quad\equiv e_k(e_je_i) + \{e_je_k\}e_i + \{e_ke_je_i\} - \{e_je_ke_i\} + e_j\{e_ie_k\}\\
&\qquad - e_k(e_je_i) - e_k\{e_ie_j\} - \{e_ie_k\}e_j + \{e_ie_j\}e_k - e_i\{e_je_k\} - \{e_ke_ie_j\}\\
&\qquad + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_ie_je_k\}\\
&\quad\equiv \{e_je_k\}e_i - e_i\{e_je_k\} + e_j\{e_ie_k\} - \{e_ie_k\}e_j + \{e_ie_j\}e_k - e_k\{e_ie_j\}\\
&\qquad + \{e_ke_je_i\} + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_je_ke_i\} - \{e_ke_ie_j\} - \{e_ie_je_k\}\\
&\quad\equiv \{e_je_k\}e_i - e_i\{e_je_k\} + \{e_ke_i\}e_j - e_j\{e_ke_i\} + \{e_ie_j\}e_k - e_k\{e_ie_j\}\\
&\qquad + \{e_ke_je_i\} + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_je_ke_i\} - \{e_ke_ie_j\} - \{e_ie_je_k\}\\
&\quad\equiv \{\{e_je_k\}e_i\} + \{\{e_ke_i\}e_j\} + \{\{e_ie_j\}e_k\}\\
&\qquad + \{e_ke_je_i\} + \{e_ie_ke_j\} + \{e_je_ie_k\} - \{e_je_ke_i\} - \{e_ke_ie_j\} - \{e_ie_je_k\}\\
&\quad\equiv 0.
\end{aligned}
$$

Thus, S is a Gröbner-Shirshov basis in $M(\{e_i\}_I)$.

(ii) follows from Theorem 2.4.

(iii) follows directly from (ii).

This completes our proof. □

4. Gröbner-Shirshov bases for free Pre-Lie algebras

In this section, we represent the free Pre-Lie algebra by considering the free non-associative algebra and give a Gröbner-Shirshov basis for a free Pre-Lie algebra. As a result, we re-show that the set of all good words in X^{**} forms a linear basis of the free Pre-Lie algebra $\mathrm{PLie}(X)$ generated by the set X (see [13]).

The proof of the next theorem is straightforward and we hence omit the details.

Theorem 4.1. *Let $M(X)$ be the free non-associative algebra generated by X and let*

$$S = \{((u)(v))(w) - (u)((v)(w)) - ((u)(w))(v) + (u)((w)(v))\colon\ (u),(v),(w) \in X^{**} \text{ and } (v) > (w)\}.$$

Then the algebra $M(X|S) = M(X)/\mathrm{Id}(S)$ is the free Pre-Lie algebra generated by X.

We now cite the definition of good words (see [13]) in X^{**} by induction on length:

1) x_i is a good word for any $x_i \in X$.

Suppose that we define good words of length $< n$.

2) A word $((v)(w))$ is called a good word if and only if
   (a) both (v) and (w) are good words,
   (b) if $(v) = ((v_1)(v_2))$, then $(v_2) \leqslant (w)$.

We denote (u) by $[u]$, if (u) is a good word. Let

$$S_0 = \{([u][v])[w] - [u]([v][w]) - ([u][w])[v] + [u]([w][v])\colon\ [u],[v],[w] \text{ are good words and } [v] > [w]\}.$$

Lemma 4.2. *Let W be the set consisting of all good words. Then*

$$\mathrm{Irr}(S_0) = \{(u) \in X^{**}\colon\ (u) \neq (a(\bar{s})b),\ a,b \in X^*,\ s \in S_0 \text{ and } (asb) \text{ is an } s\text{-word}\} = W.$$

P r o o f. Suppose that $(u) \in \mathrm{Irr}(S_0)$. We will show that (u) is a good word by using induction on $|(u)| = n$. If $n = 1$, then $(u) = x_i$ which is already a good word. Let $n > 1$ and $(u) = ((v)(w))$. This case has two subcases. By induction, we see immediately that $(v),(w)$ are both good words.

Subcase 1. If $|(v)| = 1$, then (u) is a good word.

Subcase 2. If $|(v)| > 1$ and $(v) = ((v_1)(v_2))$, then $(v_2) \leqslant (w)$ for $(u) \in \mathrm{Irr}(S_0)$. Hence (u) is a good word.

It is clear that every good word is in $\mathrm{Irr}(S_0)$ since every subword of a good word is still a good word. □

The following lemma follows from Lemmas 2.3 and 4.2.

Lemma 4.3. *In $M(X)$, any word (u) has the presentation*

$$(u) = \sum_i \alpha_i[u_i] + \sum_j \beta_j(a_js_jb_j),$$

where $\alpha_i, \beta_j \in k$, $[u_i]$ are good words, $(a_js_jb_j)$ are S_0-words, $s_j \in S_0$, $[u_i] \leqslant (u)$, $(a_j(\overline{s_j})b_j) \leqslant (u)$. Moreover, each $[u_i]$ has the same length as (u).

Lemma 4.4. *Suppose that S and S_0 are the sets defined as above. Then in $M(X)$, we have*

$$\mathrm{Id}(S) = \mathrm{Id}(S_0).$$

P r o o f. Since S_0 is a subset of S, we only need to prove that $M(X|S_0)$ is a Pre-Lie algebra. In fact, we only need to prove that the following holds in $M(X|S_0)$:

$$((u)(v))(w) - (u)((v)(w)) - ((u)(w))(v) + (u)((w)(v)) = 0$$

where $(u), (v), (w) \in X^{**}$ and $(v) > (w)$. By Lemma 4.3, it suffices to prove that for good words $[u]$, $[v]$, $[w]$ with $[v] > [w]$,

$$([u][v])[w] - [u]([v][w]) - ([u][w])[v] + [u]([w][v]) = 0.$$

This is trivial by the definition of S_0. □

Theorem 4.5. *Let the ordering $>$ be defined as before and*

$$S_0 = \{([u][v])[w] - [u]([v][w]) - ([u][w])[v] + [u]([w][v])\colon\ [v] > [w] \text{ and } [u], [v], [w] \text{ are good words}\}.$$

Then S_0 is a Gröbner-Shirshov basis in $M(X)$.

P r o o f. To simplify our notation, we shall use u for $[u]$ and $u_1u_2\ldots u_n$ for $(((u_1u_2)\ldots)u_n)$. Let

$$f_{uvw} = uvw - u(vw) - uwv + u(wv)$$

where u, v, w are good words and $v > w$. It is easy to check that $\overline{f_{uvw}} = uvw$.

Suppose that $\overline{f_{u_1v_1w_1}}$ is a subword of $\overline{f_{uvw}}$. Since u, v, w are good words, we have $u_1v_1w_1 = uv$, $u = u_1v_1$, $v = w_1$ and $v_1 > w_1 = v > w$. We will prove that the composition $(f_{uvw}, f_{u_1v_1w_1})_{uvw}$ is trivial modulo (S_0, uvw).

First, we prove that the following statements hold $\mathrm{mod}(S_0, uvw)$:

1) $u_1(v_1w)v - u_1(v_1wv) - u_1v(v_1w) + u_1(v(v_1w)) \equiv 0,$
2) $u_1wv_1v - u_1w(v_1v) - u_1wvv_1 + u_1w(vv_1) \equiv 0,$
3) $u_1(wv_1)v - u_1(wv_1v) - u_1v(wv_1) + u_1(v(wv_1)) \equiv 0,$
4) $u_1(v_1v)w - u_1(v_1vw) - u_1w(v_1v) + u_1(w(v_1v)) \equiv 0,$
5) $u_1vv_1w - u_1v(v_1w) - u_1vwv_1 + u_1v(wv_1) \equiv 0,$
6) $u_1(vv_1)w - u_1(vv_1w) - u_1w(vv_1) + u_1(w(vv_1)) \equiv 0,$
7) $u_1(vw)v_1 - u_1(vwv_1) - u_1v_1(vw) + u_1(v_1(vw)) \equiv 0,$
8) $u_1v_1(wv) - u_1(v_1(wv)) - u_1(wv)v_1 + u_1(wvv_1) \equiv 0.$

We only prove 1). 2)–8) can be proved similarly. Denote $g = u_1(v_1w)v - u_1(v_1wv) - u_1v(v_1w) + u_1(v(v_1w))$. By Lemma 4.3, we have

$$v_1w = \sum_i \alpha_i u_i + \sum_j \beta_j(a_js_jb_j)$$

where u_i are good words, $(a_js_jb_j)$ are S_0-words, $s_j \in S_0$, $u_i, (a_j(\overline{s_j})b_j) \leqslant v_1w$. Moreover, each u_i has the same length as v_1w.

By noting that $u_1(a_j\bar{s_j}b_j)v, u_1((a_j\bar{s_j}b_j)v), u_1v(a_j\bar{s_j}b_j), u_1(v(a_j\bar{s_j}b_j)) < uvw$, we have

$$g \equiv \sum_i \alpha_i g_i \bmod(S_0, uvw)$$

where $g_i = u_1u_iv - u_1(u_iv) - u_1vu_i + u_1(vu_i)$. Now $g_i = 0$ or $\bar{g_i} < uvw$ implies that $g_i \equiv 0 \bmod(S_0, uvw)$ and so $g \equiv 0 \bmod(S_0, uvw)$.

Secondly, we have

$$\begin{aligned}(f_{uvw}, f_{u_1v_1w_1})_{uvw} &= f_{uvw} - (f_{u_1v_1w_1})w \\ &= -u_1v_1(vw) - u_1v_1wv + u_1v_1(wv) + u_1(v_1v)w \\ &\quad + u_1vv_1w - u_1(vv_1)w.\end{aligned}$$

Then by 1)–6) we have, $\bmod(S_0, uvw)$,

$$\begin{aligned}-u_1v_1wv \equiv &- u_1(v_1w)v - u_1wv_1v + u_1(wv_1)v \\ \equiv &- u_1((v_1w)v) - u_1v(v_1w) + u_1(v(v_1w)) - u_1w(v_1v) - u_1wvv_1 \\ &+ u_1w(vv_1) + u_1(wv_1v) + u_1v(wv_1) - u_1(v(wv_1)) \\ \equiv &- u_1((v_1w)v) - u_1v(v_1w) + u_1(v(v_1w)) - u_1w(v_1v) \\ &- u_1wvv_1 + u_1w(vv_1) \\ &+ u_1(w(v_1v)) + u_1(wvv_1) - u_1(w(vv_1)) + u_1v(wv_1) - u_1(v(wv_1)),\end{aligned}$$

$$\begin{aligned}u_1(v_1v)w &\equiv u_1(v_1vw) + u_1w(v_1v) - u_1(w(v_1v)) \\ &\equiv u_1(v_1(vw)) + u_1(v_1wv) - u_1(v_1(wv)) + u_1w(v_1v) - u_1(w(v_1v)),\end{aligned}$$

$$\begin{aligned}u_1vv_1w &\equiv u_1v(v_1w) + u_1vwv_1 - u_1v(wv_1) \\ &\equiv u_1v(v_1w) + (u_1(vw))v_1 + u_1wvv_1 - u_1(wv)v_1 - u_1v(wv_1),\end{aligned}$$

$$\begin{aligned}-u_1(vv_1)w \equiv &- u_1(vv_1w) - u_1w(vv_1) + u_1(w(vv_1)) \\ \equiv &- u_1(v(v_1w)) - u_1(vwv_1) + u_1(v(wv_1)) - u_1w(vv_1) \\ &+ u_1(w(vv_1)).\end{aligned}$$

So, by 7)–8) we have, $\mathrm{mod}(S_0, uvw)$,

$$\begin{aligned}(f_{uvw}, f_{u_1v_1w_1})_{uvw} \equiv & -u_1v_1(vw) + u_1(v_1(vw)) + u_1(vw)v_1 - u_1(vwv_1) \\ & + u_1v_1(wv) + u_1(wvv_1) - u_1(v_1(wv)) - u_1(wv)v_1 \equiv 0.\end{aligned}$$

This completes the proof. □

The following corollary follows from Lemmas 4.2, 4.4 and Theorems 4.1, 4.5, 2.4.

Corollary 4.6. *Let* $\mathrm{PLie}(X)$ *be the free Pre-Lie algebra over the field* k *generated by* X*. Then the set of all good words in* X^{**} *is a linear basis of* $\mathrm{PLie}(X)$*.*

Acknowledgement. The authors would like to express their deepest gratitude to Professor L. A. Bokut for his kind guidance, useful discussions and enthusiastic encouragement.

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