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Communications in Mathematics, Vol. 18 (2010), No. 1, 79--82

Persistent URL: `http://dml.cz/dmlcz/141673`

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Classical Mechanics: Hamiltonian and Lagrangian Formalism by Alexei Deriglazov. Springer (2010), 308 pages, ISBN 978-3-642-14036-5, e-ISBN 978-3-642-14037-2.

Many modern books on classical mechanics are coloured by other areas of mathematical or theoretical physics. Quantum mechanics, quantum field theory, continuum mechanics and special and general relativity all exert their influence. On the positive side this means that the fundamental ideas of force, inertia and inertial frames of reference, often neglected by mathematicians, are all thoroughly explored. On the negative side it can lead to undue dependence on the historical development of the parts of the subject which the author doesn't favour. For example, in V.I. Arnold's famous *Mathematical Methods of Classical Mechanics* differential forms are not introduced until after Lagrangian dynamics is treated. Although the title of the work under review indicates a study of both the post Newtonian formalisms in mechanics the author makes it clear in the introduction that he prefers the Hamiltonian framework, not least because of its role in quantum theory. As a result Lagrangian dynamics is unfavourably compared to Hamiltonian mechanics and much of its modern formulation is untouched. In writing this review I will try and indicate some of the current trends in the Lagrangian theory.

So classical mechanics is one of those areas having multiple ownership. This can be productive because the subject has inputs from many areas which should stimulate cross fertilisation. On the other hand it has inhibited mathematicians from developing the subject as their own. We have all had the experience of learning classical mechanics as a stand alone subject with many idiosyncratic methods, not bearing any resemblance to subjects like linear algebra which have an axiomatic basis and a body of theorems applicable to a wide range of situations. Our undergraduate experience of the subject constrains us from seeing it as an area in which the beautiful theory of ordinary differential equations due to Lie, Cartan and others applies. And of course the multiple ownership of the subject will forever prevent us from teaching it as such. Nonetheless, we should at least attempt to see the differential equations aspect of classical mechanics in this light.

This book has been developed from lectures aimed at graduate students in theoretical physics, but implicitly at those with an interest in quantum theory. This may be the reason for the very limited use of differential geometric ideas, especially those of exterior calculus and the theory of connections. These two components of the calculus of manifolds are central to many of the 20^{th} century developments in Lagrangian mechanics. The reader can get a sense of the current situation (albeit that of the reviewer and this journal's editor-in-chief) from the chapter *Second Order Ordinary Differential Equations in Jet Bundles and the Inverse Problem of the Calculus of Variations* in *Handbook of Global Analysis*, edited by D. Krupka and D. Saunders, Elsevier (2007). I will describe two of these developments: Noether's theorem and progress in the inverse problem in the calculus of variations. References can be found in the aforementioned article.

The central object in the modern Lagrangian theory in one independent variable, t, and n dependent ones, x^a, is the *Cartan two-form*. For a given non-degenerate Lagrangian this form is

$$d\theta_L = d\left(Ldt + \frac{\partial L}{\partial u^a}(dx^a - u^a dt)\right) = \frac{\partial^2 L}{\partial u^a u^b}(du^a - f^a dt) \wedge (dx^b - u^b dt).$$

Here (t, x^a, u^a) are local co-ordinates on the evolution space, $E := \mathbb{R} \times TM$, M being the configuration manifold, and the Euler-Lagrange equations in normal form are

$$\ddot{x}^a = f^a(t, x^b, u^b).$$

Of course this two-form has a geometric definition which can be found in the literature, but its most important intrinsic property is that it has a one-dimensional kernel spanned by the semi spray, known as the Euler-Lagrange field,

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial u^a}.$$

Noether's theorem in this setting relates a non-trivial symmetry, $X \in \mathfrak{X}(E)$, of $d\theta_L$ to a non-trivial first integral, F, of Γ (so that $\Gamma(F) = 0$):

$$\mathcal{L}_X d\theta_L = 0 \iff X \lrcorner\, d\theta_L = dF.$$

This relation between X and F fixes X up to a multiple of Γ giving a converse to Noether's theorem which is not available if we restrict ourselves to so-called point symmetries. This remarkably simple approach to the famous theorem should be contrasted to the lengthy account given in Deriglazov's book in which the converse to the theorem is discovered in the Hamiltonian context and pulled back to the Lagrangian picture by the Legendre transformation without reference to the point symmetry issue.

The inverse problem in the calculus of variations is the problem of identifying all, if any, Lagrangians whose Euler-Lagrange field is a given semi spray Γ. The Fields medallist Jesse Douglas solved this problem for $n = 2$ in 1941. While special cases have been solved for arbitrary n the solution for $n = 3$, for example, has not yet been produced. Douglas himself, undoubtedly a modest man, said "*the problem ...... is one of the most important hitherto unsolved problems of the calculus of variations*". Apart from its intrinsic value this problem has given birth to deep results on second order differential equations. The question is of interest to physicists because if a problem admits more than one Lagrangian it may admit more than one quantisation, not all of which are equivalent.

The theorem which geometrises the *Helmholtz conditions* (due to Douglas) is

Theorem 1. *Given a semi spray, Γ, necessary and sufficient conditions for the existence of a regular Lagrangian, whose Euler-Lagrange field is Γ, are that there exists $\Omega \in \bigwedge^2(E)$:*

1. *Ω has maximal rank*
2. *$\Omega(V_1, V_2) = 0, \ \forall \ V_1, V_2 \in V(E)$, the vertical sub-bundle on E*
3. *$\Gamma \lrcorner \, \Omega = 0$*
4. *$d\Omega = 0$*

The Lagrangians are recovered from the observation that Ω is a Cartan two-form $d\theta_L$. The inverse problem is not touched upon in the book under review.

This is a comprehensive book in its own way. The chapter headings are: 1. Sketch of Lagrangian Formalism, 2. Hamiltonian Formalism, 3. Canonical Transformations of Two-Dimensional Phase Space, 4. Properties of Canonical Transformations, 5. Integral Invariants, 6. Potential Motion in a Geometric Setting, 7. Transformations, Symmetries and Noether Theorem, 8. Hamiltonian Formalism for Singular Theories. However, and as indicated earlier, the mathematical setting is not modern and the influences lie in theoretical physics outside classical mechanics. For example, of the 50 references only 10 are post the year 2000 and of those 7 are works of the author and the other 3 lie outside classical mechanics. Nonetheless, it provides interesting reading and the detailed level of discussion reflects the extensive nature of the graduate lecture course on which the book is based. For example, the discussion of Dirac's theory of constraints in chapter 8 is quite deep and provides a natural end point of the author's development of the Hamiltonian and Lagrangian frameworks. Special relativity and quantum mechanics are both represented through the examples and the quasi-Riemannian geometric formulation is developed in chapter 6 in the context of the Principle of Maupertius. Rather surprisingly this formulation of the Newtonian equations of motion as quasi-geodesic equations is developed without reference to general relativity or to Cartan's formulation of the Newtonian equations as the auto-parallel equations of a non-metric affine connection.

There are informative and serious exercises scattered throughout the text although not as many as one would find in an undergraduate text on the subject. However, I believe that the approach here is too idiosyncratic for the book to be widely accepted as a basis for an advanced course on classical mechanics. The situation is not improved by the personal mathematical style adopted, albeit consistently, by the author for dealing with the co-ordinate transformations which abound in classical mechanics along with the use of a non-standard mathematical vocabulary.

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