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SOLVABILITY OF A HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM AT RESONANCE*

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Abstract. Based on the coincidence degree theory of Mawhin, we get a new general existence result for the following higher-order multi-point boundary value problem at resonance

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad t \in (0,1),$$
$$x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad x'(0) = \ldots = x^{(n-2)}(0) = 0, \quad x^{(n-1)}(1) = \sum_{j=1}^{l} \beta_j x^{(n-1)}(\eta_j),$$

where $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $m \geqslant 2$ and $0 < \eta_1 < \ldots < \eta_l < 1$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, l$, $l \geqslant 1$. In this paper, two of the boundary value conditions are responsible for resonance.

Keywords: multi-point boundary value problem, coincidence degree theory, resonance

MSC 2010: 34B15

1. Introduction

In this paper, we are concerned with the following higher-order ordinary differential equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \quad t \in (0,1), \tag{1.1}$$

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subject to the following boundary value conditions

$$
\begin{aligned}
x(0) &= \sum_{i=1}^{m} \alpha_i x(\xi_i), \\
x'(0) &= \ldots = x^{(n-2)}(0) = 0, \\
x^{(n-1)}(1) &= \sum_{j=1}^{l} \beta_j x^{(n-1)}(\eta_j),
\end{aligned}
\tag{1.2}
$$

where $f\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $m \geqslant 2$ and $0 < \eta_1 < \ldots < \eta_l < 1$, $\beta_j \in \mathbb{R}$, $j = 1, \ldots, l$, $l \geqslant 1$.

We say that the boundary value problem (BVP for short) (1.1) and (1.2) is a resonance problem if the linear equation $Lx = x^{(n)}(t) = 0$, $t \in (0,1)$, with the boundary value conditions (1.2) has a non-trivial solution. Otherwise we call them a problem at non-resonance.

In recent years, there have been many works related to the existence of solutions for lower-order multi-point boundary value problems at resonance in the case of $\dim \operatorname{Ker} L = 1$. We refer the readers to [3], [4], [6], and the references therein. The case of $\dim \operatorname{Ker} L = 1$ for higher-order multi-point boundary value problems at resonance is mainly discussed (see [1], [2], [8]).

Recently, Kosmatov [5], Liu and Zhao [7], Meng and Du [10], Zhang et al. [12] studied the existence of solutions for some second-order multi-point boundary value problems at resonance in the case of $\dim \operatorname{Ker} L = 2$. Xue et al. [11] studied the existence of solutions for some third-order multi-point boundary value problems at resonance in the case of $\dim \operatorname{Ker} L = 2$. However, few works exist for higher-order multi-point boundary value problems at resonance in the case of $\dim \operatorname{Ker} L = 2$.

Inspired by the above mentioned papers, the goal of this paper is to study the existence of solutions for BVP (1.1) and (1.2) at resonance in the case $\dim \operatorname{Ker} L = 2$ by applying the coincidence degree theory.

The layout of this paper is as follows. In Section 2, we briefly present some notation and an abstract existence result which is due to Mawhin. In Section 3, we obtain a general existence result for BVP (1.1) and (1.2) which is marked as Theorem 3.1. Theorem 3.2 is a modification of Theorem 3.1. In Section 4, an example is given to illustrate our main results.

2. Preliminaries

Now, we briefly recall some notation and an abstract existence result due to Mawhin [9].

Let Y, Z be two real Banach spaces and $L\colon \operatorname{dom} L \subset Y \to Z$ a linear operator which is a Fredholm map of index zero. Let $P\colon Y \to Y$, $Q\colon Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}\colon \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible and we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N\colon Y \to Z$ is said to be L-compact on $\bar{\Omega}$ if the map $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N\colon \bar{\Omega} \to Y$ is compact. For more details we refer the readers to the lecture notes of Mawhin [9].

The theorem we use in this paper is Theorem IV.13 of [9].

Theorem 2.1. *Let L be a Fredholm map of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.

(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$.

(iii) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *here* $Q\colon Z \to Z$ *is a projector with* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the abstract equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

In the following, we shall use the classical spaces $C[0,1], C^1[0,1], C^2[0,1], \ldots, C^{n-1}[0,1]$, and $L^1[0,1]$. For $x \in C^{n-1}[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(n-1)}\|_\infty\},$$

and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We will use the Sobolev space $W^{n,1}(0,1)$ which may be defined by

$$W^{n,1}(0,1) = \{x\colon [0,1] \to \mathbb{R}\colon x, x', \ldots, x^{(n-1)} \text{ are absolutely continuous on } [0,1] \text{ with } x^{(n)} \in L^1[0,1]\}.$$

Throughout this paper, we will use the following resonance conditions:

$$\text{(RC)} \qquad \sum_{i=1}^{m} \alpha_i = 1, \quad \sum_{i=1}^{m} \alpha_i \xi_i^{n-1} = 0, \quad \sum_{j=1}^{l} \beta_j = 1.$$

Let $Y = C^{(n-1)}[0,1]$, $Z = L^1[0,1]$, L is the linear operator from $\operatorname{dom} L \subset Y$ to Z with

$$\operatorname{dom} L = \{x \in W^{n,1}(0,1)\colon x(t) \text{ satisfies the boundary value conditions (1.2)}\}$$

and $Lx = x^{(n)}$, $x \in \operatorname{dom} L$.

We define $N\colon Y \to Z$ by setting

$$Nx = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad t \in (0,1),$$

then BVP (1.1) and (1.2) can be written as $Lx = Nx$.

3. Main results

In this section, first of all we show a general fact as follows:

Lemma 3.1. *If* $\sum\limits_{i=1}^{m} \alpha_i = \sum\limits_{j=1}^{l} \beta_j = 1$, *then there exist* $p \in \{1, \dots, l\}$, $q \in \mathbb{Z}^+$, $q \geqslant p+1$, *such that*

$$\Lambda(p,q) = (q+1)(q+2)\dots(q+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{p+n-1}\Bigl(1 - \sum_{j=1}^{l} \beta_j \eta_j^q\Bigr) - (p+1)(p+2)\dots(p+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{q+n-1}\Bigl(1 - \sum_{j=1}^{l} \beta_j \eta_j^p\Bigr) \neq 0.$$

Proof. We can easily obtain that there exists $p \in \{1, \dots, l\}$ such that $\sum\limits_{j=1}^{l} \beta_j \eta_j^p \neq 1$. Indeed, if

$$\sum_{j=1}^{l} \beta_j \eta_j^p = 1, \quad p \in \{0, 1, \dots, l\},$$

then

$$\sum_{j=1}^{l} \beta_j \eta_j^i (1 - \eta_j) = 0, \quad i \in \{0, \dots, l-1\}.$$

This is equivalent to the following matrix equation

$$\begin{pmatrix} 1-\eta_1 & \dots & 1-\eta_l \\ \vdots & \ddots & \vdots \\ \eta_1^{l-1}(1-\eta_1) & \dots & \eta_l^{l-1}(1-\eta_l) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_l \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

However,

$$\begin{vmatrix} 1-\eta_1 & \dots & 1-\eta_l \\ \vdots & \ddots & \vdots \\ \eta_1^{l-1}(1-\eta_1) & \dots & \eta_l^{l-1}(1-\eta_l) \end{vmatrix} = \prod_{j=1}^{l}(1-\eta_j) \prod_{1 \leqslant i < j \leqslant l} (\eta_j - \eta_i) \neq 0,$$

thus $\beta_j = 0$, $j = 1, \dots, l$, which is a contradiction to $\sum\limits_{j=1}^{l} \beta_j = 1$. Similarly, for every $s \in \mathbb{Z}$, $s \geqslant 0$, there exists $k_s \in \{sm, sm+1, \dots, (s+1)m-1\}$, such that $\sum\limits_{i=1}^{m} \alpha_i \xi_i^{k_s+n-1} \neq 0$.

Set

$$S = \left\{ k_s \in \mathbb{Z}^+ \colon 1 - \sum_{j=1}^{l} \beta_j \eta_j^p = \frac{(k_s+1)\dots(k_s+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{p+n-1}\big(1 - \sum_{j=1}^{l} \beta_j \eta_j^{k_s}\big)}{(p+1)\dots(p+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{k_l+n-1}} \right\},$$

we shall prove that S is a finite set. If not, there exists a monotone sequence $\{k_{s_r}\}$, $r = 1, 2, \dots, k_{s_r} < k_{s_{r+1}}$, such that

$$1 - \sum_{j=1}^{l} \beta_j \eta_j^p = \frac{(k_{s_r}+1)\dots(k_{s_r}+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{p+n-1}\big(1 - \sum_{j=1}^{l} \beta_j \eta_j^{k_{s_r}}\big)}{(p+1)\dots(p+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{k_{s_r}+n-1}}.$$

From $\sum\limits_{j=1}^{l} \beta_j \eta_j^p \neq 1$, we obtain $\sum\limits_{i=1}^{m} \alpha_i \xi_i^{p+n-1} \neq 0$, thus

$$1 - \sum_{j=1}^{l} \beta_j \eta_j^p = \lim_{k_{s_r}\to\infty} \frac{(k_{s_r}+1)\dots(k_{s_r}+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{p+n-1}\big(1 - \sum_{j=1}^{l} \beta_j \eta_j^{k_{s_r}}\big)}{(p+1)\dots(p+n-1)\sum_{i=1}^{m} \alpha_i \xi_i^{k_{s_r}+n-1}} = \infty,$$

a contradiction. Thus there exist $p \in \{1, \dots, l\}$, $q \in \mathbb{Z}^+$, $q \geqslant p+1$, such that $\Lambda(p,q) \neq 0$. □

Lemma 3.2. *Let the condition* (RC) *hold, then* $L\colon \operatorname{dom} L \subset Y \to Z$ *is a Fredholm map of index zero. Furthermore, the linear continuous projector operator* $Q\colon Z \to Z$ *can be defined by*

$$Qy(t) = (T_1 y(t)) \cdot t^{p-1} + (T_2 y(t)) \cdot t^{q-1},$$

where

$$T_1 y = \frac{p\dots(p+n-1)}{\Lambda(p,q)} \left[(q+1)\dots(q+n-1)\left(1 - \sum_{j=1}^{l} \beta_j \eta_j^q\right) Q_1 y - \sum_{i=1}^{m} \alpha_i \xi_i^{q+n-1} Q_2 y \right],$$

$$T_2y = -\frac{q\dots(q+n-1)}{\Lambda(p,q)}\Big[(p+1)\dots(p+n-1)\Big(1-\sum_{j=1}^{l}\beta_j\eta_j^p\Big)Q_1y$$
$$-\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}Q_2y\Big],$$
$$Q_1y = \sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^{\tau_n}\dots\int_0^{\tau_2}y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\dots\mathrm{d}\tau_n,$$
$$Q_2y = \int_0^1 y(\tau_1)\,\mathrm{d}\tau_1 - \sum_{j=1}^{l}\beta_j\int_0^{\eta_j}y(\tau_1)\,\mathrm{d}\tau_1.$$

And the linear operator $K_P\colon \operatorname{Im}L \to \operatorname{dom}L\cap\operatorname{Ker}P$ *can be written as*

$$K_Py(t) = \int_0^t\int_0^{\tau_n}\dots\int_0^{\tau_2}y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\dots\mathrm{d}\tau_n,\quad y\in\operatorname{Im}L.$$

Furthermore,

$$\|K_Py\| \leqslant \|y\|_1,\quad y\in\operatorname{Im}L.$$

Proof. It is clear that $\operatorname{Ker}L = \{x\in\operatorname{dom}L\colon\ x = a+bt^{n-1},\ a,b\in\mathbb{R}\}$. Now we show that

$$\operatorname{Im}L = \{y\in Z\colon\ Q_1y = Q_2y = 0\}, \tag{3.1}$$

since the problem

$$x^{(n)}(t) = y(t) \tag{3.2}$$

has a solution $x(t)$ satisfying (1.2) if and only if

$$Q_1y = Q_2y = 0. \tag{3.3}$$

In fact, if (3.2) has a solution $x(t)$ satisfying (1.2), then from (3.2) we have

$$x(t) = x(0)+x'(0)t+\dots+\frac{1}{(n-1)!}x^{(n-1)}(0)t^{n-1}$$
$$+\int_0^t\int_0^{\tau_n}\dots\int_0^{\tau_2}y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\dots\mathrm{d}\tau_n.$$

According to the boundary value conditions (1.2) and the condition (RC), we obtain

$$Q_1y = Q_2y = 0.$$

On the other hand, if (3.3) holds, we set

$$x(t)=a+bt^{n-1}+\int_0^t\int_0^{\tau_n}\dots\int_0^{\tau_2}y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\dots\,\mathrm{d}\tau_n,$$

where a, b are arbitrary constants, then $x(t)$ is a solution of (3.2) and (1.2). Hence, (3.1) holds.

From Lemma 3.1, there exist $p\in\{1,\dots,l\}$, $q\in\mathbb{Z}^+$, $q\geqslant p+1$, such that $\Lambda(p,q)\neq 0$. Setting

$$\begin{aligned}T_1y=&\frac{p\dots(p+n-1)}{\Lambda(p,q)}\biggl[(q+1)\dots(q+n-1)\Bigl(1-\sum_{j=1}^{l}\beta_j\eta_j^q\Bigr)Q_1y\\&-\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}Q_2y\biggr],\\T_2y=&-\frac{q\dots(q+n-1)}{\Lambda(p,q)}\biggl[(p+1)\dots(p+n-1)\Bigl(1-\sum_{j=1}^{l}\beta_j\eta_j^p\Bigr)Q_1y\\&-\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}Q_2y\biggr],\end{aligned}$$

we then define

$$Qy(t)=(T_1y(t))\cdot t^{p-1}+(T_2y(t))\cdot t^{q-1},$$

and it is obvious that $\dim\operatorname{Im}Q=2$.

Again from

$$\begin{aligned}&T_1((T_1y)t^{p-1})\\&\quad=\frac{p\dots(p+n-1)}{\Lambda(p,q)}\biggl[(q+1)\dots(q+n-1)\Bigl(1-\sum_{j=1}^{l}\beta_j\eta_j^q\Bigr)Q_1((T_1y)t^{p-1})\\&\quad-\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}Q_2((T_1y)t^{p-1})\biggr]\\&\quad=\frac{1}{\Lambda(p,q)}\biggl[(q+1)\dots(q+n-1)\Bigl(1-\sum_{j=1}^{l}\beta_j\eta_j^q\Bigr)\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}\\&\quad-(p+1)\dots(p+n-1)\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\Bigl(1-\sum_{j=1}^{l}\beta_j\eta_j^p\Bigr)\biggr](T_1y)=T_1y,\end{aligned}$$

$$
\begin{aligned}
&T_1((T_2y)t^{q-1})\\
&\quad=\frac{p\ldots(p+n-1)}{\Lambda(p,q)}\bigg[(q+1)\ldots(q+n-1)\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{q}\bigg)Q_1((T_2y)t^{q-1})\\
&\qquad-\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}Q_2((T_2y)t^{q-1})\bigg]\\
&\quad=\frac{1}{\Lambda(p,q)}\bigg[\frac{p\ldots(p+n-1)}{q}\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{q}\bigg)\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\\
&\qquad-\frac{p\ldots(p+n-1)}{q}\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{q}\bigg)\bigg](T_2y)=0,\\
&T_2((T_1y)t^{p-1})\\
&\quad=-\frac{q\ldots(q+n-1)}{\Lambda(p,q)}\bigg[(p+1)\ldots(p+n-1)\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\bigg)Q_1((T_1y)t^{p-1})\\
&\qquad-\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}Q_2((T_1y)t^{p-1})\bigg]\\
&\quad=-\frac{1}{\Lambda(p,q)}\bigg[\frac{q\ldots(q+n-1)}{p}\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\bigg)\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}\\
&\qquad-\frac{q\ldots(q+n-1)}{p}\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\bigg)\bigg](T_1y)=0,\\
&T_2((T_2y)t^{q-1})\\
&\quad=-\frac{q\ldots(q+n-1)}{\Lambda(p,q)}\bigg[(p+1)\ldots(p+n-1)\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\bigg)Q_1((T_2y)t^{q-1})\\
&\qquad-\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}Q_2((T_2y)t^{q-1})\bigg]\\
&\quad=-\frac{1}{\Lambda(p,q)}\bigg[(p+1)\ldots(p+n-1)\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\bigg)\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\\
&\qquad-(q+1)\ldots(q+n-1)\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}\bigg(1-\sum_{j=1}^{l}\beta_j\eta_j^{q}\bigg)\bigg](T_2y)=T_2y,
\end{aligned}
$$

we have

$$
\begin{aligned}
Q^2y&=Q((T_1y)\cdot t^{p-1}+(T_2y)\cdot t^{q-1})\\
&=T_1((T_1y)\cdot t^{p-1}+(T_2y)\cdot t^{q-1})\cdot t^{p-1}+T_2((T_1y)\cdot t^{p-1}+(T_2y)\cdot t^{q-1})\cdot t^{q-1}
\end{aligned}
$$

$$
\begin{aligned}
&= T_1((T_1y)\cdot t^{p-1})\cdot t^{p-1} + T_1((T_2y)\cdot t^{q-1})\cdot t^{p-1} + T_2((T_1y)\cdot t^{p-1})\cdot t^{q-1}\\
&\quad + T_2((T_2y)\cdot t^{q-1})\cdot t^{q-1}\\
&= (T_1y)\cdot t^{p-1} + (T_2y)\cdot t^{q-1} = Qy,
\end{aligned}
$$

which implies that the operator Q is a projector.

Now we will show that $\operatorname{Ker} Q = \operatorname{Im} L$. If $y \in \operatorname{Ker} Q$, from $Qy = 0$ we have

$$
\begin{cases}
(q+1)\dots(q+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big)Q_1y - \sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1}Q_2y = 0,\\
(p+1)\dots(p+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big)Q_1y - \sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+n-1}Q_2y = 0.
\end{cases}
$$

Since

$$
\begin{vmatrix}
(q+1)\dots(q+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big) & -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\\
(p+1)\dots(p+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big) & -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+n-1}
\end{vmatrix} = -\Lambda(p,q) \neq 0,
$$

then we find that $Q_1y = Q_2y = 0$, which yields $y \in \operatorname{Im} L$. On the other hand, if $y \in \operatorname{Im} L$, from $Q_1y = Q_2y = 0$ and the definition of Q, it is obvious that $Qy = 0$, thus $y \in \operatorname{Ker} Q$. Hence, $\operatorname{Ker} Q = \operatorname{Im} L$.

For $y \in Z$, from $y = (y - Qy) + Qy$, $y - Qy \in \operatorname{Ker} Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$, we have $Z = \operatorname{Im} L + \operatorname{Im} Q$. And for any $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, from $y \in \operatorname{Im} Q$, there exist constants $a, b \in \mathbb{R}$, such that $y(t) = at^{p-1} + bt^{q-1}$. From $y \in \operatorname{Im} L$, we obtain

$$
\begin{cases}
q\dots(q+n-1)\sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+1}\cdot a + p\dots(p+n-1)\sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\cdot b = 0,\\
\dfrac{1}{p}\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big)\cdot a + \dfrac{1}{q}\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big)\cdot b = 0.
\end{cases}
\tag{3.4}
$$

In view of

$$
\begin{vmatrix}
q\dots(q+n-1)\sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+n-1} & p\dots(p+n-1)\sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\\
\dfrac{1}{p}\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big) & \dfrac{1}{q}\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big)
\end{vmatrix} = \Lambda(p,q) \neq 0,
$$

the equation (3.4) has a unique solution $a = b = 0$, which implies $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Since $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2$, we find that L is a Fredholm map of index zero.

Let $P\colon Y \to Y$ be defined by

$$Px(t) = x(0) + \frac{1}{(n-1)!}\, x^{(n-1)}(0)t^{n-1}, \quad t \in [0,1].$$

Then, the generalized inverse $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L can be written by

$$K_P y(t) = \int_0^t \int_0^{\tau_n} \dots \int_0^{\tau_2} y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \dots \mathrm{d}\tau_n, \quad y \in \operatorname{Im} L.$$

In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_P)y(t) = (K_P y(t))^{(n)} = y(t),$$

and for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we know that

$$\begin{aligned}(K_P L)x(t) &= (K_P)x^{(n)}(t)\\ &= \int_0^t \int_0^{\tau_n} \dots \int_0^{\tau_2} x^{(n)}(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \dots \mathrm{d}\tau_n\\ &= x(t) - \Big[x(0) + x'(0)t + \ldots + \frac{1}{(n-1)!}\, x^{(n-1)}(0)t^{n-1}\Big]\\ &= x(t) - Px(t).\end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $x'(0) = \ldots = x^{(n-2)}(0) = 0$, $Px(t) = 0$, thus

$$(K_P L)x(t) = x(t).$$

Again from the definition of K_P, we have

$$\|K_P y\|_\infty \leqslant \int_0^1 \dots \int_0^1 |y(\tau_1)|\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_n = \|y\|_1,$$

and from

$$\begin{aligned}(K_P y)'(t) &= \int_0^t \int_0^{\tau_{n-1}} \dots \int_0^{\tau_2} y(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \dots \mathrm{d}\tau_{n-1},\\ &\ \vdots\\ (K_P y)^{(n-1)}(t) &= \int_0^t y(\tau_1)\,\mathrm{d}\tau_1,\end{aligned}$$

we obtain

$$\|(K_P y)'\| \leqslant \|y\|_1, \ldots, \|(K_P y)^{(n-1)}\| \leqslant \|y\|_1,$$

that is, $\|(K_P y)\| \leqslant \|y\|_1$. This completes the proof of Lemma 3.1. □

Theorem 3.1. *Let the condition* (RC) *hold. Assume that*

(H_1) *There exist functions* $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t), \gamma(t) \in L^1[0,1]$, *such that for all* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and* $t \in [0,1]$,

$$|f(t, x_1, x_2, \ldots, x_n)| \leqslant \alpha_1(t)|x_1| + \alpha_2(t)|x_2| + \ldots + \alpha_n(t)|x_n| + \gamma(t).$$

(H_2) *There exists a constant* $A > 0$ *such that for* $x \in \operatorname{dom} L$, *if* $|x(t)| > A$ *or* $|x^{(n-1)}(t)| > A$ *for all* $t \in [0,1]$, *then*

$$Q_1 N(x(t)) \neq 0 \quad \textit{or} \quad Q_2 N(x(t)) \neq 0.$$

(H_3) *There exists a constant* $B > 0$ *such that for* $a, b \in \mathbb{R}$, *if* $|a| > B$ *or* $|b| > B$, *then either*

$$Q_1 N(a + bt^{n-1}) + Q_2 N(a + bt^{n-1}) > 0, \tag{3.5}$$

or

$$Q_1 N(a + bt^{n-1}) + Q_2 N(a + bt^{n-1}) < 0. \tag{3.6}$$

Then BVP (1.1) *and* (1.2) *has at least one solution in* $C^{n-1}[0,1]$, *provided that*

$$\sum_{i=1}^{n} \|\alpha_i\|_1 < 1.$$

P r o o f. Set

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L \colon \ Lx = \lambda N x \text{ for some } \lambda \in [0,1]\}.$$

For $x \in \Omega_1$, since $Lx = \lambda N x$, so $\lambda \neq 0$, $Nx \in \operatorname{Im} L$, hence

$$Q_1 N(x(t)) = 0 \quad \text{and} \quad Q_2 N(x(t)) = 0.$$

Thus, from (H_2), there exist $t_0, t_1 \in [0,1]$ such that $|x(t_0)| \leqslant A$, $|x^{(n-1)}(t_1)| \leqslant A$. Since x, $x^{(n-1)}$ are absolutely continuous for all $t \in [0,1]$,

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\,\mathrm{d}s, \quad x^{(n-1)}(t) = x^{(n-1)}(t_1) + \int_{t_1}^{t} x^{(n)}(s)\,\mathrm{d}s.$$

Since $x'(0) = \ldots = x^{(n-2)}(0) = 0$, we have $\|x'\|_\infty \leqslant \ldots \leqslant \|x^{(n-1)}\|_\infty$,

$$\|x\|_\infty \leqslant A + \|x'\|_\infty, \quad \|x^{(n-1)}\|_\infty \leqslant A + \|x^{(n)}\|_1.$$

From (H_1), we obtain

$$\begin{aligned}\|x^{(n)}\|_1 = \|Lx\|_1 \leqslant \|Nx\|_1 &\leqslant \sum_{i=1}^{n}\|\alpha_i\|_1\|x^{(i-1)}\|_\infty + \|\gamma\|_1\\ &\leqslant \left(\sum_{i=1}^{n}\|\alpha_i\|_1\right)\|x^{(n)}\|_1 + A\left(2\|\alpha_1\|_1 + \sum_{i=2}^{n}\|\alpha_i\|_1\right) + \|\gamma\|_1\\ &\leqslant \frac{1}{1-\sum_{i=1}^{n}\|\alpha_i\|_1}\left[A\left(2\|\alpha_1\|_1 + \sum_{i=2}^{n}\|\alpha_i\|_1\right) + \|\gamma\|_1\right],\end{aligned}$$

so there exists a constant $M_1 > 0$ such that $\|x\| \leqslant M_1$. Therefore, Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \operatorname{Ker} L\colon\ Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, $x \in \operatorname{Ker} L$ implies that x can be defined by $x = a + bt^{n-1}$, $t \in [0,1]$, a, b are arbitrary constants. Since $QNx = 0$, $Q_1N(a+bt^{n-1}) = Q_2N(a+bt^{n-1}) = 0$. It follows from ($H_3$) that $\|x\| \leqslant |a| + |b| \leqslant 2B$. So Ω_2 is bounded.

From Lemma 3.1, there exist $p \in \{1,\dots,l\}$, $q \in \mathbb{Z}^+$, $q \geqslant p+1$, such that $\Lambda(p,q) \neq 0$. For any $a, b \in \mathbb{R}$, define the linear isomorphism $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ by

$$J(a+bt^{n-1}) = \frac{1}{\Lambda(p,q)}(a'\cdot t^{p-1} + b'\cdot t^{q-1}),$$

where

$$\begin{aligned}a' &= p\dots(p+n-1)\\ &\quad\times\left[(q+1)\dots(q+n-1)\left(1-\sum_{j=1}^{l}\beta_j\eta_j^{q}\right)|a| - |b|\sum_{i=1}^{m}\alpha_i\xi_i^{q+n-1}\right],\\ b' &= -q\dots(q+n-1)\\ &\quad\times\left[(p+1)\dots(p+n-1)\left(1-\sum_{j=1}^{l}\beta_j\eta_j^{p}\right)|a| - |b|\sum_{i=1}^{m}\alpha_i\xi_i^{p+n-1}\right].\end{aligned}$$

If (3.5) holds, set

$$\Omega_3 = \{x \in \operatorname{Ker} L\colon\ -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

For any $x(t) = a + bt^{n-1} \in \Omega_3$, from $-\lambda Jx + (1-\lambda)QNx = 0$, we obtain

$$\begin{cases} (q+1)\dots(q+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big)(-\lambda|a| + (1-\lambda)Q_1N(a+bt^{n-1})) \\ \qquad -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1}(-\lambda|b| + (1-\lambda)Q_2N(a+bt)) = 0, \\ (p+1)\dots(p+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big)(-\lambda|a| + (1-\lambda)Q_1N(a+bt^{n-1})) \\ \qquad -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+n-1}(-\lambda|b| + (1-\lambda)Q_2N(a+bt^{n-1})) = 0. \end{cases}$$

Since

$$\begin{vmatrix} (q+1)\dots(q+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^q\Big) & -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{q+n-1} \\ (p+1)\dots(p+n-1)\Big(1-\sum\limits_{j=1}^{l}\beta_j\eta_j^p\Big) & -\sum\limits_{i=1}^{m}\alpha_i\xi_i^{p+n-1} \end{vmatrix} = -\Lambda(p,q) \neq 0,$$

then

$$\begin{cases} -\lambda|a| + (1-\lambda)Q_1N(a+bt^{n-1}) = 0, \\ -\lambda|b| + (1-\lambda)Q_2N(a+bt^{n-1}) = 0. \end{cases}$$

If $\lambda = 1$, then $a = b = 0$. If $\lambda \neq 1$, and $|a| > B$ or $|b| > B$, in view of the above equalities and (3.6), one has

$$\lambda(|a| + |b|) = (1-\lambda)[Q_1N(a+bt^{n-1}) + Q_2N(a+bt^{n-1})] < 0,$$

which contradicts $\lambda(|a| + |b|) \geqslant 0$, thus $\|x\| \leqslant |a| + |b| \leqslant 2B$. So Ω_3 is bounded.

If (3.5) holds, then set

$$\Omega_3 = \{x \in \operatorname{Ker} L\colon\ \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\};$$

similarly to the above argument, we can show that Ω_3 is bounded too.

In the following, we shall prove that all conditions of Theorem 2.1 are satisfied. Set Ω to be an open bounded subset of Y such that $\bigcup\limits_{i=1}^{3} \bar{\Omega}_i \subset \Omega$. By using the Arzela-Ascoli theorem, we can prove that $K_P(I-Q)N\colon \bar{\Omega} \to Y$ is compact, thus N is L-compact on $\bar{\Omega}$. Then by the above argument, we have

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.

(ii) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

Lastly, we will prove that (iii) of Theorem 2.1 is satisfied. Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. According to the above argument, we know that $H(x,\lambda) \neq 0$ for every $x \in \partial\Omega \cap \operatorname{Ker} L$. Thus, by the homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(\pm J, \Omega \cap \operatorname{Ker} L, 0) = \pm 1 \neq 0.\end{aligned}$$

Then by Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$. Thus BVP (1.1) and (1.2) has at least one solution in $C^{n-1}[0,1]$. □

From the condition (RC), $\sum_{i=1}^{m} \alpha_i \xi_i^{n-1} = 0$ implies that α_i, $i = 1,2,\ldots,m$, do not have the same sign. Assume that:

(H) There exists $s \in \{1,\ldots,m-1\}$ such that $\alpha_i < 0$ $(1 \leqslant i \leqslant s)$ and $\alpha_i > 0$ $(s+1 \leqslant i \leqslant m)$.

(H_2') There exists a constant $A > 0$ such that for $x \in \operatorname{dom} L$, if $|x(t)| > A$ for all $t \in [0,1]$, then

$$Q_1 N(x(t)) \neq 0 \quad \text{or} \quad Q_2 N(x(t)) \neq 0.$$

The following result is a modification of the previous theorem.

Theorem 3.2. *Let the conditions* (RC) *and* (H) *hold. Assume that* (H_1), (H_2'), *and* (H_3) *are fulfilled, then BVP* (1.1) *and* (1.2) *has at least one solution in* $C^{n-1}[0,1]$, *provided that*

$$\sum_{i=1}^{n} \|\alpha_i\|_1 < 1.$$

Proof. Set

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L \colon\ Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}.$$

For $x \in \Omega_1$, since $Lx = \lambda Nx$, so $\lambda \neq 0$, $Nx \in \operatorname{Im} L$, hence

$$Q_1 N(x(t)) = 0 \quad \text{and} \quad Q_2 N(x(t)) = 0.$$

Thus, from (H_2'), there exists $t_0 \in [0,1]$ such that $|x(t_0)| \leqslant A$.

By (H), there exist $t_1 \in [0,\eta_s]$ and $t_2 \in [\eta_{s+1},\eta_m]$ such that

$$x(t_1) = \frac{1}{1-\sum_{i=1}^{s}\alpha_i}\left[x(0) - \sum_{i=1}^{s}\alpha_i x(\xi_i)\right] = \frac{1}{\sum_{i=s+1}^{m}\alpha_i}\sum_{i=s+1}^{m}\alpha_i x(\xi_i) = x(t_2).$$

It follows that there exists $\tau_1 \in (t_1, t_2)$, such that $x'(\tau_1) = 0$. Taking note of $x'(0) = 0$, thus there exists $\tau_2 \in (0, \tau_1)$, such that $x''(\tau_2) = 0$. Since also $x''(0) = 0$, there exists $\tau_3 \in (0, \tau_2)$, such that $x'''(\tau_3) = 0$. Continuing like this, there exists $\tau_{n-1} \in (0, \tau_{n-2})$ $(n \geqslant 3)$, such that $x^{(n-1)}(\tau_{n-1}) = 0$. Since x, $x^{(n-1)}$ are absolutely continuous for all $t \in [0, 1]$,

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)\,\mathrm{d}s,$$
$$x^{(n-1)}(t) = x^{(n-1)}(\tau_{n-1}) + \int_{\tau_{n-1}}^{t} x^{(n)}(s)\,\mathrm{d}s.$$

Since $x'(0) = \ldots = x^{(n-2)}(0) = 0$, we get $\|x'\|_\infty \leqslant \ldots \leqslant \|x^{(n-1)}\|_\infty$,

$$\|x\|_\infty \leqslant A + \|x'\|_\infty, \quad \|x^{(n-1)}\|_\infty \leqslant \|x^{(n)}\|_1.$$

From (H_1), we obtain

$$\begin{aligned}\|x^{(n)}\|_1 = \|Lx\|_1 \leqslant \|Nx\|_1 &\leqslant \sum_{i=1}^{n} \|\alpha_i\|_1 \|x^{(i-1)}\|_\infty + \|\gamma\|_1 \\ &\leqslant \left(\sum_{i=1}^{n} \|\alpha_i\|_1\right) \|x^{(n)}\|_1 + A\|\alpha\|_1 + \|\gamma\|_1 \\ &\leqslant \frac{1}{1 - \sum_{i=1}^{n} \|\alpha_i\|_1} (A\|\alpha\|_1 + \|\gamma\|_1),\end{aligned}$$

so there exists a constant $M_1 > 0$ such that $\|x\| \leqslant M_1$. Therefore Ω_1 is bounded.

The rest of the proof is similar to Theorem 3.1. □

4. Example

Example 4.1. Consider the boundary value problem

$$x''' = \frac{1}{2\pi} h(t)(\sin x + \sin x' + x'' + 2\pi \mathrm{e}^t), \quad t \in (0, 1), \tag{4.1}$$

$$x(0) = \frac{9}{5} x\Big(\frac{1}{3}\Big) - \frac{4}{5} x\Big(\frac{1}{2}\Big), \quad x'(0) = 0, \quad x''(1) = -2x''\Big(\frac{1}{2}\Big) + 3x''\Big(\frac{2}{3}\Big) \tag{4.2}$$

where

$$h(t) = \begin{cases} \frac{1}{2}, & 0 \leqslant t \leqslant \frac{1}{2}, \\ \frac{1}{10}, & \frac{1}{2} < t \leqslant 1. \end{cases}$$

Let $\alpha_1 = \frac{9}{5}$, $\alpha_2 = -\frac{4}{5}$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, $\beta_1 = -2$, $\beta_2 = 3$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{2}{3}$, $f(t,x,y,z) = \frac{1}{2}\pi^{-1}h(t)(\sin x + \sin y + z + 2\pi e^t)$, then $\alpha_1 + \alpha_2 = 1$, $\alpha_1\xi_1^2 + \alpha_1\xi_2^2 = 0$, $\beta_1 + \beta_2 = 1$. Thus the condition (RC) holds.

Again

$$|f(t,x,y,z)| \leqslant \frac{1}{2\pi}(|x| + |y| + |z|).$$

Taking $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \frac{1}{2}\pi^{-1}$, $t \in (0,1)$, we have

$$\|\alpha_1\|_1 + \|\alpha_2\|_1 + \|\alpha_3\|_1 = \frac{3}{2\pi} < 1.$$

Now setting $A = 3+2\pi e$, for any $x \in \operatorname{dom} L$, assume $|x''(t)| > A$ holds for $t \in (0,1)$, from the continuity of x'', either $x''(t) > A$ or $x''(t) < -A$ holds for $t \in (0,1)$.

If $x''(t) > A$ holds for $t \in (0,1)$, then

$$\begin{aligned}
Q_1N(x) &= \alpha_1 \int_0^{\xi_1}\int_0^{\tau_3}\int_0^{\tau_2} Nx(\tau_1)\,d\tau_1\,d\tau_2\,d\tau_3 \\
&\quad + \alpha_2 \int_0^{\xi_2}\int_0^{\tau_3}\int_0^{\tau_2} Nx(\tau_1)\,d\tau_1\,d\tau_2\,d\tau_3 \\
&= \frac{1}{2}\alpha_1 \int_0^{\xi_1} (\xi_1 - s)^2 Nx(s)\,ds + \frac{1}{2}\alpha_2 \int_0^{\xi_2} (\xi_2 - s)^2 Nx(s)\,ds \\
&= -\frac{1}{5}\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)Nx(s)\,ds - \frac{1}{10}\int_{1/3}^{1/2}(1 - 4s + 4s^2)Nx(s)\,ds \\
&= -\frac{1}{10}\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)\Big[\frac{1}{2\pi}\sin x(s) + \sin x'(s) + \frac{1}{2\pi}x''(x) + e^s\Big]\,ds \\
&\quad - \frac{1}{20}\int_{1/3}^{1/2}(1 - 4s + 4s^2)\Big[\frac{1}{2\pi}\sin x(s) + \sin x'(s) + \frac{1}{2\pi}x''(x) + e^s\Big]\,ds \\
&< -\frac{1}{10}\Big(\frac{-1}{2\pi} + \frac{-1}{2\pi} + \frac{A}{2\pi} + 1\Big) \\
&\quad \times \Big[\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)\,ds + \frac{1}{2}\int_{1/3}^{1/2}(1 - 4s + 4s^2)\Big]\,ds \\
&= -\frac{1}{6480\pi}[9A + 18\pi - 18] < 0.
\end{aligned}$$

If $x''(t) < -A$ holds for $t \in (0,1)$, then

$$\begin{aligned}
Q_1N(x) &= \alpha_1 \int_0^{\xi_1}\int_0^{\tau_3}\int_0^{\tau_2} Nx(\tau_1)\,d\tau_1\,d\tau_2\,d\tau_3 \\
&\quad + \alpha_2 \int_0^{\xi_2}\int_0^{\tau_3}\int_0^{\tau_2} Nx(\tau_1)\,d\tau_1\,d\tau_2\,d\tau_3
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\alpha_1 \int_0^{\xi_1} (\xi_1 - s)^2 Nx(s)\,\mathrm{d}s + \frac{1}{2}\alpha_2 \int_0^{\xi_2} (\xi_2 - s)^2 Nx(s)\,\mathrm{d}s \\
&= -\frac{1}{5}\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big) Nx(s)\,\mathrm{d}s - \frac{1}{10}\int_{1/3}^{1/2}(1 - 4s + 4s^2)Nx(s)\,\mathrm{d}s \\
&= -\frac{1}{10}\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)\Big[\frac{1}{2\pi}\sin x(s) + \sin x'(s) + \frac{1}{2\pi}x''(x) + \mathrm{e}^s\Big]\,\mathrm{d}s \\
&\quad - \frac{1}{20}\int_{1/3}^{1/2}(1 - 4s + 4s^2)\Big[\frac{1}{2\pi}\sin x(s) + \sin x'(s) + \frac{1}{2\pi}x''(x) + \mathrm{e}^s\Big]\,\mathrm{d}s \\
&> -\frac{1}{10}\Big(\frac{1}{2\pi} + \frac{1}{2\pi} + \frac{-A}{2\pi} + \mathrm{e}\Big) \\
&\quad \times \Big[\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)\,\mathrm{d}s + \frac{1}{2}\int_{1/3}^{1/2}(1 - 4s + 4s^2)\Big]\,\mathrm{d}s \\
&= \frac{1}{6480\pi}[9A - 18 - 18\pi e] > 0.
\end{aligned}
$$

Thus the condition (H_2) holds.

Finally taking $B = 25 + \pi(13\mathrm{e} - 12)$, for any $a, b \in \mathbb{R}$, when $|b| > B$, then either $b > B$, or $b < -B$. If $b > B$, then

$$
\begin{aligned}
&Q_1 N(a + bt^2) + Q_2 N(a + bt^2) \\
&= \alpha_1 \int_0^{\xi_1}\int_0^{\tau_3}\int_0^{\tau_2} N(a + b\tau_1^2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3 + \alpha_2 \int_0^{\xi_2}\int_0^{\tau_3}\int_0^{\tau_2} N(a + b\tau_1^2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3 \\
&\quad + \int_0^1 N(a + b\tau_1^2)\,\mathrm{d}\tau_1 - \beta_1 \int_0^{\eta_1} N(a + b\tau_1^2)\,\mathrm{d}\tau_1 - \beta_2 \int_0^{\eta_2} N(a + b\tau_1^2)\,\mathrm{d}\tau_1 \\
&= -\frac{1}{5}\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)N(a + bs^2)\,\mathrm{d}s - \frac{1}{10}\int_{1/3}^{1/2}(1 - 4s + 4s^2)N(a + bs^2)\,\mathrm{d}s \\
&\quad - 2\int_{1/2}^{2/3} N(a + bs^2)\,\mathrm{d}s + \int_{2/3}^{1} N(a + bs^2)\,\mathrm{d}s \\
&< -\frac{1}{10}\Big(\frac{-1}{2\pi} + \frac{-1}{2\pi} + \frac{2b}{2\pi} + 1\Big) \\
&\quad \times \Big[\int_0^{1/3}\Big(s - \frac{5}{2}s^2\Big)\,\mathrm{d}s + \frac{1}{2}\int_{1/3}^{1/2}(1 - 4s + 4s^2)\,\mathrm{d}s + 2\int_{1/2}^{2/3}\,\mathrm{d}s\Big] \\
&\quad + \frac{1}{10}\Big(\frac{1}{2\pi} + \frac{1}{2\pi} + \frac{2b}{2\pi} + \mathrm{e}\Big)\int_{2/3}^{1}\,\mathrm{d}s \\
&= -\frac{9b - 225 - \pi(108\mathrm{e} - 117)}{3240\pi} < 0.
\end{aligned}
$$

If $b < -B$, then

$$
\begin{aligned}
&Q_1N(a+bt^2)+Q_2N(a+bt^2)\\
&=\alpha_1\int_0^{\xi_1}\int_0^{\tau_3}\int_0^{\tau_2}N(a+b\tau_1^2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\\
&\quad+\alpha_2\int_0^{\xi_2}\int_0^{\tau_3}\int_0^{\tau_2}N(a+b\tau_1^2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\\
&\quad+\int_0^1 N(a+b\tau_1^2)\,\mathrm{d}\tau_1-\beta_1\int_0^{\eta_1}N(a+b\tau_1^2)\,\mathrm{d}\tau_1-\beta_2\int_0^{\eta_2}N(a+b\tau_1^2)\,\mathrm{d}\tau_1\\
&=-\frac{1}{5}\int_0^{1/3}\Big(s-\frac{5}{2}s^2\Big)N(a+bs^2)\,\mathrm{d}s-\frac{1}{10}\int_{1/3}^{1/2}(1-4s+4s^2)N(a+bs^2)\,\mathrm{d}s\\
&\quad-2\int_{1/2}^{2/3}N(a+bs^2)\,\mathrm{d}s+\int_{2/3}^{1}N(a+bs^2)\,\mathrm{d}s\\
&>-\frac{1}{10}\Big(\frac{1}{2\pi}+\frac{1}{2\pi}+\frac{2b}{2\pi}+\mathrm{e}\Big)\\
&\quad\times\Big[\int_0^{1/3}\Big(s-\frac{5}{2}s^2\Big)\,\mathrm{d}s+\frac{1}{2}\int_{1/3}^{1/2}(1-4s+4s^2)\,\mathrm{d}s+2\int_{1/2}^{2/3}\,\mathrm{d}s\Big]\\
&\quad+\frac{1}{10}\Big(\frac{-1}{2\pi}+\frac{-1}{2\pi}+\frac{2b}{2\pi}+1\Big)\int_{2/3}^{1}\,\mathrm{d}s\\
&=\frac{-9b-225-\pi(117\mathrm{e}-108)}{3240\pi}>0.
\end{aligned}
$$

So the condition (H_3) holds. Hence, from Theorem 3.1, the BVP (4.1) and (4.2) has at least one solution in $C^2[0,1]$. □

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