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A STUDY OF GALERKIN METHOD FOR THE HEAT CONVECTION EQUATIONS

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Abstract. The paper investigates the Galerkin method for an initial boundary value problem for heat convection equations. New error estimates for the approximate solutions and their derivatives in strong norm are obtained.

Keywords: approximate solution, error estimate, Galerkin method, heat convection equation, orthogonal projection

MSC 2010: 65J10, 65M60, 35K90

1. Introduction

A large number of works is devoted to the study of various convection problems. It is possible to recognize two important directions of the study of convection phenomena. The first is the experimental and theoretical study of the convective stability. In detail these questions are considered, for example, in the monograph [7]. The other important direction is the numerical modeling of convection processes (see, for example, [7], [6], [5], [16], [2]). It allows to calculate the modes of convection at various meanings of Rayleigh, Reynolds numbers and at other parameters of the model. It is known that the main theoretical basis of numerical methods is the proof of convergence of the approximate solution to the exact one of the corresponding differential problem. In this connection we point out the monograph [14], where a thorough research of numerical methods for solving the Navier-Stokes equations is carried out. The order of the convergence speed of approximate solutions of a nonlinear problem much depends on the kind of the nonlinear terms. It is often difficult to establish the convergence. In this case the basic information on the convergence of the computing procedure is found by numerical experiments.

In the present paper we study the Galerkin method for the approximate solution of an initial boundary value problem for a non-stationary quasi-linear system which describes the motion of the non-uniformly heated viscous incompressible fluid. The convergence of the Galerkin approximations in a strong norm is established, and also the asymptotic error estimates for the solutions and their derivatives in the uniform norm are obtained.

2. Statement of the problem and auxiliary assertions

Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, $Q = \Omega \times (0, T)$, $S = \partial\Omega \times (0, T]$, where $T < \infty$.

The initial boundary value problem for the heat convection in Boussinesq approximation is formulated in the following way ([7], [12], [3]): We seek a vector-function $u(x,t)\colon\ \Omega \times [0,T] \to \mathbb{R}^2$ and scalar functions $p(x,t), \theta(x,t)\colon\ \Omega \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} - \nu\Delta u + \varrho_0^{-1}\nabla p + (u\cdot\nabla)u - g\beta k_3\theta = f \quad \text{in } Q, \tag{1}$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + u\cdot\nabla\theta = \varphi \quad \text{in } Q, \tag{2}$$

$$\operatorname{div} u = 0 \quad \text{in } Q, \tag{3}$$

$$u = 0,\ \ \theta = 0 \quad \text{on } S, \tag{4}$$

$$u(x,0) = 0,\ \ \theta(x,0) = 0,\ \ x \in \overline{\Omega}. \tag{5}$$

These equations model the motion of the non-uniformly heated viscous incompressible fluid, where u is the velocity vector, θ is the temperature, p is the pressure, ν is the kinematical viscosity, κ is the thermal diffusivity, ϱ_0 is the constant density, g is the free fall acceleration, $\beta > 0$ is the temperature-expansion coefficient, f is the apparent density of the external forces, φ is the apparent density of the heat source, k_3 is the up-ward vertically directed along the unit vector.

Let $L_p(\Omega)$, $1 < p < \infty$, $(L_\infty(\Omega))$ be the space of real functions absolutely integrable on Ω with the power of p according to Lebesgue measure $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2$ (respectively, essentially bounded). These spaces with the norms

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p\,\mathrm{d}x\right)^{1/p}$$

and

$$\|u\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |u(x)|$$

are Banach spaces. The space $L_p(Q)$ is defined similarly. The Sobolev space $W_p^m(\Omega)$ is the space of functions from $L_p(\Omega)$ whose all generalized partial derivatives up to order m inclusively belong to $L_p(\Omega)$ (m is a nonnegative integer). It is a Banach space with the norm

$$\|u\|_{W_p^m(\Omega)} = \Big(\sum_{|j|\leqslant m} \|D^j u\|^p_{L_p(\Omega)}\Big)^{1/p}.$$

The space $W_p^{2m,m}(Q)$ (see [11]) with m being a nonnegative integer is the Banach space of functions from $L_p(Q)$, which have generalized derivatives $D_t^r D_x^s$ with arbitrary nonnegative integers r and s satisfying the inequality $2r+s \leqslant 2m$. The norm in $W_p^{2m,m}(Q)$ is defined as

$$\|u\|_{W_p^{2m,m}(Q)} = \sum_{j=0}^{2m} \sum_{2r+s=j} \|D_t^r D_x^s u\|_{L_p(Q)}.$$

We put

$$W_p^{1,0}(Q) = \{u \in L_p(Q)\colon\ D_x u \in L_p(Q)\},$$
$$\overset{\circ}{W}{}_2^1(\Omega) = \{u \in W_2^1(\Omega)\colon\ u = 0 \ \text{ on } \ \partial\Omega \ \text{ in the sense of traces}\}.$$

The symbol $\overset{\circ}{W}{}_2^{2,1}(Q)$ denotes the set of functions belonging to $W_2^{2,1}(Q)$ satisfying zero initial conditions and vanishing on S.

We shall deal with two-dimensional vector-functions, each component of which belongs to one of the above defined spaces. We set $[L_p(\Omega)]^2 = L_p(\Omega) \times L_p(\Omega)$, $[L_p(Q)]^2 = L_p(Q) \times L_p(Q)$, etc. The norm, for example, in $[L_p(\Omega)]^2$ ($p > 2$) is denoted by $[\cdot]_{L_p(\Omega)}$. A similar notation is used for the norms in the spaces $[W_2^2(\Omega)]^2$, $[L_p(Q)]^2$, $[W_2^{2,1}(Q)]^2$.

Let $\|\cdot\|$ and $[\cdot]$ stand for the norm in $L_2(\Omega)$ and in $[L_2(\Omega)]^2$, respectively. The inner product in $L_2(\Omega)$ and in $[L_2(\Omega)]^2$ will be denoted by $(\cdot,\cdot)$.

The solution of the problem (1)–(5) is a triple of functions (u, p, θ) from $[W_2^{2,1}(Q)]^2 \times W_2^1(Q) \times W_2^{2,1}(Q)$ that satisfy equations (1)–(3) for almost all t and also the boundary and initial conditions (4)–(5) in the sense of traces.

Let $J(\Omega)$ be the space of solenoidal infinitely differentiable and finite on Ω vectors $v(x) = (v_1(x), v_2(x))$, let $\overset{\circ}{J}(\Omega)$ be the closure with respect to the norm of the space $[W_2^1(\Omega)]^2$. The elements of $\overset{\circ}{J}(Q)$ are the vectors $v(x,t)$ that belong to $\overset{\circ}{J}(\Omega)$ for almost all t. Let P_J be the orthogonal projection of $[L_2(\Omega)]^2$ onto $\overset{\circ}{J}(\Omega)$.

It is known (see [10]) that $[L_2(\Omega)]^2 = \overset{\circ}{J}(\Omega) \oplus G(\Omega)$, where the subspace $G(\Omega)$ contains the gradients of all single-valued functions in Ω. Acting on (1) by the operator P_J and taking into account $P_J \nabla p = 0$, we come to the problem

$$\frac{\partial u}{\partial t} - \nu P_J \Delta u + P_J((u \cdot \nabla)u) - g\beta P_J(k_3\theta) = P_J f \quad \text{in } Q, \tag{6}$$

$$\frac{\partial \theta}{\partial t} - \kappa \Delta \theta + u \cdot \nabla \theta = \varphi \quad \text{in } Q, \tag{7}$$

$$u = 0, \;\; \theta = 0 \quad \text{on } S, \tag{8}$$

$$u(x,0) = 0, \;\; \theta(x,0) = 0 \quad \text{for a.e. } x \in \overline{\Omega}. \tag{9}$$

On the other hand, if functions $u \in \overset{\circ}{J}(\Omega)$ and $\theta \in W_2^{2,1}(Q)$ are solutions of problem (6)–(9), then equation (6) can be written as

$$P_J\Big(\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u - g\beta k_3\theta - f\Big) = 0$$

and hence,

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u - g\beta k_3\theta - f = \nabla p_1$$

for almost all t, where $\nabla p_1 \in [L_2(\Omega)]^2$. Thus, problems (1)–(5) and (6)–(9) are equivalent.

We consider the spectral problems

$$-\nu P_J \Delta e = \lambda e, \quad e \in \overset{\circ}{J}(\Omega),$$
$$e(x) = 0, \quad x \in \partial\Omega$$

and

$$-\kappa \Delta m = \mu m,$$
$$m(x) = 0, \quad x \in \partial\Omega.$$

By λ_i we denote the eigenvalue corresponding to the eigenvector $e_i(x)$, by μ_i we denote an eigenvalue, corresponding to the eigenvector $m_i(x)$. The existence and completeness of the eigenfunctions $e_i(x) \in [W_2^2(\Omega)]^2 \cap \overset{\circ}{J}(\Omega)$, $m_i(x) \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ in the spaces $[L_2(\Omega)]^2$ and $L_2(\Omega)$ are proved in [10], [1].

Let P_{n1} be the orthogonal projection of $[L_2(\Omega)]^2$ onto the linear span of the vector-functions $\{e_i(x)\}_{i=1}^n$, let P_{n2} be the orthogonal projection of $L_2(\Omega)$ onto the linear span of the functions $\{m_i(x)\}_{i=1}^n$.

The approximate solutions for the problem (6)–(9) are defined as

$$u_n(x,t) = \sum_{i=1}^n \alpha_i(t) e_i(x),$$

$$\theta_n(x,t) = \sum_{i=1}^n \gamma_i(t) m_i(x),$$

where the unknown functions $\alpha_i(t)$ and $\gamma_i(t)$ $(i = 1, 2, \ldots, n)$ are the exact solution of the following problem:

$$(10)\quad \frac{\partial u_n}{\partial t} - \nu P_J \Delta u_n + P_{n1} P_J ((u_n \cdot \nabla) u_n) - g\beta P_{n1} P_J (k_3 \theta_n) = P_{n1} P_J f \quad \text{in } Q,$$

$$(11)\quad \frac{\partial \theta_n}{\partial t} - \kappa \Delta \theta_n + P_{n2}(u_n \cdot \nabla \theta_n) = P_{n2}\varphi \quad \text{in } Q,$$

$$(12)\quad u_n(x, 0) = 0,\ \ \theta_n(x, 0) = 0, \quad x \in \overline{\Omega}.$$

Here we have used the facts that $P_{n1}(\nu P_J \Delta u_n) = \nu P_J \Delta u_n$ and $P_{n2}(\kappa P_J \Delta \theta_n) = \kappa P_J \Delta \theta_n$.

From now on, by C we denote the so-called generic positive constant. It is independent of n and can have different values at different occurrences.

Later the following multiplicative inequalities will be used very often (see, for example, [8]). Let $v \in W_{p_1}^{l_1}(\Omega)$,

$$\nu_1 = \frac{\kappa_3 - \kappa_2}{\kappa_3 - \kappa_1}, \quad \kappa_i = \frac{2}{p_i} - l_i, \quad \kappa_1 < \kappa_2 < \kappa_3, \quad l_3 \leqslant l_2 < l_1.$$

Then

$$(13)\quad \|v\|_{W_{p_2}^{l_2}(\Omega)} \leqslant C \|v\|_{W_{p_3}^{l_3}(\Omega)}^{1-\nu_1} \|v\|_{W_{p_1}^{l_1}(\Omega)}^{\nu_1}.$$

This inequality also holds with $l_2 = l_3 = 0$ and $p_2 = \infty$.

In addition, we shall use the following well-known fact (see, for example, [13]).

Let $u \in W_p^{2m,m}(Q)$ and $p(2m - 2h - s) > 4$. Then any derivative $D_t^h D_x^\alpha u$ with $|\alpha| = s$ belongs to the space $L_r(Q)$ with any $r \geqslant p$ including $r = \infty$ and the inequality

$$\|D_t^h D_x^\alpha u\|_{L_r(Q)} \leqslant \varepsilon^{2m-2h-s-4(1/p-1/r)} \|u\|_{W_p^{2m,m}(Q)} + \varepsilon^{-2h-s-4(1/p-1/r)} \|u\|_{L_p(Q)}$$

holds for any $\varepsilon > 0$.

In the multiplicative form this inequality can be written as

$$(14)\quad \|D_t^h D_x^\alpha u\|_{L_r(Q)} \leqslant C \|u\|_{W_p^{2m,m}(Q)}^{\beta} \|u\|_{L_p(Q)}^{1-\beta},$$

where $\beta = (2h + s + 4(1/p - 1/r))/(2m)$.

Lemma 2.1. *Let $f(x,t) \in [L_2(Q)]^2$, $\varphi(x,t) \in L_2(Q)$. Then problem* (10)–(12) *has a unique solution $u_n(x,t) \in [W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$, $\theta_n(x,t) \in \overset{\circ}{W}{}_2^{2,1}(Q)$ for each n and the inequalities*

$$(15)\quad [u_n(x,t)]_{W_2^{2,1}(Q)} \leqslant C,$$

$$(16)\quad \|\theta_n(x,t)\|_{W_2^{2,1}(Q)} \leqslant C$$

hold.

P r o o f. We take the $L_2(\Omega)$-inner product of (11) and $\theta_n(x,\tau)$ and integrate the resulting relation over the interval $[0,t]$, $t \leqslant T$. Then, using the equality

$$(P_{n2}(u_n \cdot \nabla\theta_n), \theta_n) = 0,$$

we obtain

$$\sup_{0\leqslant t\leqslant T} \|\theta_n\|^2 + \int_0^t \|\nabla\theta_n\|^2 \,\mathrm{d}\tau \leqslant C. \tag{17}$$

Similarly, using the equality

$$(P_{n1}P_J((u_n \cdot \nabla)u_n), u_n) = 0$$

and (17), we have

$$\sup_{0\leqslant t\leqslant T} [u_n]^2 + \int_0^t [\nabla u_n]^2 \,\mathrm{d}\tau \leqslant C. \tag{18}$$

We multiply equation (10) in $[L_2(\Omega)]^2$ by $-P_J\Delta u_n$ and integrate the resulting relation over the interval $[0,t]$, $t \leqslant T$. Then

$$\begin{aligned}
&\frac{1}{2}[\nabla u_n(x,t)]^2 + \nu\int_0^t [P_J\Delta u_n(x,\tau)]^2 \,\mathrm{d}\tau \\
&\qquad \leqslant \int_0^t [f(x,\tau)][P_J\Delta u_n(x,\tau)] \,\mathrm{d}\tau + g\beta\int_0^t \|\theta_n(x,\tau)\|[P_J\Delta u_n(x,\tau)] \,\mathrm{d}\tau \\
&\qquad\quad + \int_0^t [(u_n(x,\tau)\cdot\nabla)u_n(x,\tau)][P_J\Delta u_n(x,\tau)] \,\mathrm{d}\tau.
\end{aligned}$$

Now, using the Cauchy inequality $|a||b| \leqslant \frac{1}{2}\varepsilon|a|^2 + \frac{1}{2}\varepsilon^{-1}|b|^2$ for sufficiently small $\varepsilon > 0$, we obtain

$$[\nabla u_n]^2 + \int_0^t [P_J\Delta u_n]^2 \,\mathrm{d}\tau \leqslant C([f]^2_{L_2(Q)} + \|\theta_n\|^2_{L_2(Q)} + \int_0^t [(u_n\cdot\nabla)u_n]^2 \,\mathrm{d}\tau).$$

This and (17) yield

$$[\nabla u_n]^2 + \int_0^t [P_J\Delta u_n]^2 \,\mathrm{d}\tau \leqslant C\Big(1 + \int_0^t [(u_n\cdot\nabla)u_n]^2 \,\mathrm{d}\tau\Big). \tag{19}$$

Let us estimate the integral on the right-hand side of (19). Applying the Hölder inequality, we find that

$$J \equiv \int_0^t [(u_n\cdot\nabla)u_n]^2 \,\mathrm{d}\tau \leqslant \int_0^t [u_n]^2_{L_4(\Omega)}[\nabla u_n]^2_{L_4(\Omega)} \,\mathrm{d}\tau.$$

By using (13) for the spaces $W_4^1(\Omega)$, $W_2^2(\Omega)$ and $W_2^1(\Omega)$, we get

$$J \leqslant C\int_0^t [u_n]^2_{L_4(\Omega)}[u_n]_{W_2^1(\Omega)}[u_n]_{W_2^2(\Omega)}\,\mathrm{d}\tau.$$

Next,

$$[u_n]^2_{W_2^1(\Omega)} = [u_n]^2 + [\nabla u_n]^2.$$

Since $u_n = 0$ on $\partial\Omega$, using Friedrichs inequality, we have

$$[u_n] \leqslant C[\nabla u_n].$$

Hence,

$$[u_n]_{W_2^1(\Omega)} \leqslant C[\nabla u_n].$$

Therefore,

$$J \leqslant C\int_0^t [u_n]^2_{L_4(\Omega)}[\nabla u_n][u_n]_{W_2^2(\Omega)}\,\mathrm{d}\tau.$$

From this and the coercive inequality (see, for example, [10])

$$[z(x)]_{W_2^2(\Omega)} \leqslant C[-P_J\Delta z(x)], \quad \forall\, z(x) \in [W_2^2(\Omega)]^2 \cap \overset{\circ}{J}(\Omega) \tag{20}$$

we obtain

$$J \leqslant C\int_0^t [u_n]^2_{L_4(\Omega)}[P_J\Delta u_n][\nabla u_n]\,\mathrm{d}\tau.$$

Therefore, from (19) it follows that

$$[\nabla u_n]^2 + \int_0^t [P_J\Delta u_n]^2\,\mathrm{d}\tau \leqslant C\biggl(1 + \varepsilon\int_0^t [P_J\Delta u_n]^2\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_0^t [u_n]^4_{L_4(\Omega)}[\nabla u_n]^2\,\mathrm{d}\tau\biggr).$$

Choosing sufficiently small $\varepsilon > 0$, we have

$$[\nabla u_n]^2 \leqslant C\biggl(1 + \int_0^t [u_n]^4_{L_4(\Omega)}[\nabla u_n]^2\,\mathrm{d}\tau\biggr).$$

Now, applying the Gronwall inequality (see [4]), we come to an estimate

$$[\nabla u_n]^2 \leqslant C\exp\biggl(C\int_0^t [u_n]^4_{L_4(\Omega)}\,\mathrm{d}\tau\biggr). \tag{21}$$

We use the inequality (13) for the spaces $L_4(\Omega)$, $W_2^1(\Omega)$, $L_2(\Omega)$, arriving at

$$\int_0^T [u_n]^4_{L_4(\Omega)}\,\mathrm{d}\tau \leqslant C\int_0^T [u_n]^2[\nabla u_n]^2\,\mathrm{d}\tau \leqslant C, \tag{22}$$

where the last inequality comes from (18). Therefore, from (21) we have

$$\sup_{0\leqslant t\leqslant T}[\nabla u_n(x,t)] \leqslant C. \tag{23}$$

By applying estimate (23), from equation (11) it follows that

$$\sup_{0\leqslant t\leqslant T}\|\nabla \theta_n(x,t)\| \leqslant C. \tag{24}$$

From the coercive inequality (see [10])

$$[z]_{W_2^{2,1}(Q)} \leqslant C\Big[\Big(\frac{\partial}{\partial t} - \nu P_J \Delta\Big) z\Big]_{L_2(Q)} \quad \forall\, z \in [W_2^{2,1}(Q)]^2 \cap \mathring{J}(Q) \tag{25}$$

we have

$$[u_n]_{W_2^{2,1}(Q)} \leqslant C([f]_{L_2(Q)} + \|\theta_n\|_{L_2(Q)} + [(u_n\cdot\nabla)u_n]_{L_2(Q)}).$$

Then

$$[u_n]_{W_2^{2,1}(Q)} \leqslant C(1 + [u_n]_{L_6(Q)}[\nabla u_n]_{L_3(Q)}).$$

By using inequality (14) for the spaces $W_3^{1,0}(Q)$, $W_2^{2,1}(Q)$, $L_2(Q)$, we get

$$[u_n]_{W_2^{2,1}(Q)} \leqslant C(1 + [u_n]_{L_6(Q)}[u_n]^{5/6}_{W_2^{2,1}(Q)}[u_n]^{1/6}_{L_2(Q)}).$$

Since the space $W_2^1(\Omega)$ is embedded in $L_6(\Omega)$ and inequalities (18), (23) hold, it follows that

$$[u_n]_{W_2^{2,1}(Q)} \leqslant C(1 + [u_n]^{5/6}_{W_2^{2,1}(Q)}).$$

Thus, estimate (15) holds. By analogy we obtain (16).

From estimates (15), (16) and the Leray-Schauder principle it follows that the solution of problem (10)–(12) exists. By analogy with the proof of the uniqueness of the solution of the initial boundary value problem for the Navier-Stokes equations (see [10]) the uniqueness of the solution to problem (10)–(12) is proved. □

3. Error estimates for Galerkin method

In this section we establish error estimates for the approximate solutions, for the gradient of the approximate solutions and for the derivative with respect to t.

Theorem 3.1. *Suppose that $f(x,t) \in [L_2(Q)]^2$ and $\varphi(x,t) \in L_2(Q)$. Then*

$$\sup_{0\leqslant t\leqslant T} [u_n(x,t) - u(x,t)] \leqslant C(\lambda_{n+1}^{-1/2} + \mu_{n+1}^{-1/2}), \tag{26}$$

$$\sup_{0\leqslant t\leqslant T} \|\theta_n(x,t) - \theta(x,t)\| \leqslant C(\lambda_{n+1}^{-1/2} + \mu_{n+1}^{-1/2}), \tag{27}$$

$$\lim_{n\to\infty} [u_n(x,t) - u(x,t)]_{W_2^{2,1}(Q)} = 0, \tag{28}$$

$$\lim_{n\to\infty} \|\theta_n(x,t) - \theta(x,t)\|_{W_2^{2,1}(Q)} = 0, \tag{29}$$

where $u(x,t)$ and $\theta(x,t)$ are the solution of problem (6)–(9).

P r o o f. For the differences $u_n - u$ and $\theta_n - \theta$ we have

$$\begin{aligned}\frac{\partial(u_n - u)}{\partial t} &- \nu P_J \Delta(u_n - u)\\ &= (I - P_{n1})P_J((u_n\cdot\nabla)u_n - f) + P_J((u\cdot\nabla)u - (u_n\cdot\nabla)u_n)\\ &\quad - g\beta(I - P_{n1})P_J(k_3\theta_n) + g\beta P_J(k_3\theta_n - k_3\theta),\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial(\theta_n - \theta)}{\partial t} &- \kappa\Delta(\theta_n - \theta)\\ &= (I - P_{n2})(\nabla\theta_n\cdot u_n - \varphi) - \nabla\theta_n\cdot u_n + \nabla\theta\cdot u.\end{aligned} \tag{31}$$

We take the $[L_2(\Omega)]^2$-inner product of (30) and $u_n - u$, take the $L_2(\Omega)$-inner product of (31) and $\theta_n - \theta$ and integrate over the interval $[0,s]$, $s \leqslant T$. Then

$$\begin{aligned}\frac{1}{2}[u_n - u]^2 &+ \nu\int_0^s [\nabla(u_n - u)]^2\,\mathrm{d}t\\ &\leqslant \int_0^s |(P_J(-f + (u_n\cdot\nabla)u_n - g\beta k_3\theta_n), (I - P_{n1})(u_n - u))|\,\mathrm{d}t\\ &\quad + \int_0^s |(-(u_n\cdot\nabla)u_n + (u\cdot\nabla)u, u_n - u)|\,\mathrm{d}t\\ &\quad + g\beta\int_0^s |(k_3\theta_n - k_3\theta, u_n - u)|\,\mathrm{d}t,\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{1}{2}\|\theta_n - \theta\|^2 &+ \kappa\int_0^s \|\nabla(\theta_n - \theta)\|^2\,\mathrm{d}t\\ &\leqslant \int_0^s |(\nabla\theta_n\cdot u_n - \varphi, (I - P_{n2})(\theta_n - \theta))|\,\mathrm{d}t\\ &\quad + \int_0^s |(\nabla\theta\cdot u - \nabla\theta_n\cdot u_n, \theta_n - \theta)|\,\mathrm{d}t.\end{aligned} \tag{33}$$

Since $\theta_n - \theta$ belongs to the space $\overset{\circ}{W}{}_2^{2,1}(Q)$, from (33) we obtain

$$\begin{aligned}\frac{1}{2}\|\theta_n-\theta\|^2 + \kappa\int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t\\ \leqslant \mu_{n+1}^{-1/2}\int_0^s \|\nabla(\theta_n-\theta)\|\|\nabla\theta_n\cdot u_n-\varphi\|\,\mathrm{d}t\\ +\int_0^s |(\nabla\theta_n\cdot(u_n-u),\theta_n-\theta)|\,\mathrm{d}t.\end{aligned}$$

Because the space $W_2^{2,1}(Q)$ is embedded in $L_4(Q)$ and in $W_4^{1,0}(Q)$, by using the Hölder and Cauchy inequalities and (15), (16), we get

$$\begin{aligned}\frac{1}{2}\|\theta_n-\theta\|^2 + \kappa\int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t\\ \leqslant \frac{C}{\varepsilon}\,\mu_{n+1}^{-1} + \frac{\varepsilon}{2}\int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t\\ +\int_0^s \|\nabla\theta_n\|[u-u_n]_{L_4(\Omega)}^2\|\theta_n-\theta\|_{L_4(\Omega)}^2\,\mathrm{d}t.\end{aligned}$$

This and (24) for sufficiently small $\varepsilon > 0$ imply

$$\|\theta_n-\theta\|^2 + \int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t \leqslant C\Big(\mu_{n+1}^{-1} + \int_0^s [u-u_n]_{L_4(\Omega)}^2\|\theta_n-\theta\|_{L_4(\Omega)}^2\,\mathrm{d}t\Big).$$

Using inequality (13) for the spaces $L_4(\Omega)$, $W_2^1(\Omega)$ and $L_2(\Omega)$, we find that

$$\begin{aligned}\|\theta_n-\theta\|^2 + \int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t\\ \leqslant C\Big(\mu_{n+1}^{-1} + \int_0^s [u-u_n]_{L_4(\Omega)}^2\|\nabla(\theta_n-\theta)\|\|\theta_n-\theta\|\,\mathrm{d}t\Big).\end{aligned}$$

Now, by applying the Cauchy inequality, we obtain

$$\begin{aligned}\|\theta_n-\theta\|^2 + \int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t\\ \leqslant C\Big(\mu_{n+1}^{-1} + \frac{\varepsilon}{2}\int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t + \frac{1}{2\varepsilon}\int_0^s [u-u_n]_{L_4(\Omega)}^4\|\theta_n-\theta\|^2\,\mathrm{d}t\Big).\end{aligned}$$

Hence, for sufficiently small $\varepsilon > 0$ it follows that

$$\|\theta_n-\theta\|^2 + \int_0^s \|\nabla(\theta_n-\theta)\|^2\,\mathrm{d}t \leqslant C\Big(\mu_{n+1}^{-1} + \int_0^s [u-u_n]_{L_4(\Omega)}^4\|\theta_n-\theta\|^2\,\mathrm{d}t\Big). \tag{34}$$

By analogy, from equation (32) we have

$$(35)\qquad [u_n-u]^2+\int_0^s[\nabla(u_n-u)]^2\,\mathrm{d}t \leqslant C\Big(\lambda_{n+1}^{-1}+\int_0^s[u-u_n]^4_{L_4(\Omega)}[u_n-u]^2\,\mathrm{d}t+\int_0^s\|\theta_n-\theta\|^2\,\mathrm{d}t\Big).$$

Adding (34) and (35), we come to the inequality

$$[u_n-u]^2+\|\theta_n-\theta\|^2 \leqslant C\Big(\mu_{n+1}^{-1}+\lambda_{n+1}^{-1}+\int_0^s([u-u_n]^4_{L_4(\Omega)}+1)(\|\theta-\theta_n\|^2+[u-u_n]^2)\,\mathrm{d}t\Big).$$

We use the Gronwall inequality; then

$$(36)\quad [u_n-u]^2+\|\theta_n-\theta\|^2\leqslant C\Big(\mu_{n+1}^{-1}+\lambda_{n+1}^{-1}\Big)\exp\Big(C\int_0^T([u-u_n]^4_{L_4(\Omega)}+1)\,\mathrm{d}t\Big).$$

Similarly to (22) we obtain

$$\int_0^T[u-u_n]^4_{L_4(\Omega)}\,\mathrm{d}t\leqslant C\int_0^T[\nabla(u-u_n)]^2[u_n-u]^2\,\mathrm{d}t\leqslant C,$$

where the last inequality comes from (18) and (23). Therefore, from (36) we obtain estimates (26) and (27).

Now we shall prove the relations (28) and (29). We put

$$\delta_n^1=\frac{\partial u_n}{\partial t}-\nu P_J\Delta u_n+P_J((u_n\cdot\nabla)u_n)-g\beta P_J(k_3\theta_n)-P_Jf,$$
$$\delta_n^2=\frac{\partial\theta_n}{\partial t}-\kappa\Delta\theta_n+\nabla\theta_n\cdot u_n-\varphi.$$

Since u_n and θ_n are the solution of problem (10)–(12), we have

$$\delta_n^1=-(I-P_{n1})P_Jf+(I-P_{n1})P_J((u_n\cdot\nabla)u_n)-g\beta(I-P_{n1})P_J(k_3\theta_n).$$

Hence,

$$[\delta_n^1]_{L_2(Q)}\leqslant[(I-P_{n1})P_Jf]_{L_2(Q)}+[(I-P_{n1})P_J((u_n\cdot\nabla)u_n)]_{L_2(Q)} +[g\beta(I-P_{n1})P_J(k_3\theta_n)]_{L_2(Q)}.$$

From (15), (16) and from the embedding theorems (see [13]) it follows that the sets $\{P_J(u_n\cdot\nabla)u_n\}$ and $\{\theta_n\}$ are compact in $L_2(Q)$. It is known that a sequence of bounded operators converges uniformly on a compact set, and therefore

$$(37)\qquad [\delta_n^1]_{L_2(Q)}\to0,\quad n\to\infty.$$

By analogy, we establish that

$$\|\delta_n^2\|_{L_2(Q)} \to 0, \quad n \to \infty.$$

For the difference $u_n - u$ we have the identity

$$\frac{\partial(u_n - u)}{\partial t} - \nu P_J \Delta(u_n - u) = \delta_n^1 + P_J((u \cdot \nabla)u - (u_n \cdot \nabla)u_n) + g\beta P_J(k_3(\theta_n - \theta)).$$

From this and from (25) it follows that

$$\begin{aligned}(38)\quad & [u_n - u]_{W_2^{2,1}(Q)} \\ & \leqslant C([\delta_n^1]_{L_2(Q)} + [(u \cdot \nabla)u - (u_n \cdot \nabla)u_n]_{L_2(Q)} + [g\beta k_3(\theta_n - \theta)]_{L_2(Q)}).\end{aligned}$$

From (27) we obtain

$$[g\beta k_3(\theta_n - \theta)]_{L_2(Q)} \to 0, \quad n \to \infty. \tag{39}$$

Now we estimate the second summand on the right-hand side of (38)

$$[(u \cdot \nabla)u - (u_n \cdot \nabla)u_n]_{L_2(Q)} \leqslant [u_n - u]_{L_4(Q)}[\nabla u]_{L_4(Q)} + [u_n]_{L_6(Q)}[\nabla(u_n - u)]_{L_3(Q)}.$$

By using the inequality (15) and the embedding of the space $W_2^{2,1}(Q)$ in $L_6(Q)$ and in $W_4^{1,0}(Q)$, we come to the estimate

$$[(u \cdot \nabla)u - (u_n \cdot \nabla)u_n]_{L_2(Q)} \leqslant C([u_n - u]_{L_4(Q)} + [\nabla(u_n - u)]_{L_3(Q)}).$$

Further, applying inequality (14) for spaces $L_4(Q)$, $W_2^{2,1}(Q)$, and $L_2(Q)$, and also for $W_3^{1,0}(Q)$, $W_2^{2,1}(Q)$, and $L_2(Q)$, we have

$$\begin{aligned}& [(u \cdot \nabla)u - (u_n \cdot \nabla)u_n]_{L_2(Q)} \\ & \quad \leqslant C([u_n - u]^{1/2}_{W_2^{2,1}(Q)}[u_n - u]^{1/2}_{L_2(Q)} + [u_n - u]^{5/6}_{W_2^{2,1}(Q)}[u_n - u]^{1/6}_{L_2(Q)}).\end{aligned}$$

From this, (15), and (26) it follows that

$$[(u \cdot \nabla)u - (u_n \cdot \nabla)u_n]_{L_2(Q)} \to 0, \quad n \to \infty. \tag{40}$$

Using (37), (38), (39), and (40), we obtain (28). By analogy, we come to estimate (29). □

In order to obtain error estimates for the derivatives of the approximate solutions we introduce an auxiliary problem and prove its unique solvability. Let the vector-function $z_1(x,t)$ belong to the space $[W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$ and let the function $z_2(x,t)$ belong to the space $\overset{\circ}{W}{}_2^{2,1}(Q)$. We consider the problem

$$\frac{\partial v}{\partial t} - \nu P_J \Delta v + P_J((z_1 \cdot \nabla)v + (v \cdot \nabla)z_1) - g\beta P_J(k_3 w) = P_J h \quad \text{in } Q, \tag{41}$$

$$\frac{\partial w}{\partial t} - \kappa \Delta w + \nabla w \cdot z_1 + \nabla z_2 \cdot v = h_1 \quad \text{in } Q, \tag{42}$$

$$\operatorname{div} v = 0 \quad \text{in } Q, \tag{43}$$

$$v = 0, \ \ w = 0 \quad \text{on } S, \tag{44}$$

$$v(x,0) = 0, \ \ w(x,0) = 0, \quad x \in \overline{\Omega}. \tag{45}$$

Lemma 3.1. *Suppose that* $h(x,t) \in [L_2(Q)]^2$, $h_1(x,t) \in L_2(Q)$. *Then problem* (41)–(45) *has a unique solution* $v \in [W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$, $w \in \overset{\circ}{W}{}_2^{2,1}(Q)$ *for any functions* $z_1 \in [W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$ *and* $z_2 \in \overset{\circ}{W}{}_2^{2,1}(Q)$ *such that*

$$[z_1]_{W_2^{2,1}(Q)} \leqslant R, \quad \|z_2\|_{W_2^{2,1}(Q)} \leqslant R,$$

where R *is a positive constant.*

P r o o f. We put

$$K(z_1,z_2)\psi = \begin{pmatrix} P_J((z_1 \cdot \nabla)I + (I \cdot \nabla)z_1) & -g\beta P_J(k_3 I) \\ \nabla z_2 \cdot I & \nabla I \cdot z_1 \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix}.$$

Then

$$\begin{aligned} &\|K(z_1,z_2)\psi\|_{[L_2(\Omega)]^3} \\ &\qquad \leqslant [(z_1 \cdot \nabla)v] + \|\nabla w \cdot z_1\| + [(v \cdot \nabla)z_1] + \|\nabla z_2 \cdot v\| + g\beta[k_3 w]. \end{aligned} \tag{46}$$

By using the Hölder inequality and (13), we obtain

$$I_1 = [(z_1 \cdot \nabla)v] \leqslant [z_1]_{L_4(\Omega)} [\nabla v]_{L_4(\Omega)} \leqslant C[z_1]_{L_4(\Omega)} [v]^{3/4}_{W_2^2(\Omega)} [v]^{1/4}.$$

From (20) we have

$$I_1 \leqslant C[z_1]_{L_4(\Omega)} [-P_J \Delta v]^{3/4} [v]^{1/4}. \tag{47}$$

Because $z_1(x,0) = 0$, we see that

$$\int_0^s \int_\Omega \frac{\partial z_1}{\partial t} z_1^3 \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{4} \int_\Omega \int_0^s \frac{\partial}{\partial t}(z_1^4) \, \mathrm{d}t \, \mathrm{d}x = \frac{1}{4} \int_\Omega z_1^4 \, \mathrm{d}x.$$

This implies that

$$\int_\Omega z_1^4 \,\mathrm{d}x \leqslant C[z_1]_{W_2^{2,1}(Q)}[z_1]^3_{L_6(Q)}.$$

The space $W_2^{2,1}(Q)$ is embedded in $L_6(Q)$ (see [13]). Therefore,

$$\int_\Omega z_1^4 \,\mathrm{d}x \leqslant C[z_1]^4_{W_2^{2,1}(Q)}.$$

Hence, from (47) we have

$$I_1 \leqslant CR[-P_J\Delta v]^{3/4}[v]^{1/4}. \tag{48}$$

By analogy, we come to the estimate

$$I_2 = \|\nabla w \cdot z_1\| \leqslant CR\|\Delta w\|^{3/4}\|w\|^{1/4}. \tag{49}$$

The space $W_2^2(\Omega)$ is embedded in $L_\infty(\Omega)$ (see [13]). Therefore,

$$I_3 = [(v \cdot \nabla)z_1] \leqslant C[v]_{L_\infty(\Omega)}[\nabla z_1].$$

By using inequality (13) for the spaces $L_\infty(\Omega)$, $W_2^2(\Omega)$, and $L_2(\Omega)$, we get

$$I_3 \leqslant C[\nabla z_1][-P_J\Delta v]^{1/2}[v]^{1/2}. \tag{50}$$

It is obvious that

$$\begin{aligned}\int_0^s\int_\Omega \frac{\partial z_1}{\partial t}\Delta z_1 \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^s\int_\Omega \frac{\partial \nabla z_1}{\partial t}\nabla z_1 \,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_\Omega\int_0^s \frac{\partial|\nabla z_1|^2}{\partial t}\,\mathrm{d}t\,\mathrm{d}x = -\frac{1}{2}[\nabla z_1]^2.\end{aligned}$$

From this and from (50) it follows that

$$I_3 \leqslant CR[-P_J\Delta v]^{1/2}[v]^{1/2}. \tag{51}$$

Likewise, we come to the estimate

$$I_4 = \|\nabla z_2 \cdot v\| \leqslant CR[-P_J\Delta v]^{1/2}[v]^{1/2}. \tag{52}$$

From (46), (48), (49), (51), and (52) we obtain

$$\begin{aligned}&\|K(z_1,z_2)\psi\|_{[L_2(\Omega)]^3} \\ &\quad\leqslant CR([-P_J\Delta v]^{3/4}[v]^{1/4} + \|\Delta w\|^{3/4}\|w\|^{1/4} + [-P_J\Delta v]^{1/2}[v]^{1/2}) + C\|w\|.\end{aligned} \tag{53}$$

Setting

$$A = \begin{pmatrix} -\nu P_J\Delta & 0 \\ 0 & -\kappa\Delta \end{pmatrix},$$

we get

$$\|A\psi\|_{[L_2(\Omega)]^3} = [-\nu P_J\Delta v] + \|-\kappa\Delta w\|.$$

From this and (53), by virtue of the positive definiteness of the operators $(-P_J\Delta)$ and $(-\Delta)$, it follows that

$$\|K(z_1,z_2)\psi\|_{[L_2(\Omega)]^3} \leqslant C\|A\psi\|^{3/4}_{[L_2(\Omega)]^3}\|\psi\|^{1/4}_{[L_2(\Omega)]^3}. \tag{54}$$

Thus, the operator $K(z_1,z_2)$ is subordinated to the operator A with order $\frac{3}{4}$. Therefore, the statement of the lemma follows from Theorem 2.1 of paper [15]. □

Theorem 3.2. *Let $f(x,t) \in [C^1([0,T];L_2(\Omega))]^2$, $f(x,0)=0$, $\varphi(x,t) \in C^1([0,T];L_2(\Omega))$, $\varphi(x,0)=0$. Then*

$$\sup_{0\leqslant t\leqslant T}\Big[\frac{\partial(u_n-u)}{\partial t}\Big] + \sup_{0\leqslant t\leqslant T}\Big\|\frac{\partial(\theta_n-\theta)}{\partial t}\Big\| \leqslant C(\lambda_{n+1}^{-1/8}+\mu_{n+1}^{-1/8}), \tag{55}$$

$$\sup_{0\leqslant t\leqslant T}[\nabla(u_n-u)] + \sup_{0\leqslant t\leqslant T}\|\nabla(\theta_n-\theta)\| \leqslant C(\lambda_{n+1}^{-1/4}+\mu_{n+1}^{-1/4}). \tag{56}$$

P r o o f. In problem (41)–(45) we put $z_1 = u$, $z_2 = \theta$, where u and θ are the solution of problem (6)–(9), $h = \partial f/\partial t$, $h_1 = \partial\varphi/\partial t$. Then, according to Lemma 3.1, problem (41)–(45) has a unique solution $v \in [W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$, $w \in \overset{\circ}{W}{}_2^{2,1}(Q)$. Now, we set $z_1 = u_n$, $z_2 = \theta_n$, where u_n and θ_n are the solution of the problem (10)–(12). Again, using Lemma 3.1, we obtain that problem (41)–(45) has a unique solution $v(n) = v(n;x,t) \in [W_2^{2,1}(Q)]^2 \cap \overset{\circ}{J}(Q)$, $w(n) = w(n;x,t) \in \overset{\circ}{W}{}_2^{2,1}(Q)$ for each n.

If $\psi = \begin{pmatrix} v \\ w \end{pmatrix}$, $\psi(n) = \psi(n;x,t) = \begin{pmatrix} v(n) \\ w(n) \end{pmatrix}$, $F = \begin{pmatrix} P_J(\partial f/\partial t) \\ \partial\varphi/\partial t \end{pmatrix}$, then

$$\frac{\partial\psi}{\partial t} + A\psi + K(u,\theta)\psi = F, \quad \psi(x,0)=0, \tag{57}$$

$$\frac{\partial\psi(n)}{\partial t} + A\psi(n) + K(u_n,\theta_n)\psi(n) = F, \quad \psi(n;x,0)=0. \tag{58}$$

Let $P_n = \begin{pmatrix} P_{n1} & 0 \\ 0 & P_{n2} \end{pmatrix}$ be the orthogonal projection of $[L_2(\Omega)]^3$ onto the linear span of the elements $\{e_i, m_i\}_{i=1}^n$. Then from (57), (58) we have

$$\begin{aligned} \frac{\partial(\psi-\psi(n))}{\partial t} &+ A(\psi-\psi(n)) + P_nK(u,\theta)(\psi-\psi(n)) \\ &= (P_n - I)K(u,\theta)(\psi-\psi(n)) + (K(u_n,\theta_n)-K(u,\theta))\psi(n). \end{aligned} \tag{59}$$

According to inequality (54), the operator $K(u,\theta)$ is subordinated to the operator A with order $\frac{3}{4}$. Hence, from [15] it follows that the operator $((\partial/\partial t)+A+P_nK(u,\theta))$ is invertible and

$$\Big(\frac{\partial}{\partial t}+A+P_nK(u,\theta)\Big)^{-1} = \Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big(I-P_n\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1}K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big).$$

Therefore, from (59) we have

$$\begin{aligned}(60)\quad \psi-\psi(n) &= \Big(\frac{\partial}{\partial t}+A\Big)^{-1} \\ &\times\Big(I-P_n\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1}K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big) \\ &\times\big((P_n-I)K(u,\theta)(\psi-\psi(n))+(K(u_n,\theta_n)-K(u,\theta))\psi(n)\big).\end{aligned}$$

We estimate each summand on the right-hand side of (60):

$$\begin{aligned}\|J_1\|_{[L_2(\Omega)]^3} &= \Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big(I-P_n\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1} \\ &\quad\times K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big)(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(\Omega)]^3} \\ &\leqslant \Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(\Omega)]^3} \\ &\quad+\Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1} \\ &\quad\times K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(\Omega)]^3}.\end{aligned}$$

In [15] the estimates were established, which with reference to the given problem can be written as

$$\begin{aligned}(61)\qquad &\Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)g\Big\|_{[L_2(\Omega)]^3} \\ &\qquad\leqslant C(\lambda_{n+1}^{-1/2}+\mu_{n+1}^{-1/2})\|g\|_{[L_2(Q)]^3},\quad \forall\, g\in[L_2(Q)]^3,\end{aligned}$$

$$(62)\qquad \Big\|\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1}\Big\|_{[L_2(Q)]^3\to[L_2(Q)]^3}\leqslant C,$$

$$(63)\qquad \Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}g\Big\|_{[L_2(\Omega)]^3}\leqslant C\|g\|_{[L_2(Q)]^3},\quad \forall\, g\in[L_2(Q)]^3.$$

Taking into account (61)–(63), we come to the estimate

$$\begin{aligned}(64)\quad \|J_1\|_{[L_2(\Omega)]^3} &\leqslant C(\lambda_{n+1}^{-1/2}+\mu_{n+1}^{-1/2})\|K(u,\theta)(\psi-\psi(n))\|_{[L_2(Q)]^3} \\ &\quad+C\Big\|K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(Q)]^3}.\end{aligned}$$

From (54) we have

$$\|K(u,\theta)(\psi-\psi(n))\|_{[L_2(Q)]^3} \leqslant C\|A(\psi-\psi(n))\|^{3/4}_{[L_2(Q)]^3}\|\psi-\psi(n)\|^{1/4}_{[L_2(Q)]^3}.$$

From Lemma 3.1 it follows that

$$\sup_{0\leqslant t\leqslant T}\|\psi(n)\|_{[L_2(\Omega)]^3} \leqslant C,$$

$$\|A\psi(n)\|_{[L_2(Q)]^3} \leqslant C. \tag{65}$$

Therefore, from the last three inequalities we obtain

$$\|K(u,\theta)(\psi-\psi(n))\|_{[L_2(Q)]^3} \leqslant C. \tag{66}$$

Now we shall consider the second summand of (64):

$$\begin{aligned}
&\Big\|K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(Q)]^3}\\
&\leqslant C\Big\|A\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|^{3/4}_{[L_2(Q)]^3}\\
&\quad\times\Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|^{1/4}_{[L_2(Q)]^3}.
\end{aligned}$$

Thus, taking into account inequalities (61) and (66), we derive

$$\Big\|K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}(P_n-I)K(u,\theta)(\psi-\psi(n))\Big\|_{[L_2(Q)]^3} \leqslant C(\lambda_{n+1}^{-1/8}+\mu_{n+1}^{-1/8}).$$

This, together with (64) and (66), leads to the estimate

$$\|J_1\|_{[L_2(\Omega)]^3} \leqslant C(\lambda_{n+1}^{-1/8}+\mu_{n+1}^{-1/8}). \tag{67}$$

We estimate the second summand on the right-hand side of (60):

$$\begin{aligned}
\|J_2\|_{[L_2(\Omega)]^3} &= \Big\|\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big(I-P_n\Big(I+K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}P_n\Big)^{-1}\\
&\quad\times K(u,\theta)\Big(\frac{\partial}{\partial t}+A\Big)^{-1}\Big)\big((K(u_n,\theta_n)-K(u,\theta))\psi(n)\big)\Big\|_{[L_2(\Omega)]^3}\\
&\leqslant C\|(K(u_n,\theta_n)-K(u,\theta))\psi(n)\|_{[L_2(Q)]^3}.
\end{aligned} \tag{68}$$

It is obvious that

$$\begin{aligned}
&\|(K(u_n,\theta_n)-K(u,\theta))\psi(n)\|_{[L_2(Q)]^3}\\
&\quad\leqslant [P_J(((u_n-u)\cdot\nabla)v(n))]_{L_2(Q)}+\|\nabla w(n)(u_n-u)\|_{L_2(Q)}\\
&\qquad+[P_J((v(n)\cdot\nabla)(u_n-u))]_{L_2(Q)}+\|\nabla(\theta_n-\theta)\cdot v(n)\|_{L_2(Q)}.
\end{aligned} \tag{69}$$

Now we shall consider each summand on the right-hand side of (69). Applying the Hölder inequality, we get

$$I_1 = [P_J(((u_n - u) \cdot \nabla)v(n))]_{L_2(Q)} \leqslant [u_n - u]_{L_4(Q)}[\nabla v(n)]_{L_4(Q)}.$$

Next, we use the estimate (65) and inequality (13) for the spaces $L_4(\Omega)$, $W_2^2(\Omega)$, $L_2(\Omega)$. Therefore,

$$I_1 \leqslant C\bigg(\int_0^T [u_n - u]^2_{W_2^2(\Omega)}[u_n - u]^3 \,\mathrm{d}t\bigg)^{1/4}.$$

From this and from Theorem 3.1 we obtain

$$I_1 \leqslant C\lambda_{n+1}^{-3/8}. \tag{70}$$

By analogy, we find that

$$\begin{aligned} I_2 &= \|\nabla w(n)(u_n - u)\|_{L_2(Q)} \leqslant C\lambda_{n+1}^{-3/8}, \\ I_3 &= [P_J((v(n)\cdot\nabla)(u_n - u))]_{L_2(Q)} \leqslant [v(n)]_{L_6(Q)}[\nabla(u_n - u)]_{L_3(Q)}. \end{aligned} \tag{71}$$

Now we apply estimate (65) and inequality (13) for the spaces $W_3^1(\Omega)$, $W_2^2(\Omega)$, $L_2(\Omega)$. Then

$$I_3 \leqslant C\bigg(\int_0^T [u_n - u]^2_{W_2^2(\Omega)}[u_n - u] \,\mathrm{d}t\bigg)^{1/3}.$$

From the preceding relation and also from Lemma 2.1 and Theorem 3.1 it follows that

$$I_3 \leqslant C\lambda_{n+1}^{-1/6}. \tag{72}$$

Likewise, we obtain

$$I_4 = \|\nabla(\theta_n - \theta)v(n)\|_{L_2(Q)} \leqslant C\mu_{n+1}^{-1/6}. \tag{73}$$

According to inequalities (68)–(73), we have

$$\|J_2\|_{[L_2(\Omega)]^3} \leqslant C(\lambda_{n+1}^{-1/6} + \mu_{n+1}^{-1/6}). \tag{74}$$

From (60), (67), and (74) it follows that

$$\|\psi - \psi(n)\|_{[L_2(\Omega)]^3} \leqslant C(\lambda_{n+1}^{-1/8} + \mu_{n+1}^{-1/8}). \tag{75}$$

Putting $\psi_n = \begin{pmatrix} v_n \\ w_n \end{pmatrix}$, $z_n = \begin{pmatrix} u_n \\ \theta_n \end{pmatrix}$, $z = \begin{pmatrix} u \\ \theta \end{pmatrix}$, we get

$$\frac{\partial \psi_n}{\partial t} + A\psi_n + P_n K(u_n, \theta_n)\psi_n = P_n F, \quad \psi_n(x,0) = 0. \tag{76}$$

Equations (76) and (58) lead to the relation

$$\begin{aligned}\frac{\partial(\psi_n - \psi(n))}{\partial t} + A(\psi_n - \psi(n)) + P_n K(u_n, \theta_n)(\psi_n - \psi(n))& \\ = (I - P_n)(K(u_n, \theta_n)\psi(n) - F).&\end{aligned}$$

By analogy with the proof of (75), it is easy to establish that

$$\|\psi_n - \psi(n)\|_{[L_2(\Omega)]^3} \leqslant C(\lambda_{n+1}^{-1/8} + \mu_{n+1}^{-1/8}). \tag{77}$$

From (28) and (29) it follows that

$$\lim_{n\to\infty} \Big\|\psi_n - \frac{\partial z}{\partial t}\Big\|_{[L_2(Q)]^3} = 0. \tag{78}$$

It is obvious that

$$\Big\|\psi - \frac{\partial z}{\partial t}\Big\|_{[L_2(Q)]^3} \leqslant \|\psi - \psi(n)\|_{[L_2(Q)]^3} + \|\psi_n - \psi(n)\|_{[L_2(Q)]^3} + \Big\|\psi_n - \frac{\partial z}{\partial t}\Big\|_{[L_2(Q)]^3}.$$

The last inequality, together with (75), (77), and (78) leads to the equality $\psi = \partial z/\partial t$ almost everywhere in Q. Since

$$\Big\|\frac{\partial(z_n - z)}{\partial t}\Big\|_{[L_2(\Omega)]^3} \leqslant \Big\|\frac{\partial z}{\partial t} - \psi(n)\Big\|_{[L_2(\Omega)]^3} + \|\psi(n) - \psi_n\|_{[L_2(\Omega)]^3},$$

from (75) and (77) we obtain estimate (55).

Because z_n is the solution of problem (10)–(12), we see that

$$\begin{aligned}\|Az_n\|_{[L_2(\Omega)]^3} \leqslant \Big\|\frac{\partial z_n}{\partial t}\Big\|_{[L_2(\Omega)]^3} &+ [P_{n1}P_J((u_n \cdot \nabla)u_n)] + \|P_{n2}(\nabla\theta_n \cdot u_n)\| \\ &+ g\beta[P_{n1}P_J(k_3\theta_n)] + [P_{n1}P_J f] + \|P_{n2}\varphi\|.\end{aligned}$$

Hence, from (55) and (17) we have

$$\|Az_n\|_{[L_2(\Omega)]^3} \leqslant C + [(u_n \cdot \nabla)u_n] + \|\nabla\theta_n \cdot u_n\|.$$

Applying the Hölder inequality, we obtain

$$\|Az_n\|_{[L_2(\Omega)]^3} \leqslant C + [u_n]_{L_6(\Omega)}[\nabla u_n]_{L_3(\Omega)} + [u_n]_{L_6(\Omega)}\|\nabla\theta_n\|_{L_3(\Omega)}.$$

Since the space $W_2^1(\Omega)$ is embedded in $L_6(\Omega)$, from (23) and (24) it follows that

$$\|Az_n\|_{[L_2(\Omega)]^3} \leqslant C(1 + [\nabla u_n]_{L_3(\Omega)} + \|\nabla\theta_n\|_{L_3(\Omega)}).$$

Now we use inequality (13) for the spaces $W_3^1(\Omega), W_2^2(\Omega), L_2(\Omega)$, and inequalities (17) and (18). Then

$$\|Az_n\|_{[L_2(\Omega)]^3} \leqslant C(1 + [u_n]^{2/3}_{W_2^2(\Omega)} + \|\theta_n\|^{2/3}_{W_2^2(\Omega)}).$$

This and coercive inequality (20) yield

$$\|Az_n\|_{[L_2(\Omega)]^3} \leqslant C(1 + \|Az_n\|^{2/3}_{[L_2(\Omega)]^3}).$$

Therefore,

$$\sup_{0\leqslant t\leqslant T} \|Az_n\|_{[L_2(\Omega)]^3} \leqslant C.$$

We use the moment inequality (see [9]), concluding that

$$[\nabla(u_n - u)] + \|\nabla(\theta_n - \theta)\| \leqslant \|A(z_n - z)\|^{1/2}_{[L_2(\Omega)]^3} \|z_n - z\|^{1/2}_{[L_2(\Omega)]^3}.$$

Hence, the desired estimate (56) follows. The proof of the theorem is complete. □

Remark. In this paper, the unique solvability of the approximate problem for the heat convection equations was proved. The convergence estimates for the Galerkin approximations and their derivatives in the uniform norm were obtained. In this case the eigenfunctions are not represented in an explicit form. However, in numerical realization of the method the eigenfunctions can be found approximately.

The technique can be applied to the study of other initial boundary value problems for the heat convection (e.g. with a free boundary). Then the eigenfunctions can be written explicitly for certain types of domains.

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