Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 3 (1962), No. 1, 15--16

Persistent URL: http://dml.cz/dmlcz/142141

1962 ACTA UNIVERSITATIS CAROLINAE — MATHEMATICA ET PHYSICA NO. 1 PAG. 15—16

A NOTE ON AN INEQUALITY

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Received October 30, 1961.

In [1] the authors use the following lemma (see [1], page 1233, Lemma 1, proof):

If $0 < y < y + \varepsilon \leq 1$ *and* $0 < \varepsilon \leq \frac{1}{e}$, *then*

$$|(y + \varepsilon) \lg (y + \varepsilon) - y \lg y| \leq \varepsilon^{1/2}$$

However, it may be of some interest perhaps even when applications are considered that the condition $0 < \varepsilon \leq \frac{1}{e}$ is unnecessary and that it may be generalised as follows:

LEMMA: *For every* $\alpha \in (0, 1)$ *there is a constant* a_α *such that if* $0 < y < y + \varepsilon \leq 1$ *then*

$$|(y + \varepsilon) \lg (y + \varepsilon) - y \lg y| \leq a_\alpha \varepsilon^\alpha .$$

Proof: Observe first that there are constants $\beta \geq 1$, $\gamma \geq 1$, $\delta \geq 1$ such that $(1 + u)^{\frac{1}{u}} \leq \beta$, $(1 + u)^{-\frac{1}{u}} \leq \gamma$ and $v^{-v^{1-\alpha}} \leq \delta$ for every $u \in (0, \infty)$ and for every $v \in (0, 1\rangle$. We conclude that $-v^{1-\alpha} \lg v \leq \lg \delta$, $-v \lg v \leq v^\alpha \lg \delta$ and finally $v^{-v} \leq \delta^{v^\alpha}$ for every $v \in (0, 1\rangle$.

Now put $\lambda_\alpha = \max [\beta, \gamma\delta]$ and consider $0 < y < y + \varepsilon \leq 1$. Let $(y + \varepsilon)^{y+\varepsilon} \geq y^y$.
Then

$$(y + \varepsilon)^{y+\varepsilon} y^{-y} = \left(1 + \frac{\varepsilon}{y}\right)^{\frac{y}{\varepsilon} \cdot \varepsilon} (y + \varepsilon)^\varepsilon \leq \beta^\varepsilon . 1 \leq \lambda_\alpha^\varepsilon \leq \lambda_\alpha^{\varepsilon^\alpha}.$$

In case that $y^y > (y + \varepsilon)^{y+\varepsilon}$ we have $y^y . (y + \varepsilon)^{-(y+\varepsilon)} =$

$$= \left(1 + \frac{\varepsilon}{y}\right)^{-\frac{y}{\varepsilon} \cdot \varepsilon} (y + \varepsilon)^{-\varepsilon} \leq \gamma^\varepsilon . \varepsilon^{-\varepsilon} \leq \gamma^{\varepsilon^\alpha} \delta^{\varepsilon^\alpha} = (\gamma\delta)^{\varepsilon^\alpha} \leq \lambda_\alpha^{\varepsilon^\alpha}.$$

In any case $|(y + \varepsilon) \lg (y + \varepsilon) - y \lg y| \leq \varepsilon^\alpha \lg \lambda_\alpha$.
Now by putting $a_\alpha = \lg \lambda_\alpha$ our lemma is proved.

Note: By elementary considerations best value for β, γ, δ and a_α may be found:

$$\beta = e, \gamma = 1, \delta = \exp\left\{\frac{1}{e(1-\alpha)}\right\}, a_\alpha = \max\left\{1, \frac{1}{e(1-\alpha)}\right\}$$

For every fixed ε, $0 < \varepsilon < 1$, the function $a_\alpha \varepsilon^\alpha$ takes its minimum for $\alpha_0 = 1 - e^{-1} = 0{,}63212056\ldots$ which is the best value for α's. In this case yet $a_{\alpha_0} = 1$.

COROLARY:

If $0 < y < y + \varepsilon \leq 1$, *then*

$$|(y + \varepsilon)\lg(y + \varepsilon) - y \lg y| \leq \varepsilon^{\alpha_0},$$

where $\alpha_0 = 1 - e^{-1}$.

POZNÁMKA K JEDNÉ NEROVNOSTI

Souhrn

V práci se dokazuje tvrzení: Ke každému $\alpha \in (0, 1)$ existuje a_α takové, že je-li $0 < y < y + \varepsilon \leq 1$ pak $|(y + \varepsilon)\lg(y + \varepsilon) - y \lg y| \leq a_\alpha \varepsilon^\alpha$. Nejlepší odhad tohoto typu nastává pro $\alpha_0 = 1 - e^{-1}$ kde jest $a_{\alpha_0} = 1$.

REFERENCES

[1] DAVID BLACKWELL, LEO BREIMAN, A. J. THOMASIAN: The capacity of a class of channels, The Annals of Math. Statistics vol 30 (1959), p. 1229—1241.

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