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## A NOTE ON AN INEQUALITY

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In [1] the authors use the following lemma (see [1], page 1233, Lemma 1, proof):

If  $0 < y < y + \varepsilon \leq 1$  and  $0 < \varepsilon \leq \frac{1}{e}$ , then

$$|(y + \varepsilon) \lg(y + \varepsilon) - y \lg y| \leq \varepsilon^{1/2}.$$

However, it may be of some interest perhaps even when applications are considered that the condition  $0 < \varepsilon \leq \frac{1}{e}$  is unnecessary and that it may be generalised as follows:

LEMMA: For every  $\alpha \in (0, 1)$  there is a constant  $a_\alpha$  such that if  $0 < y < y + \varepsilon \leq 1$  then

$$|(y + \varepsilon) \lg(y + \varepsilon) - y \lg y| \leq a_\alpha \varepsilon^\alpha.$$

Proof: Observe first that there are constants  $\beta \geq 1$ ,  $\gamma \geq 1$ ,  $\delta \geq 1$  such that  $(1 + u)^{\frac{1}{u}} \leq \beta$ ,  $(1 + u)^{-\frac{1}{u}} \leq \gamma$  and  $v^{-v^{1-\alpha}} \leq \delta$  for every  $u \in (0, \infty)$  and for every  $v \in (0, 1)$ . We conclude that  $-v^{1-\alpha} \lg v \leq \lg \delta$ ,  $-v \lg v \leq v^\alpha \lg \delta$  and finally  $v^{-v} \leq \delta^{v^\alpha}$  for every  $v \in (0, 1)$ .

Now put  $\lambda_\alpha = \max[\beta, \gamma\delta]$  and consider  $0 < y < y + \varepsilon \leq 1$ . Let  $(y + \varepsilon)^{y+\varepsilon} \geq y^y$ .

Then

$$(y + \varepsilon)^{y+\varepsilon} y^{-y} = \left(1 + \frac{\varepsilon}{y}\right)^{\frac{y}{\varepsilon}} (y + \varepsilon)^\varepsilon \leq \beta^\varepsilon \cdot 1 \leq \lambda_\alpha^\varepsilon \leq \lambda_\alpha^{y+\varepsilon}.$$

In case that  $y^y > (y + \varepsilon)^{y+\varepsilon}$  we have  $y^y \cdot (y + \varepsilon)^{-(y+\varepsilon)} =$

$$= \left(1 + \frac{\varepsilon}{y}\right)^{-\frac{y}{\varepsilon}} (y + \varepsilon)^{-\varepsilon} \leq \gamma^\varepsilon \cdot \varepsilon^{-\varepsilon} \leq \gamma^\varepsilon \delta^{\varepsilon^\alpha} = (\gamma\delta)^{\varepsilon^\alpha} \leq \lambda_\alpha^{y+\varepsilon}.$$

In any case  $|(y + \varepsilon) \lg(y + \varepsilon) - y \lg y| \leq \varepsilon^\alpha \lg \lambda_\alpha$ .

Now by putting  $a_\alpha = \lg \lambda_\alpha$  our lemma is proved.

Note: By elementary considerations best value for  $\beta$ ,  $\gamma$ ,  $\delta$  and  $a_\alpha$  may be found:

$$\beta = e, \gamma = 1, \delta = \exp \left\{ \frac{1}{e(1-\alpha)} \right\}, a_\alpha = \max \left\{ 1, \frac{1}{e(1-\alpha)} \right\}$$

For every fixed  $\varepsilon$ ,  $0 < \varepsilon < 1$ , the function  $a_\alpha e^{\alpha\varepsilon}$  takes its minimum for  $\alpha_0 = 1 - e^{-1} = 0,63212056\dots$  which is the best value for  $\alpha$ 's. In this case yet  $a_{\alpha_0} = 1$ .

**COROLARY:**

If  $0 < y < y + \varepsilon \leq 1$ , then

$$|(y + \varepsilon) \lg(y + \varepsilon) - y \lg y| \leq \varepsilon^{\alpha_0},$$

where  $\alpha_0 = 1 - e^{-1}$ .

#### POZNÁMKA K JEDNÉ NEROVNOSTI

##### Souhrn

V práci se dokazuje tvrzení: Ke každému  $\alpha \in (0, 1)$  existuje  $a_\alpha$  takové, že je-li  $0 < y < y + \varepsilon \leq 1$  pak  $|(y + \varepsilon) \lg(y + \varepsilon) - y \lg y| \leq a_\alpha \varepsilon^\alpha$ . Nejlepší odhad tohoto typu nastává pro  $\alpha_0 = 1 - e^{-1}$  kde jest  $a_{\alpha_0} = 1$ .

##### REFERENCES

- 11] DAVID BLACKWELL, LEO BREIMAN, A. J. THOMASIAN: The capacity of a class of channels, The Annals of Math. Statistics vol 30 (1959), p. 1229—1241.

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