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A NOTE ON THE MINIMUM-IDEAL OF A GIVEN MATRIX

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The aim of the present note is to show how the notion of the minimum-ideal and related methods introduced by N. H. McCoy [1] may be used for generalising a theorem on singular matrices due to O. BORŮVKA [2].

Let A be a given square matrix with elements in an arbitrary commutative ring $\mathfrak{R}$ with unit element. The *minimum-ideal* $\mathfrak{J}$ of A consists of all polynomials $\varphi(x) \in \mathfrak{R}[x]$ with the property $\varphi(A) = O$. It was already proved by McCoy that, given any $\psi(x) \in \mathfrak{R}[x]$, the matrix $\psi(A)$ has its inverse matrix over $\mathfrak{R}$ if and only if $(\mathfrak{J}, \psi(x)) = (1)$. From that it follows immediatly that $\mathfrak{J}$ is not a maximal ideal of $\mathfrak{R}[x]$ if and only if there is a polynomial $\psi(x) \in \mathfrak{R}[x]$ such that $\psi(A) \neq O$ and such that $\psi(A)$ has no inverse matrix over $\mathfrak{R}$.

However, a generalisation of Borůvka's theorem (see [2], Theorem 2 or the end of this paper) may be got as well as follows:

THEOREM: *Let $\mathfrak{R}$ be an arbitrary commutative ring with unit element and let A be a square matrix of order n with elements in $\mathfrak{R}$. The minimum-ideal $\mathfrak{J}$ of A is not prime if and only if there is a polynomial $\psi(x) \in \mathfrak{R}[x]$ such that $\psi(A) \neq O$ and such that at least one of the following two conditions is satisfied*:

α) $\det((\psi(A)) = 0$,

β) *there exists a square matrix B of order n with elements in $\mathfrak{R}$ such that* $\det(B) \neq 0$ *and* $\psi(A)B = O$.

Proof: Suppose first of all that $\mathfrak{J}$ is not prime and consider $\varphi(x)\lambda(x) \in \mathfrak{J}$, $\varphi(x) \notin \mathfrak{J}$, $\lambda(x) \notin \mathfrak{J}$. It follows that $\varphi(A)\lambda(A) = O$, $\varphi(A) \neq O$, $\lambda(A) \neq O$. If $\det(\lambda(A)) = 0$ the condition α) holds for $\psi(x) = \lambda(x)$. In the opposite case we put $\psi(x) = \varphi(x)$ and $B = \lambda(A)$ and we get β).

To make the proof in the opposite direction consider at first the case $\psi(A) \neq O$ and $\det(\psi(A)) = 0$. We denote by E the identity matrix of order n and we put $M(x) = \psi(x)E - \psi(A)$, $\delta(x) = \det(M(x))$. From $M(A) = O$ follows (see [1], Theorem 3') $\delta(A) = O$ and $\delta(x) \in \mathfrak{J}$. Further from $\det(\psi(A)) = 0$, using the well-known expansion of $\det(M(x))$ in powers of $\psi(x)$, we get $\delta(x) = [\psi(x)]^m[a + \psi(x)\lambda(x)]$ with $a \neq 0$, $\lambda(x) \in \mathfrak{R}[x]$, $1 \leq m \leq n$. Now the assumption for $\mathfrak{J}$ to be prime leads to a contradiction as follows. It would be $a + \psi(x)\lambda(x) \in \mathfrak{J}$, $aE + \psi(A)\lambda(A) = O$. From $a \neq 0$ we have $\lambda(x) \notin \mathfrak{J}$. It follows

$$\psi(A)\lambda(A) = -aE,\ 0 = \det(\psi(A))\det(\lambda(A)) = (-a)^n,$$

$$[\psi(A)\lambda(A)]^n = (-a)^n E = O.$$

It must be $\lambda(x) \in \mathfrak{J}$ what is a contradiction with $\lambda(x) \notin \mathfrak{J}$.

Consider now the case $\psi(A) \neq O$, $\det(B) \neq 0$ and $\psi(A)B = O$. Put $M(x) = \psi(x)B$ so that $M(A) = O$. If $\delta(x) = \det(M(x))$, it follows again $\delta(A) = O$. But $\delta(x) = [\psi(x)]^n \det(B) \in \mathfrak{J}$ and the assumption for $\mathfrak{J}$ to be prime would give $\det(B) \in \mathfrak{J}$ and $\det(B) = 0$ contrary to our hypothesis. Thus our theorem is proved.

In the case that $\mathfrak{R}$ is a field the condition β) never holds and we get Borůvka's theorem (see [2], Theorem 2): *The minimal polynomial $\varphi(x)$ of A is reducible over $\mathfrak{R}$ if and only if there is a polynomial $\psi(x) \in \mathfrak{R}[x]$ such that $\psi(A) \neq O$ and* $\det(\psi(A)) = 0$.

POZNÁMKA O M-IDEÁLU DANÉ MATICE

Souhrn

V práci se používá pojmu *M-ideálu (minimum-ideal)* čtvercové matice nad libovolným komutativním okruhem s jednotkovým prvkem, zavedeného N. H. McCoyem [1], ke zobecnění jedné věty O. Borůvky ([2], věta 2). Uvádí se nutná a postačující podmínka k tomu, aby M-ideál dané matice nebyl prvoideál.

REFERENCES

[1] N. H. McCoy: Concerning matrices with elements in a commutative ring, Bull. Amer. Math. Soc. vol. 45., (1939), p. 280—284.

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