Jan Vlček

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FINITE DIFFERENCE METHOD FOR THE CAUCHY PROBLEM FOR LINEAR HYPERBOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

ŘEŠENÍ CAUCHYOVY ÚLOHY PRO LINEÁRNI HYPERBOLICKOU SOUSTAVU S NESPOJITÝMI KOEFICIENTY DIFERENČNÍ METODOU

РЕШЕНИЕ ЗАДАЧИ КОШИ ДЛЯ ЛИНЕЙНОЙ ГИПЕРБОЛИЧЕСКОЙ СИСТЕМЫ С РАЗРЫВНЫМИ КОЭФФИЦИЕНТАМИ МЕТОДОМ КОНЕЧНЫХ РАЗНОСТЕЙ

JAN VLČEK

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1. ENERGY INEQUALITIES

Denotations and definitions.

We consider in the domain $M = E_1 x \langle o, T \rangle, o < T < + \infty$ in the plane $\{x, t\}$ the system

$$\frac{\partial u}{\partial t} + \frac{\partial (Au)}{\partial x} + Bu = f \tag{1}$$

where $A = (a^{kl})$, $B = (b^{k,l})$, $k, l = 1, 2, \ldots, r$ are matrices, $f = (f^{(1)}, f^{(2)}, \ldots, f^{(r)})$, $u = (u^{(1)}, u^{(2)}, \ldots, u^{(r)})$ are vector-functions. We suppose, during all the paper, all the functions to be real. We say that the function $u(x, t)$ is the generalized solution (solution almost everywhere) of (1) with initial condition

$$u(o, x) = \varphi(x) \tag{2}$$

where $\varphi(x) = (\varphi^{(1)}(x), \varphi^{(2(}(x), \ldots, \varphi^{(r)}_{(x)}$ is a given function, if

1. $u^{(m)}, \dfrac{\partial u^{(m)}}{\partial t}, \dfrac{\partial (Au)^{(m)}}{\partial x} \in L_2(M)$, $m = 1, 2, \ldots, r$,
2. u satisfies the system (1) almost everywhere,
3. u admits initial values (2) in the following sense:

$$\lim_{t \to 0+} \int_{E_1} | u^{(m)}(x, t) - \varphi^{(m)}(x) |^2 \, dx = 0, m = 1, 2, \ldots, r.$$

We say that the function μ, defined on M, has the compact support with respect to x in M if there exists a positive constant L such that $\mu = 0$ for $t \in \langle o, T \rangle$, $| x | > L$. We denote by $\varepsilon^{(k)}(M)$ the set of functions with compact support with respect to x in M such that all their derivatives up to the order k are continuous in M. We denote, for arbitrary open domain Ω in the plane $\{x, t\}$, by $\varepsilon^{(k)}(\Omega)$ the set of the functions with continuous derivatives up to the order k in Ω, having compact support in Ω.

Suppose that

(α) the elements of A are bounded piecewise continuous functions on M with unique discontinuity line $x = 0$, and they have bounded first derivatives with respect to t, satisfying the Lipschitz condition with respect to t. A is symmetric and positive definite uniformely in M. The elements of B are bounded piecewise continuous in M, satisfying the Lipschitz condition with respect to t.

(β) The components of f are piecewise continuous in M, satisfying the Lipschitz condition with respect to t and they have the compact support with resp. to x.

The components of φ are piecewise continuous with compact support in E_1 and such that the derivative $\partial \dfrac{d(A(x, o)\varphi(x)}{dx}$ is bounded.

We construct, in the $\{x, t\}$ plane, the net with mesh sizes $\Delta x = h$, $\Delta t = \tau$ and netpoints $(ih, j\tau)$, i, j integers; if μ is a function defined in the net points, we write $\mu_{ij} = \mu\,(ih, j\tau)$. We introduce the operators

$$\Delta_1 \mu_{ij} = \frac{\mu_{i+1j} - \mu_{i-1j}}{2h}, \quad \Delta_2 \mu_{ij} = \frac{\mu_{ij+1} - \mu_{ij-1}}{2\tau}, \quad \mathcal{J}\mu_{ij} = \mu_{ij+1} + \mu_{ij-1}.$$

We shall use, for solving the probleme (1), (2) the following explicit difference scheme:

In the net points we define the function u as follows:

(3) $$u_{i1} = u_{i0} = \varphi(ih) \text{ for every integer } i.$$

For every integer i and $0 \leq j \leq \left[\frac{T}{\tau}\right]$ we write the difference equations

(4) $$\Delta_2 u_{ij} + \Delta_1 v_{ij} + (Bu)_{ij} = f_{ij},$$

where $v = Au$.

(3) and (4) determine u_{ij} for every integer i and $j = -1, 0, \ldots, \left[\frac{T}{\tau}\right] + 1$: we take in the discontinuity points the left-hand (or right-hand) limit values. The symbol $\sum_i$ signifies the summation over all the integers i. We denote by Ω_p, for a positive integer p, the set of all the couples of integers i, j satisfying the inequalities $1 \leq j \leq p - 1$; $u \,.\, v$ is the scalar product of the vectors u, v, $u^2 = u.u$, $(u, v)_{L_2(G)} = \int_G u \,.\, vdG$, $\| u \|^2_{L_2(G)} = (u, u)_G$ for an arbitrary set G and for vector functions u, v defined on G. We denote by $L_2(G)$ the set of the vector functions u such that $\| u \|_{L_2(G)} < +\infty$.

Lemma 1.1 There exists a positive integer q such that for all integer i satisfying the inequalities $i \geq q$, $i \leq -q$, $0 \leq j \leq \left[\frac{T}{\tau}\right] + 1$ holds $u_{ij} = 0$, where u_{ij} is the function defined by (3), (4).

Proof. There exists a q_0 such that for $i \geq q_0$, $i \leq -q_0$ $0 \leq j \leq \left[\frac{T}{\tau}\right] f_{ij} = 0$, $\varphi(ih) = 0$. Hence we can take $q = q_0 + \left[\frac{T}{\tau}\right]$.

Lemma 1.2 For arbitry mesh functions μ_{ij}, ξ_{ij}, satisfying the conditions

$\mu_{ij} = \xi_{ij} = 0$ for $0 \leq j \leq \left[\frac{T}{\tau}\right]$ and $i = q_1$, $i = q_2$ ($q_1 < q_2$ are two integers) the relation

(5) $$\sum_{i=q_1}^{q_2} \Delta_1 \mu_{ij} \xi_{ij} = -\sum_{i=q_1}^{q_2} \mu_{ij} \Delta_1 \xi_{ij}$$

holds for every integer j, satisfying the inequalities $0 \leq j \leq \left[\frac{T}{\tau}\right]$.

Proof. The formula (5) follows immediately from the evident formula

(6) $$\sum_{k=1}^{l} [\mu(k+1) - \mu(k-1)]\, \xi(k) = -\sum_{k=1}^{l} \mu_k [\xi(k+1) - \xi(k-1)] + $$
$$+ \mu(l+1)\xi(l) - \mu(1)\xi(0) + \mu(l)\xi(l+1) - \mu(0)\xi(1),$$

which is true for arbitrary functions μ, ξ of integer argument.

Theorem 1. There exists a positive constant K, independant on τ, h, f, φ such that for u_{ij} defined by (3) and (4) the inequality

(7) $$h \sum_i u_{ip}^2 \leq K\{h \sum_i (u_{i0}^2 + u_{i1}^2) + \tau h \sum_{\Omega_p} f_{ij}^2\}$$

holds for τ, h, $\varkappa = \frac{\tau}{h}$ sufficiently small and for arbitrary p, $1 \leq p \leq \left[\frac{T}{\tau}\right]$.

Proof. Multipliing (4) by Jv_{ij} and summing over Ω_p, we get

(8) $$\sum_{\Omega_p} \Delta_2 u_{ij} \,.\, Jv_{ij} + \sum_{\Omega_p} \Delta_1 v_{ij} Jv_{ij} + \sum_{\Omega_p} (Bu)_{ij} Jv_{ij} = \sum_{\Omega_p} f_{ij} \,.\, Jv_{ij}\,.$$

Let us arrange the expressions in (8) as follows:

$$\sum_{\Omega_p} \Delta_2 u_{ij} Jv_{ij} = \sum_{\Omega_p} \Delta_2 (u_{ij} v_{ij}) + \sum_{\Omega_p} \frac{u_{ij+1} v_{ij-1} - u_{ij-1} v_{ij+1}}{2\tau}$$

Put $\vartheta(j) = \sum_i u_{ij}^2$. We have also

$$\sum_{\Omega_p} \Delta_2 (u_{ij} v_{ij}) = \sum_i \frac{u_{ip} v_{ip} + u_{ip-1} v_{ip-1}}{2\tau} - \sum_i \frac{u_{i1} v_{i1} + v_{i0} v_{i0}}{2\tau}$$

By the boundedness, the symmetry and the positivity of A we obtain (the Kj are positive constants)

(9) $$\sum_{\Omega_p} \Delta_2 (u_{ij} v_{ij}) \geq \frac{c}{2\tau} (\vartheta(p) + \vartheta(p_{-1})) - \frac{K_1}{2\tau} (\vartheta(o) + \vartheta(1))$$

It holds also

$$p \sum_{\Omega_p} \frac{u_{i\,j+1} v_{i\,j-1} - u_{ij-1} v_{ij+1}}{2\tau} = \sum_{\Omega_p} u_{ij+1} \left(\frac{A_{ij-1} - A_{ij+1}}{2t}\right) u_{ij-1},$$

(10) $$\left| \sum_{\Omega_p} \frac{u_{ij+1} v_{ij-1} - u_{ij-1} v_{ij+1}}{2\tau} \right| \leq K_2 \sum_{j=o}^{p} \vartheta(j),$$

where the Lipschitz continuity of A with respect to t was also used.

It follows by the lemma 1.2:

$$\sum_{\Omega_p} \Delta_1 v_{ij} \mathcal{J} v_{ij} = - \sum_{\Omega_p} v_{ij} \Delta_1 (\mathcal{J} v_{ij}) = - \sum_{\Omega_p} v_{ij-1} \Delta_1 v_{ij} - \sum_{\Omega_p} v_{ij+1} \Delta_1 v_{ij} + \\ + \sum_i v_{ip} \Delta_1 v_{ip-1}$$

$$- \sum_i v_{ip-1} \Delta_1 v_{ip} + \sum_i v_{i0} \Delta_1 v_{i1} + \sum_i \Delta_1 v_{i1} v_{i0} = - \sum_{\Omega_p} \Delta_1 v_{ij} \mathcal{J} v_{ij} + \\ + 2 \sum_i v_{ip} \Delta_1 v_{ip-1} + 2 \sum_i v_{i0} \Delta_1 v_{i1}$$

hence we get

$$(11) \qquad \left| \sum_{\Omega_p} \Delta_1 v_{ij} \mathcal{J} v_{ij} \right| \leq \frac{K_3 \varkappa}{2\tau} \{\vartheta(p) + \vartheta(p-1) + \vartheta(0) + \vartheta(1)\}$$

It can be easily proved now that

$$(12) \qquad \left| \sum_{\Omega_p} (Bu)_{ij} \mathcal{J} v_{ij} \right| \leq K_4 \sum_{j=0}^{p} \vartheta(j),$$

$$(13) \qquad \left| \sum_{\Omega_p} f_{ij} \, . \, \mathcal{J} v_{ij} \right| \leq \sum_{\Omega_p} f_{ij}^2 + K_5 \sum_{j=0}^{p} \vartheta(j).$$

Suppose

(*) $\varkappa \leq \dfrac{c}{2k_3}$ and substitute the relations (9) — (13) in (8). We obtain

$$(14) \qquad \vartheta(p) \leq K_6 (\vartheta(0) + \vartheta(1) + \tau \sum_{\Omega_p} f^2{}_{ij}) + K_6 \tau \sum_{j=0}^{p} \vartheta(j).$$

Let $\gamma = K_6 (\vartheta(0) + \vartheta(1) + \tau \sum_{\Omega_p} f^2{}_{ij})$, $R(p) = \sum_{j=0}^{p} \vartheta(j)$.

In these expressions the relation (14) gets the form

$$(15) \qquad R(p) (1 - \tau K_6) \leq \gamma + R (p-1).$$

Let τ be so small, that $1 - \tau K_6 \geq \frac{1}{2}$. If we denote $E = (1 - \tau K_6)^{-1}$, we easily prove by mathematical induction

$$(16) \qquad R(p) \leq \gamma E \, . \, \frac{E^{p-1} - 1}{E - 1} + E^{p-1} R(1).$$

Substituing this in (14) and using the relations

$$E_p = \left(1 + \frac{\tau K_6}{1 - \tau K_6}\right)^p \leq e^{\frac{\eta K_6}{1 - \eta K_6}} \leq e^{2TK_6}, \ \tau K_6 = \frac{E-1}{E}, R(1) = \\ = \vartheta(0) + \vartheta(1),$$

we obtain immediately the statement of the theorem.

Theorem 2. There exists a positive constant L, independent on τ, h, f, φ such that for u_{ij} defined by (3), (4) and $\tau, h, \varkappa = \frac{\tau}{h}$ sufficiently small the inequality

$$(17) \qquad h \sum_i | \Delta_2 u_{ip} |^2 \leq L \{ h \sum_i (u_{i0}^2 + u_{i1}^2) + h \sum_i (|\Delta_2 u_{i0}^2 + \Delta_2 u_{i0}^2) + \tau h [\sum_i \sum_{j=0}^{p-2} f_{ij}^2 + \sum_{\Omega_p} (\Delta_2 f_{ij})^2] \}$$

holds for arbitrary p, $1 \leq p \leq \left[\frac{T}{\tau}\right]$

Proof. Denoting $\Delta_2 u_{ij} = \mu_{ij}$ and appliing to the system (4) the operator Δ_2 we obtain

$$(18) \quad \Delta_2 \mu_{ij}, + \Delta_1 (\Delta_{ij+1} \mu_{ij} + \Delta_2 A_{ij} u_{ij-1}) + B_{ij} + 1 \mu_{ij} = \Delta_2 f_{ij} - \Delta_2 B_{ij} u_{ij-1}$$

for $1 \leq j \leq \left[\frac{T}{t}\right] - 1$. Let us define the matrix A in $E_2 > M$ as follows:

$$A(x, t) = \frac{\partial A}{\partial t}(x, T) . (t - T) + A(x, T) \text{ for } t > T, x \in E_1$$

$$A(x, t) = \frac{\partial A}{\partial t}(x, o)\, t + A(x, o) \quad \text{for } t < 0, x \in E_1.$$

We note that the functions $\Delta_2 A_{ij}$, $\Delta_2^2 A_{ij}$ remain bounded by a constant independent on τ, h.
Denoting $A_{ij+1} \mu_{ij} + \Delta_2 A_{ij} u_{ij-1} = \xi_{ij}$,
multipliing (18) by $\mathcal{J} \xi_{ij}$ and summing over Ω_p we get

$$(19) \qquad \sum_{\Omega_p} \Delta_2 \mu_{ij} \mathcal{J} \xi_{ij} + \sum_{\Omega_p} \Delta_1 \xi_{ij} \mathcal{J} \xi_{ij} + \sum_{\Omega_p} B_{ij+1} \mu_{ij} \mathcal{J} \xi_{ij} = = \sum_{\Omega_p} \Delta_2 f_{ij} \mathcal{J} \xi_{ij} - \sum_{\Omega_p} \Delta_2 B_{ij} u_{ij-1} \mathcal{J} \xi_{ij}.$$

Let us transform the expressions in (19) as follows:
It is

$$(20) \qquad \sum_{\Omega_p} \Delta_2 \mu_{ij} \mathcal{J} \xi_{ij} = \sum_{\Omega_p} \Delta_2 \mu_{ij} [\mathcal{J}(A_{ij+1} \mu_{ij}) + \mathcal{J}(\Delta_2 A_{ij} \mu_{ij-1})].$$

We shall estimate separately the two terms in the right-hand side of (20).
We have

$$(21) \qquad \sum_{\Omega_p} \Delta_2 \mu_{ij} \mathcal{J}(A_{ij+1} \mu_{ij}) = \sum_{\Omega_p} \Delta_2 (\mu_{ij} A_{ij+1} \mu_{ij}) + + \sum_{\Omega_p} \frac{\mu_{ij+1} A_{ij} \mu_{ij-1} - \mu_{ij-1} A_{ij+2} \mu_{ij+1}}{2\tau}.$$

Denoting $\vartheta(j) = \sum_i \mu_{ij}^2$, $\psi(j) = \sum_i u_{ij}^2$, we obtain

$$\sum_{\Omega_p} \Delta_2 (\mu_{ij} A_{ij+1} \mu_{ij}) = \sum_i \frac{\mu_{ip} A_{ip+1} \mu_{ip} + \mu_{ip-1} A_{ip-1} \mu_{ip-1}}{2\tau} -$$

$$- \sum_i \frac{\mu_{i1} A_{i2} \mu_{i1} + \mu_{i0} A_{i1} \mu_{i0}}{2\tau} = \sum_\tau \frac{\mu_{ip} A_{ip} \mu_{ip} + \mu_{ip-1} A_{ip-1} \mu_{ip-1}}{2\tau} +$$

$$+\sum_i \mu_{ip}\frac{A_{ip+1}-A_{ip}}{2\tau}\mu_{ip}+\sum_i \mu_{ip-1}\frac{A_{ip}-A_{ip-1}}{2\tau}\mu_{ip-1}-$$
$$-\sum_i \frac{\mu_{i1}A_{i2}\mu_{i1}+\mu_{i0}A_{i1}\mu_{i0}}{2\tau}.$$

According to the assumptions (α) on A the following inequalities hold

$$(22)\qquad \sum_{\Omega_p}\frac{\mu_{ip}A_{ip}\mu_{ip}+\mu_{ip-1}A_{ip-1}\mu_{ip-1}}{2\tau}\geq\frac{C}{2\tau}(\vartheta(p)+\vartheta(p-1)),$$

$$(23)\qquad \Bigg|\sum_i \mu_{ip}\frac{A_{ip+1}-A_{ip}}{2\tau}\mu_{ip}+\sum_i \mu_{ip-1}\frac{A_{ip-1}-A_{ip-1}}{2\tau}\mu_{ip-1}-$$
$$-\sum_i \frac{\mu_{i1}A_{i2}\mu_{i1}+\mu_{i0}A_{i1}\mu_{i0}}{2\tau}\Bigg|\leq L_1(\vartheta(p)+\vartheta(p-1)+\frac{L_2}{2\tau}(\vartheta(1)+\vartheta(o)),$$

where L_j are positive constants.
Let us estimate now the second term in the right-hand side of (21). By the symmetry of A we derive simillarly as in the proof of the theorem 1 the inequality

$$(24)\qquad \Bigg|\sum_{\Omega_p}\frac{\mu_{ij+1}A_{ij}\mu_{ij-1}-\mu_{ij-1}A_{ij+2}\mu_{ij+1}}{2\tau}\Bigg|\leq L_3\sum_{j=0}\vartheta(j).$$

Now, we estimate the second term in the right-hand side of (20). We shall use the obvious identity

$$2\,\Delta_2(u_{ij}v_{ij})=\mathcal{J}u_{ij}\,\Delta_2 v_{ij}+\mathcal{J}v_{ij}\,\Delta_2 u_{ij},$$

which is true for arbitrary mesh functions u, v. Using this formula, we get

$$(25)\qquad \sum_{\Omega_p}\Delta_2\mu_{ij}\mathcal{J}(\Delta_2 A_{ij}\mu_{ij-1})=2\sum_{\Omega_p}\Delta_2(\mu_{ij}\Delta_2 A_{ij}u_{ij-1})-$$
$$-\sum_{\Omega_p}\mathcal{J}\mu_{ij}(\Delta_2^2 A_{ij}u_{ij}+\Delta_2 A_{ij-1}\mu_{ij-1}).$$

First, we transform the first term of the right-hand side of (25):

$$(26)\ 2\sum_{\Omega_p}\Delta_2(\mu_{ij}\Delta_2 A_{ij}u_{ij-1})=2\sum_i\frac{u_{ip}\Delta_2 A_{ip}u_{ip-1}+\mu_{ip-1}\Delta_2 A_{ip-1}u_{ip-2}}{2\tau}-$$
$$-2\sum_i\frac{\mu_{i1}\Delta_2 A_{i1}u_{i0}+\mu_{i0}\Delta_2 A_{20}u_{i-1}}{2\tau}.$$

Using the inequality $2|ab|\leq a^2\varepsilon^2+\frac{b^2}{\varepsilon^2}$ (a, b, ε arbitrary real numbers), we obtain

$$\Bigg|2\sum_{\Omega_p}\Delta_2(\mu_{ij}\Delta_2 A_{ij}u_{ij-1})\Bigg|\leq\frac{L_4}{2\tau}\Bigg[\frac{c}{2L_4}(\vartheta(p)+\vartheta(p-1))+$$
$$+\frac{2L_4}{c}(\psi(p-1)+\psi(p\,.\,2)+\vartheta(o)+\vartheta(1)+\psi(0)+\psi(-1)\Bigg]$$

(with c arbitrory small).

The second term in the right-hand side of (25) can be estimated as follows:

(27) $$\left| \sum_{\Omega_p} \mathcal{J}\mu_{ij} (\Delta_2^2 A_{ij} u_{ij} + \Delta_2 A_{ij-1} \mu_{ij-1}) \right| \leq L_5 \sum_{j=0}^{p} \vartheta(j) + L_6 \sum_{j=1}^{p-1} \psi(j).$$

We obtain similarly as in the proof of the theorem 1

$$\sum_{\Omega_p} \Delta_1 \xi_{ij} \mathcal{J}\xi_{ij} = \sum_i \xi_{ip} \Delta_1 \xi_{ip-1} + \sum_i \xi_{i0} \Delta_1 \xi_{i1} =$$
$$= \sum_i (A_{ip+1} \mu_{ip} + \Delta_2 A_{ip} u_{ip-1}) \Delta_1 (A_{ip} \mu_{ip-1}) +$$
$$+ \sum_i (A_{ip+1}\mu_{ip} + \Delta_2 A_{ip} u_{ip-1}) \Delta_1 (\Delta_2 A_{ip-1} u_{ip-2}) +$$
$$+ \sum_i (A_{i1} \mu_{i0} + \Delta_2 A_{i0} u_{i-1}) \Delta_1 (A_{i2} \mu_{i1}) +$$
$$+ \sum_i (A_{i1} \mu_{i0} + \Delta_2 A_{i0} u_{i-1}) \Delta_1 (\Delta_2 A_{i1} u_{i0}).$$

It follows.

(28) $$\left| \sum_{\Omega_p} \Delta_1 \xi_{ij} \mathcal{J}\xi_{ij} \right| \leq \frac{L_7 \varkappa}{2\tau} [\vartheta(p) + \vartheta(p-1 + \vartheta(1) +$$
$$+ \vartheta(o) + \psi(p-1) + \psi(p-2) + \psi(o) + \psi(-1)].$$

It is easility to prove that

(29) $$\left| \sum_{\Omega_p} B_{ij+1} \mu_{ij} \mathcal{J}\xi_{ij} \right| \leq L_8 \left[\sum_{j=0}^{p} \vartheta(j) + \sum_{j=-1}^{p-1} \psi(j)\right],$$

(30) $$\left| \sum_{\Omega_p} \Delta_2 f_{ij} \mathcal{J}\xi_{ij} \right| \leq \sum_{\Omega_p} | \Delta_2 f_{ij} |^2 + L_9 \left[\sum_{j=0}^{p} (\vartheta(j) + \sum_{j=-1}^{p-1} \psi(j)\right],$$

(31) $$\left| \sum_{\Omega_p} \Delta_2 B_{ij} u_{ij-1} \mathcal{J}\xi_{ij} \right| \leq L_{10} \left[\sum_{j=0}^{p} \vartheta(j) + \sum_{j=-1}^{p-1} \psi(j)\right].$$

Suppose now $\varkappa \leq \dfrac{C}{4L_7}$, $\tau \leq \dfrac{C}{16 L_1}$
and substitute (22) — (31) in (19). We obtain

(32) $$\vartheta(p) \leq L_{11} \{\vartheta(1) + \vartheta(o) + \psi(o) + \psi(-1) + \psi(p-1) + \psi(p-2)$$
$$+ \tau \left[\sum_{\Omega_p} (\Delta_2 f_{ij})^2 + \sum_{j=0}^{p} \vartheta(j) + \sum_{j=-1}^{p-1} \psi(j)\right]\}.$$

It follows by (7) and (16)

(33) $$\vartheta(p) \leq L_{11} \{\vartheta(1) + \vartheta(o) + \psi(o) +$$
$$+ \tau \left[\sum_{\Omega_p} (\Delta_2 f_{ij})^2 + \sum_i \sum_{j=0}^{p-2} f_{ij}]^2\right\} + L_{11} \tau \sum_{j=0}^{p} \vartheta(j)].$$

From (33) we derive by the same arguments as in the proof of the theorem 1 the inequality (17).

Theorem 3. There exists a constant M such that under the assumptions of the theorem 2

$$(34) \qquad \tau h \sum_i \sum_{j=1}^{p} (\Delta_1 v_{ij})^2 \leq M \{h \sum_i [u_{i0}^2 + u_{i1}^2 + (\Delta_2 u_{i0})^2 + (\Delta_2 u_{i1})^2] + \\ + \tau h [\sum_i \sum_{j=0}^{p} f_{ij}^2 + \sum_{\Omega_p} (\Delta_2 f_{ij})^2]\}$$

holds for an arbitrary integer p, $1 \leq p \leq \left[\frac{T}{\tau}\right]$.

Proof. (34) follows immediately from (4) using the theorems 1,2.

2. EXISTENCE THEOREM

Let the matrices A, B and functions f, φ satisfy the conditions (α) and (β) respectively. Let us define, for an arbitrary mesh function μ_{ij}, the function $\tilde{\mu}(x, t)$ by

$$(36) \qquad \tilde{\mu}(x, t) = \mu_{ij} \text{ for } hi \leq x < h(i + 1), j\tau \leq t < (j + 1)\tau.$$

Theorem 4. There exists a constant K independent on τ, h, φ, f such that for sufficiently small τ, h, $\frac{\tau}{h} = \varkappa$

$$(37) \qquad \| \tilde{u} \|_{L_2(M)} \leq K\Theta, \ \| \widetilde{\Delta_2 u} \| \leq K\Theta, \ \| \widetilde{\Delta_1 v} \| \leq K\Theta$$

holds with $\Theta = \tau h \sum_{\Omega_p} [f_{ij}^2 + (\Delta_2 f_{ij})^2] + h \sum_i \varphi(ih)^2 + h \sum_i [\Delta_1 (A_{i0} \varphi(ib))]^2$.

Proof. Since the expresions $h \sum_i (\Delta_2 u_{i0})^2$, $h \sum_i (\Delta_2 u_{i1})^2$ may be estimated by (4), we get the statement easily by theorems 1—3.
For instance

$$\sum_i (\Delta_2 u_{i1})^2 \leq C \sum_i \{(\Delta_1 A_{i1} \varphi(ih)|^2 + (B_{i0} \varphi(ih))^2 + f_{i1}^2\} \leq$$

$$\leq C_1 \sum_i \left\{(\Delta_1 A_{i0} \varphi(ih))^2 + \varkappa^2 \left[\frac{A_{i1} - A_{i0}}{\tau} \varphi(ih)\right]^2 + (B_{i0} \varphi(ih))^2 + f_{i1}^2\right\}$$

We also use the obvious estimate $h \sum_i (f_{i0}^2 + f_{i1}^2) \leq C_2 \tau h \sum_{\Omega_p} (f_{ij}^2 + (\Delta_2 f_{ij})^2)$.

Let τ_n, h_n be sequences of sufficiently small positive numbers such that $\tau_n, h_n \to 0$ as $n \to \infty$ and $\varkappa_n = \frac{\tau_n}{h_n}$ satisfy (*). Let $\tilde{u}_n$, $\tilde{v}_n$ be the respective functions. The weak compactness (in $L_2(M)$) of the sequences $\tilde{u}_n$, $\widetilde{\Delta_2 u_n}$, $\widetilde{\Delta_1 v_n}$ follows by the theorem 4. Hence we may choose weakly convergent subsequences which will be denoted by

$$(38) \qquad \tilde{u}_\nu, \widetilde{\Delta_2 u_\nu}, \widetilde{\Delta_1 v_\nu}:$$

Let

$$(38') \qquad u, u_2, v_1 \in L_2(M),$$

be the respective limits and $v = Au$

It holds:

Theorem 5.

$$\frac{\partial u}{\partial t} = u_2, \quad \frac{\partial v}{\partial x} = v_1,$$

where the derivatives are to be taken in the sense of distributions.

Proof: Let $\psi = (\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(2)}) \in \varepsilon^{(1)} (M_0)$

(M_0 is the interior of M). Let $\widetilde{\psi}_\nu^{(m)}, \widetilde{f}_\nu^{(m)}, \widetilde{a}_\nu^{kl}, \widetilde{b}_\nu^{kl}$, $m, k, l = 1, 2, \ldots, r$ be the functions coinciding with $\psi^{(m)}, f^{(m)}, a^{kl}, b^{kl}$ respectively at the points of the nets (with mesh sizes τ_ν, k_ν) and defined in the other points by the formula (36). It is easy to prove

(39) $$(\widetilde{\psi}_\nu, \widetilde{\Delta_2 u_\nu})_{L_2(M)} = -(\widetilde{\Delta_2 \psi_\nu}, \widetilde{u}_\nu)_{L_2(M)}$$

(40) $$(\widetilde{\psi}_\nu, \widetilde{\Delta_1 v_\nu})_{L_2(M)} = -(\widetilde{\Delta_1 \psi_\nu}, \bar{v}_\nu)_{L_2(M)}.$$

Now, limiting the equalities (39), (40) we get the statement of the theorem.
Theorem 6. It holds

1. $u, \frac{\partial u}{\partial_t} \in L_2 (M), v \in W_2^1 (M)$.
2. The functions u, v have a compact support with respect to x in M.

Proof. The assertion 1. is an immediate consequence of the for (38), (38′) and theorem 5. The assertion 2. follows by the lemma 1,2.

In the following, we shall use the embeding

Theorem 7. Let $u, \frac{\partial u}{\partial t} \in L_2 (M)$. Then there exists $\hat{u} (x, t)$ such that $\hat{u} (x, t) = u (x, t), \frac{\partial \hat{u}}{\partial t} = \frac{\partial u}{\partial t}$ allmost every where in M, and for every $t \in \langle o, T \rangle$

$$\int_{E_1} (\hat{u} (x, t) - \hat{u} (x, \tau))^2 \, d x \to o \text{ as } \tau \to t \text{ and}$$

$$\int_{E_1} (\hat{u} (x, t))^2 \, dx \leq K \left\{ \| \hat{u} \|^2_{L_2(M)} + \left\| \frac{\partial \hat{u}}{\partial t} \right\|^2_{L_2(M)} \right\}.$$

where K depends only on t.

(A similar theorem may be found e. g. in (1) p. 35).

Theorem 8. The function u is a generalized solution of (1), (2) in M.

Proof. Let $\Phi = \Phi^{(1)}, \ldots, \Phi^{(1)} \in \varepsilon^1 (M_0)$. Multipliing the equation (4) by $\tau h . \widetilde{\Phi}_\nu$ and summing over the net-points in M, we get

(41) $$(\Delta_2 \widetilde{u}_\nu + \widetilde{\Delta_1 v_\nu} + \widetilde{B}_\nu \widetilde{u}_\nu - \widetilde{f}_\nu, \widetilde{\Phi}_\nu)_{L_2(M)} = 0.$$

Limitting in (41) gives:

$$\left(\frac{\partial u}{\partial t} + \frac{\partial (Au)}{\partial x} + Bu - f, \Phi \right)_{L_2(M)} = 0.$$

Since the Φ was arbitrarily chosen in $\varepsilon^1(M^0)$, and since $\varepsilon^1(M^0)$ is dense in $L_2(M)$, we get

$$\frac{\partial u}{\partial t} + \frac{\partial (Au)}{\partial x} + Bu = f \text{ almost everywhere in } M.$$

Let us, to prowe that the function u satisfies the condition (2), define the functions

$$u_\nu^* = \begin{cases} \widetilde{\varphi}_\nu \text{ for } t \leq 0 \\ \widetilde{u}_\nu \text{ for } t \geq 0 \end{cases}$$

where $\widetilde{u}_\nu$ are the functions of (38) and $\widetilde{\varphi}_\nu(x) = \varphi(kh_\nu)$ for $x \in \langle k\, h_\nu, (k+1)\, h_\nu\rangle$, k integer.
As $\Delta_2 u_\nu^* = 0$ for $t \leq 0$, we have (by (38) and the theorem 5)

$$u_\nu^* \rightharpoonup u^*, \Delta_2 u_\nu^* \rightharpoonup \frac{\partial u^*}{\partial t} \text{ in } L_2(N), \text{ where } N = E_1 \times \langle -1, T\rangle.$$

Obviously $u^* = u$ for $t \geq 0$, $u^* = \varphi$ for $t \leq 0$. By the embedding theorem 7 there exists $\psi(x) \in L_2(E_1)$ such that

$$\int_{E_1} (u^*(x,t) - \psi)^2\, dx \to 0 \text{ for } t \to 0+.$$

As $u^* = \varphi$ for $t \leq 0$ it follows $\psi = \varphi$ allmost everywhere. Hence u satisfies the condition (2).

3. GENERAL EXISTENCE THEOREM

We shall prove in the present section the existence theorem for the problem (1), (2) under more general assumptions. We suppose that

$$(\gamma) \qquad f, \frac{\partial f}{\partial t} \in L_2(M)$$

$$A(o, x)\,\varphi(x) \in W_2^1(M)$$

Than there exist sequences f_n, φ_n satisfying the condition (β) such that

$$(\varepsilon) \qquad \|f_n - f\|_{E_2^1(M)} \to 0, \left\| \frac{\partial t_n}{\partial t} - \frac{\partial f}{\partial t} \right\|_{L_2(M)} \to 0$$

$$\|A(\alpha, o)\,\varphi_n(x) - A(x, 0)\,\varphi(x)\|_{W_2^1(E_1)} \to 0, \|\varphi_n - \varphi\|_{L_2(M)} \to 0 \text{ as } n \to \infty.$$

Let τ_n, h_n be two suquences of positive numbers such that $\tau_n, h_n \to 0$ such as $n \to \infty$ and such that $\frac{\tau_n}{h_m} = \varkappa_n$ satisfies the condition (*) and finally that

$$(\eta) \qquad \|\widetilde{f}_n - f\|_{L_2(M)} \to 0, \left\| \Delta_2 \widetilde{f}_n - \frac{\partial f}{\partial t} \right\|_{L_2(M)} \to 0$$

$$\|\widetilde{\varphi}_n - \varphi\|_{L_2(M)} \to 0, \left\| \Delta_1 \widetilde{A}_0\, \widetilde{\varphi}_n - \frac{d(A(0,x)\,\varphi(x))}{dx} \right\|_{L_2(E_1)} \to 0$$

with $n \to \infty$.
Here $\widetilde{f}_n$, $\widetilde{\varphi}_n$, $\widetilde{A}_0\, \widetilde{\varphi}_n$ are piecewise constant functions which coincide in the points of the net with mesh sizes τ_n, h_n with the functions f_n, φ_n, $A(x, 0)\,\varphi_n$ respectively, and defined in the other points by the formule (36).

Now we are able to prove the

Theorem 9. There exists, under the assumptions (α), (γ) a generalized solution (solution almost everywhere) of the probleme (1), (2). Moreover, if u_n is the solution of the difference equations (3), (4) on the net with mesh sizes τ_n, h_n, with f_n, φ_n defined by (δ) and (η), than

$$\widetilde{u}_n \to u, \widetilde{\Delta_2 u_n} \to \frac{\partial u}{\partial t}, \widetilde{\Delta_1 A u_n} \to \frac{\partial A u}{\partial x}$$

where u is a solution of (1), (2).

Proof: The weak compactness of the sets of functions $\widetilde{u}_n$, $\widetilde{\Delta_2 u_n}$, $\widetilde{\Delta_1 A}\ u_n = \Delta_1 \widetilde{v}_n$ in $L_2(M)$ follows from the assumptions (δ), (η) by the theorem 4. Let us denote by u, u_2, v_2 the limits of some convergent subsequences.

We get

$$u_2 = \frac{\partial u}{\partial t}, v_1 = \frac{\partial (Au)}{\partial x} \text{ and}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (Au)}{\partial x} + Bu = f$$

almost everywhere in M, similarly as in the proof of the theorems 5, 8. It can be proved in the same manner that u satisfies the condition (2). The convergence of the whole sequences $\widetilde{u}_n$, $\widetilde{\Delta_2 u_n}$, $\widetilde{\Delta_1 A u_n}$ follows from the uniqueness theorem, stated in the next section.

4. UNIQUENESS THEOREM

Theorem 10. Let A, B and f, φ satisfy the assumptions (α) and (γ) respectively. Than the generalized solution of (1), (2) is unique.

Proof: Suppose u_1 and u_2 be two solutions of (1), (2) with the same values f, φ. The difference $v = u_1 - u_2$ satisfies the conditions

(42) $$\frac{\partial v}{\partial t} + \frac{\partial (Av)}{\partial x} + Bv = 0$$

(43) $$\int_{E_1} v^2(x, t)\, dx \to 0 \text{ as } t \to 0+,$$

$$v, \frac{\partial v}{\partial t} \in L_2(M), Av \in W_2^1(M).$$

Put $w = Av$. There exists a sequence $w_n \in \varepsilon_x^2(M)$ such that

(44) $$\| w_n - w \|_{W_2^1(M)} \to 0.$$

If $v_n = A^{-1} w_n$, we get

(45) $$\| v_n - v \|_{L_2(M)} \to 0, \left\| \frac{\partial v_n}{\partial t} - \frac{\partial v}{\partial t} \right\|_{L_2(M)} \to 0$$

according to the boundedness A, A^{-1} and their first derivatives with rsepect to t, Using also the theorem 7, we obtain from (42)—(45)

(46) $$\frac{\partial v_n}{\partial t} + \frac{\partial (A v_n)}{\partial x} + B v_n = f_n$$

where

(47) $$\| f_n \|_{L_2(M)} \to 0$$

and

(48) $$\| v_n (x, 0) \|_{L_2(E_1)} \to 0 \text{ as } n \to \infty.$$

Multiplying (46) by Av_n and integrating over $Mt = E_1 x \langle o, t \rangle \; 0 \leq t \leq T$, we obtain

$$\int_M \frac{\partial v_n}{\partial t} A\, v_n\, dx\, dt + \int_M \frac{\partial (Av_n)}{\partial x} A\, v_n\, dx\, dt + \int_{M_t} Bv_n\, A\, v_n\, dx\, dt =$$

$$= \int_{M_t} f_n\, Av_n\, dx\, dt.$$

Integration by parts leads to the equation

$$\frac{1}{2} \int_{E_1} A(x, t)\, v_n(x, t) \,.\, v_n(x, t)\, dx = \frac{1}{2} \int_{E_1} A\,(x, 0)\, v_n(x, 0) \,.\, v_n(x, 0)\, dx +$$

$$+ \int_{M_t} f_n\, Av_n\, dx\, dt - \int_{M_t} B\, v_n\, Av_n\, dx\, dt$$

and we get (using the positivity of A)

$$\| v_n \|^2_{L_2(S_t)} \leq K \left\{ \| v_n \|^2_{L_2(S_0)} + \int_0^t \| v_n \|^2_{L_2(S_\tau)}\, dt + \| f \|^2_{L_2(M_t)} \right\},$$

where $S\tau$ is the plane $t = \tau$.

Hence, we get by the well known lemma, the inequalities

(49) $$\| v_n \|^2_{L_2(S_t)} \leq K_1 \left\{ \| v_n \|^2_{L_2(S_0)} + \| f_n \|^2_{L_2(M_t)} \right\}$$

$$\| v_n \|^2_{L_2(M)} \leq K_2 \left\{ \| v_n \|^2_{L_2(S_0)} + \| f_n \|^2_{L_2(M)} \right\}.$$

Now, if $n \to \infty$, the left-hand side of (49) tends to $\| v \|^2_{L_2(M)}$ and the right-hand side tends to zero, which proves the theorem.

Remark. The results of the paper holds if A admits finite number of discontinuity lines $x = x_i = \text{const}$. Also B can be taken more general.

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SOUHRN

Pro řešení Cauchyovy úlohy pro soustavu

(1) $$\frac{\partial u}{\partial t} + \frac{\partial (Au)}{\partial x} + Bu = f$$

(2) $$u\,(x, o) = \varphi\,(x)$$

v oblasti $M = E_1 x \langle 0, T \rangle, o < T < +\infty$,

kde A má konečný počet přímek nespojitosti, rovnoběžných s osou $x = 0$, $\frac{\partial A}{\partial t}$ splňuje Lipschitzovu podmínku podle t, A je symetrická a positivně definitivní, je použito následujícího diferenční-

ho schématu. Na síti s kroky $\tau = \Delta t$, $h = \Delta x$ s $\varkappa = \frac{\tau}{h}$ dostatečně malým určíme v uzlových bodech sítě funkci $u_{ij} = u\,(ih, j\tau)$ z rovnic

$$u_{i0} = u_{i1} = \varphi\,(ih)$$
$$\Delta_2 u_{ij} + \Delta_1\,(Au)_{ij} + B_{ij}\, u_{ij} = f_{ij}$$

pro libovolná celá i a celá j, $0 \leq j \leq \left[\frac{T}{\tau}\right]$, kde Δ_1 resp. Δ_2 jsou symetrické diferenční operátory

$$\Delta_1 u_{ij} = \frac{u_{i+1\,j} - u_{i-1\,j}}{2h} \text{ resp. } \Delta_2 u_{ij} = \frac{v_{ij+1} - v_{ij-1}}{2\tau}.$$

Je dokázána stabilita tohoto schématu a existence a jednoznačnost zobecněného řešení u úlohy (1), (2); splňujícího podmínky u, $\frac{\partial u}{\partial t}$, $\frac{\partial(Au)}{\partial x} \in L_2(M)$, které nabývá počátečních hodnot (2) v průměru.

РЕЗЮМЕ

Для решения задачи Коши для системы

(1) $\frac{\partial u}{\partial t} + \frac{\partial(Au)}{\partial x} + Bu = f$

(2) $u\,(x, 0) = \varphi\,(x)$,

в области $M = E_1 \times \langle 0, T\rangle$ $0 < T < +\infty$,

где A имеет конечное число линий разрыва, параллельных оси $x = 0$, $\frac{\partial A}{\partial t}$ удовлетворяет условию Липшида по t, A симметрическая и положительно-определенная, предлагается следующая явная разностная схема. На сетке с шагами $\tau = \Delta t$, $h = \Delta x$ с $\varkappa = \frac{\tau}{h}$ достаточно малым определяется сеточная функция $u_{ij} = u\,(ih, j\tau)$ из уравнений

$$u_{i0} = u_{i1} = \varphi\,(ih)$$
$$\Delta_2 u_{ij} + \Delta_1\,(Au)_{ij} + B_{ij} u_{ij} = f_{ij}$$

для любых целых i и целых j, удовлетворяющих неравенствам $0 \leq j \leq \left[\frac{T}{\tau}\right]$, где Δ_1, Δ_2 симметричные разностные операторы

$$\Delta_1 u_{ij} = \frac{u_{ij+1} - u_{ij-1}}{2h},\ \Delta_2 u_{ij} = \frac{u_{ij+1} - u_{ij-1}}{2\tau}$$

соответственно.

Доказывается устойчивость этой схемы и существование и единственность обобщенного решения задачи (1), (2), удовлетворяющего условиям

$$u, \frac{\partial u}{\partial t}, \frac{\partial Au}{\partial x} \in L_2\,(M),$$

и принимающего начальные данные (2) в среднем.