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ON THE CROSSING NUMBERS OF GRAPHS

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In the last ten years a number of authors have been concerned with different questions arising from the problem of embedding a given graph into a given topological surface. (See e.g. the summarizing paper of Beineke /1/.) One of these questions has led to the study of crossing numbers of different graphs.

The crossing number of the graph G for a given topological surface equals the minimum number of crossings of edges which a model (drawing) of the graph G on this surface can have.

The crossing number of the graph G for a closed orientable surface having genus i will be denoted $c_i^+(G)$, and for a closed nonorientable surface having genus i, it will be denoted $c_i(G)$.

So far, all the authors concerned with the problems of crossing numbers have only paid attention to two topological surfaces - a plane (sphere) and torus. For the first time, this paper will pay attention to two non-orientable surfaces - the projective plane and Klein's bottle.

The article is divided into two parts. In the first part I shall give a summary of the most important results known so far, concerning orientable surfaces. Some of these results are, however, only under print and will be quoted here for the first time.In the second part the crossing numbers of complete graphs for non - orientable surfaces - the projective plane and Klein's bottle - are investigated.

1.-A. P L A N E (S P H E R E)

1,1. The regular bicomplete graph $K_{m,n}$, is the graph with two chromatic classes with m and n vertices in which each two vertices belonging to different classes are joined by one edge.

The regular bicomplete graph $K_{m,n}$ was the first graph ever for which the crossing number was investigated. The impulse to study this problem came from P. Turán and the first to study it was Zarankiewicz in /2/. Here the equality

$$c_o^+ (K_{m,n}) = \left[\frac{m}{2}\right] \cdot \left[\frac{m-1}{2}\right] \cdot \left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right],$$

which is now known as Zarankiewicz's crossing number hypothesis or first crossing number hypothesis, was published.

This hypothesis has, however, not yet been proved. Only some partial results are known. E.g. Blažek and Koman (see /3/) verified Zarankiewicz s hypothesis for all the graphs $K_{m,n}$ for which either

$$\min (m,n) \leqq 4$$

or

$$\max (m,n) \leqq 6.$$

The following table of the numbers $c_o^+ (K_{m,n})$ for $3 \leqq m \leqq 7$, $3 \leqq n \leqq 10$ can be obtained from there.

TABLE 1.

$c_o^+(K_{m,n})$	3	4	5	6	7	8	9	10
3	1	2	4	6	9	12	16	20
4	2	4	8	12	18	24	32	40
5	4	8	16	24	$\leqq 36$ $\geqq 34$	$\leqq 48$ $\geqq 46$	$\leqq 64$ $\geqq 60$	$\leqq 80$ $\geqq 75$
6	6	12	24	36	$\leqq 54$ $\geqq 51$	$\leqq 72$ $\geqq 69$	$\leqq 96$ $\geqq 90$	$\leqq 120$ $\geqq 113$
7	9	12	$\leqq 36$ $\geqq 34$	$\leqq 54$ $\geqq 51$	$\leqq 81$ $\geqq 72$	$\leqq 108$ $\geqq 97$	$\leqq 144$ $\geqq 127$	$\leqq 180$ $\geqq 150$

For the other cases these inequalities can be proved:

(1) $$c_0^+(K_{m,n}) \leqq H_0^+(m,n) = \left[\frac{m}{2}\right] \cdot \left[\frac{m-1}{2}\right] \cdot \left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right],$$

(2) $$c_0^+(K_{m,n}) \geqq \frac{1}{3}\binom{n}{2}\left[\frac{m}{2}\right] \cdot \left[\frac{m-1}{2}\right], \qquad (n \geqq 5);$$

(3) $$c_0^+(K_{m,n}) \geqq \frac{1}{3}\binom{m}{2}\left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right], \qquad (m \geqq 5).$$

According to R. K. Guy the above hypothesis has been proved by D. Kleitman for all regular complete graphs $K_{m,n}$ for which

$$\min(m,n) \leqq 6.$$

From there the lower estimates follow:

$$c_0^+(K_{m,n}) \geqq \frac{2}{5}\binom{n}{2}\left[\frac{m}{2}\right] \cdot \left[\frac{m-1}{2}\right], \qquad (n \geqq 7);$$

$$c_0^+(K_{m,n}) \geqq \frac{2}{5}\binom{m}{2}\left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right], \qquad (m \geqq 7).$$

Using these inequalities and the upper estimate (1) we get

$$\limsup c_0^+(K_{m,n}) \binom{m}{2}^{-1}\binom{n}{2}^{-1} \leqq \frac{1}{4},$$

$$\liminf c_0^+(K_{m,n}) \binom{m}{2}^{-1}\binom{n}{2}^{-1} \geqq \frac{1}{5}.$$

<u>1,2. The complete graph</u> K_n, i.e. the graph with n vertices each two of which are joined by a single edge. The problem of finding the crossing number of the complete graph K_n was given by P. Erdös. R. K. Guy /4/, F. Harary and A. Hill /5/, J. Blažek and M. Koman /6/, A. Saaty /7/ and others have been concerned with this problem. It is, however, not yet solved.

For $5 \leqq n \leqq 16$ the following results are known:

TABLE 2.

n	5	6	7	8	9	10	11	12	13	14	15	16
$c_0^+(K_n)$	1	3	9	18	36	60	$\leqq$ 100 $\geqq$ 95	$\leqq$ 150 $\geqq$ 143	$\leqq$ 225 $\geqq$ 207	$\leqq$ 315 $\geqq$ 290	$\leqq$ 441 $\geqq$ 396	$\leqq$ 558 $\geqq$ 528

R. K. Guy gives the equations $c_0^+(K_9) = 36$, $c_0^+(K_{10}) = 60$ (not published). In some papers (e.g. /4/) it has been stated - with reference to F. Harary - that with the help of computers it has been found that for $n \leqq 16$, the crossing numbers of the graph K_n equal the upper estimates of the table 2. According to

R. K. Guy, however, Harary's information turned out to be wrong.

To obtain the crossing numbers of graphs K_n for small n it is usually also necessary to determine the number $d_o^+(K_n)$ of topologically different models of the graph K_n with the minimal number of crossings. For these numbers R. K. Guy gives the table 3.

TABLE 3.

n	1	2	3	4	5	6	7	8
$d_o^+(K_n)$	1	1	1	1	1	1	5	3

For the crossing numbers $c_o^+(K_n)$ the so-called second crossing number hypothesis seems to be valid:

$$(4)\quad c_o^+(K_n) = H_o^+(n) = \frac{1}{4} \cdot \left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right] \cdot \left[\frac{n-2}{2}\right] \cdot \left[\frac{n-3}{2}\right].$$

This hypothesis has so far been verified for $n \leqq 10$. For the other n it is possible by two constructions giving topologically different models of the graph K_n (see e.g. /6/) to prove the inequality

$$(5)\qquad c_o^+(K_n) \leqq H_o^+(n).$$

The lower estimate is given by the inequality

$$(6)\qquad c_o^+(K_n) \geqq D_o^+(n) = \frac{1}{84}\, n(n-1)(n-2)(n-3),$$

which follows from the equality $c_o^+(K_{10}) = 60$ and the inequality

$$c(K_n) \geqq c(K_r)\binom{n}{4}\binom{r}{4}^{-1}, \qquad n \geqq r \geqq 4.$$

This inequality is valid not only for a plane, but for all topological surfaces; we therefore only write $c(K_n)$ instead of $c_o^+(K_n)$.

Because the majorant polynomial of the function $c_o^+(K_n)$ is of the 4^{th} degree, it is natural to study the function $c_o^+(K_n).n^{-4}$. Using the upper and lower estimates (5), (6) we easily obtain

$$\limsup c_o^+(K_n).n^{-4} \leqq 64^{-1},$$

$$\liminf c_o^+(K_n).n^{-4} \geqq 84^{-1}.$$

P. Kainen /8/ has moreover proved that, supposing that Zarankiewicz's hypothesis is valid,

$$\lim c_o^+(K_n).n^{-4} = 64^{-1}.$$

I shall also mention three problems related to the problem of crossing numbers of the graph K_n. J. W. Moon has put himself an interesting question. He investigated the socalled geodetic crossing number of the graph K_n for a sphere. We ottain the geodetic crossing number as follows: We choose at random n vertices on the surface of the sphere and join them pairwise by the shorter arcs of great circles. The expected number of crossings is called geodetic crossing number $g_o^+(K_n)$. J. W. Moon has shown in /9/ that

$$g_o^+(K_n) = 2^{-6}.n(n-1)(n-2)(n-3).$$

G. Ringel proved /10/, that any model of graph K_n (where $n \geq 5$) in a plane has

$$h(n) = 2n-2$$

edges without crossings at the most. However, all models having just h(n) edges without crossings are homeomorphic.

H. Harborth investigated the interesections of diagonals in convex polygons. He was interested specially in multiple intersections. For instance, in the paper /11/ the following is proved:

For $n \equiv \pm 2 \pmod 6$ in a regular n-gon no point except its centre and vertices can be the intersecting point of more than three diagonals. While the number of triple points of intersection is given by the formula

$$E_3(n) = \frac{1}{48} n(n-2)(5n-38), \qquad \text{if } n \equiv \pm 2 \pmod{12};$$
$$E_3(n) = \frac{1}{48} n(n-4)(5n-28), \qquad \text{if } n \equiv \pm 4 \pmod{12}.$$

<u>1,3. The complete k-chromatic graph</u> $Kn_1n_2...n_k$, that is the graph with k chromatic classes having $n_i (i=1,2,...,k; k \geq 2)$ vertices in which each two vertices belonging to different classes are joined by a single edge.

The problem of crossing numbersof these graphs was studied e.g. by J. Blažek, M. Koman and H. Harborth. I shallnemtion the results of papers /3/ and /12/ here. To simplify the expression of the main theorem concerning crossing numbers of the graphs $K_{n_1 n_2 \dots n_k}$, I shall give some denotations beforehand.

Let $K_{n_1 n_2 \dots n_k}$ be the given graph. For all $i = 1,2,\dots k$ let us denote

$$N_i = (n_1 + n_2 + \dots + n_k) - n_i.$$

For all $1 \leq r, s, t, u \leq k$ let us define

$$L(r,s,t,u) = a_r a_s a_t a_u + a_r a_s b_t b_u + a_r b_s b_t a_u + b_r b_s b_t b_u + b_r b_s a_t a_u + b_r a_s a_t b_u,$$

where a_i, b_i $(i = 1,2,\dots,k)$ are non-negative integers for which

$$a_i + b_i = n_i, \quad 0 \leq (a_i - b_i)(-1)^{n_1 + n_2 + \dots + n_i} \leq 1$$

holds.

We can now give the upper bound for the crossing numbers of the graphs $K_{n_1 n_2 \dots n_k}$. We shall consider only the cases when $k \geq 3$, because the case of $k = 2$ has already been discussed.

1) For $k = 3$ the crossing number $c_o^+(K_{n_1 n_2 \dots n_k})$ has the upper bound

$$(7) \qquad H_o^+(n_1,n_2,n_3) = \sum_{i=1}^{3} H_o^+(n_i,N_i) - \sum_{\substack{i,j=1\\ i<j}}^{3} H_o^+(n_i,n_j).$$

2) For $k \geq 4$ the crossing number $c_o^+(K_{n_1 n_2 \dots n_k})$ we have obtained the upper bound

$$(8) \qquad H_o^+(n_1,n_2,\dots,n_k) = \sum_{i=1}^{k} H_o^+(n_i,N_i) - \sum_{\substack{i,j=1\\ i<j}}^{k} H_o^+(n_i,n_j) + \sum_{\substack{r,s,t,u=1\\ r<s<t<u}}^{k} L(r,s,t,u).$$

It can be proved that the function (8) is symmetric, i.e. its values are independent of the order of the numbers $n_1,n_2,\dots,n_k$.

If all n_i $(i = 1,2,\dots,k)$ are even, we can rewrite the function (8) in this form

$$H_o^+(n_1,n_2,\dots,n_k) = \sum_{i=1}^{k} H_o^+(n_i,N_i) - \sum_{\substack{i,j=1\\ i<j}}^{k} H_o^+(n_i,n_j) +$$

$$+ \frac{3}{8} \sum_{\substack{r,s,t,u=1 \\ r<s<t<u}}^{k} n_r n_s n_t n_u .$$

For some special graphs $K_{n_1 n_2 \dots n_k}$ the summations of the function (8) can be enumerated. In the paper /13/ the following three results were found:

$$H_o^+(2,4,\dots,2k) = \frac{1}{24} \binom{k+1}{4} (3k^2 + 11k + 4)(3k^2 - k - 6);$$

If $n_1 = n_2 = \dots = n_k = n$, then

$$H_o^+(n,n,\dots,n) = k.H_o^+(n,kn-n) - \binom{k}{2} H_o^+(n,n) +$$

$$+ 4H_o^+(k).H_o^+(n+1,n+1) + H_o^+(k) \left(\left[\frac{n+1}{4}\right]^4 + \left[\frac{n}{2}\right]^4\right)$$

$$+ H_o^+(k+1,k-2)\left(\left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right]^3 + \left[\frac{n}{2}\right]^3 \cdot \left[\frac{n+1}{2}\right]\right) +$$

$$+ \frac{1}{2}H_o^+(n+1,n+1) \cdot \left[\frac{k+3}{2}\right] \cdot \left[\frac{k+1}{2}\right] \cdot \left[\frac{k}{2}\right] \cdot \left[\frac{k-2}{2}\right]$$

holds, specially for even n we have

$$H_o^+(n,n,\dots,n) = \frac{3}{16} n^3(n-2) \binom{k}{3} + \frac{3}{8} n^4 \binom{k}{4}.$$

If $n_1 = n, n_2 = \dots = n_k = 1$, then

$$H_o^+(n,1,1,\dots,1) = H_o^+(n,k-1) + H_o^+(k-1) + \left[\frac{n}{2}\right] \cdot \left[\binom{r}{3} + 2\binom{s}{3}\right] +$$

$$+ \left[\frac{n+1}{2}\right] \cdot \left[\binom{r}{3} + \binom{s}{3} + \binom{r-1}{3}\right],$$

where $r = \left[\frac{k+1}{2}\right]$, $s = \left[\frac{k}{2}\right]$.

For the crossing numbers of graphs $K_{n_1 n_2 \dots n_k}$ we can give the socalled third crossing number hypothesis

$$c_o^+(K_{n_1 n_2 \dots n_k}) = H_o^+(n_1, n_2, \dots, n_k).$$

This hypothesis is, however, proved only for a few cases. Using the inequality

$$\sum_{i=1}^{k} n_i . c(K_{n_1, n_2, \dots n_{i-1}, n_i - 1, n_{i+1}, \dots n_k}) =$$

$$= (n_1 + n_2 + \dots + n_k - 4) c \, (K_{n_1 n_2 \dots n_k})$$

which is true not only for the Euclidean plane but for all topological surfaces (we therefore write only c instead of c_o^+) and from some further considerations we can obtain:

1) If $k = 3$, then for $n_1 \leq n_2 \leq 2$ or $n_1 + n_2 + n_3 \leq 7$

$$c_o^+(K_{n_1 n_2 n_3}) = H_o^+(n_1, n_2, n_3)$$

holds. For $8 \leq n_1 + n_2 + n_3 \leq 10$ the estimates from these tables are true:

TABLE 4.

n_1	n_2	n_3	lower b.	upper b.	n_1	n_2	n_3	lower b.	upper b.
1	3	4	5	6	1	3	6	14	15
2	3	3	5	7	1	4	5	17	20
1	3	5	10	10	2	3	5	16	21
1	4	4	10	12	2	4	4	17	20
2	3	4	10	12	3	3	4	16	25
3	3	3	9	15					

2) If $k = 4$, then for $n_1 = n_2 = n_3 = 1$ or $n_1 + n_2 + n_3 + n_4 \leq 7$

$$c_o^+(K_{n_1 n_2 n_3 n_4}) = H_o^+(n_1, n_2, n_3, n_4)$$

holds. For $8 \leq n_1 + n_2 + n_3 + n_4 \leq 10$ the estimates from the following tables are true:

TABLE 5.

n_1	n_2	n_3	n_4	lower b.	upper b.	n_1	n_2	n_3	n_4	lower b.	upper b.
1	1	2	4	6	6	2	2	2	3	11	15
1	1	3	3	6	8	1	1	2	6	12	15
1	2	2	3	6	8	1	1	3	5	18	23
2	2	2	2	6	6	1	1	4	4	16	24
1	1	2	5	10	10	1	2	2	5	18	23
1	1	3	4	11	14	1	2	3	4	19	27
1	2	2	4	11	14	2	2	2	4	19	24
1	2	3	3	11	19	2	2	3	3	19	30

H. Harborth has, according to M. Fiedler, also proved the equality

$$c_o^+(K_{2,2,\dots,2}) = 6.\binom{k}{4}.$$

In the paper /3/ there is one more limit theorem for crossing numbers:

For a given $k \geq 2$ let $G_i = K_{n_1^{(i)} n_2^{(i)} \dots n_k^{(i)}}$ be such a sequence of graphs, that $\lim_{i\to\infty} N_i = \infty$ holds for the numbers $N_i = n_1^{(i)} + n_2^{(i)} + \dots + n_k^{(i)}$. Then

$$\alpha_k = \limsup N_i^{-4}.c_o^+(G_i) = \frac{1}{32k^3}\left[2\binom{k}{3}+\binom{k-1}{3}\right],$$

$$\limsup_{k\to\infty} \alpha_k = 64^{-1}.$$

Supposing Zarankiewicz's hypothesis is true we can obtain

$$\lim_{k\to\infty} \alpha_k = 64^{-1}.$$

In my paper /12/ I have been interested in two problems: For a given $N \geq k \geq 2$ I have investigated the maximal and minimal crossing numbers

$$P(N,k) = \max_{D(N,k)} c_o^+(K_{n_1 n_2 \ldots n_k}), \quad p(N,k) = \min_{D(N,k)} c_o^+(K_{n_1 n_2 \ldots n_k}),$$

where $D(N,k)$ denotes the set of all decompositions of the number N into k integer addends $n_1, n_2, \ldots, n_k$.

I have found the decompositions for which, for the given $N \geq k \geq 2$ the function

(9) $\quad H_o^+(n_1, n_2, \ldots, n_k)$, where $N = n_1 + n_2 + \ldots + n_k$

reaches its extremes. These extremes then give the upper bounds for the numbers $P(N,k)$ and $p(N,k)$.

Order not considered, there exist three different decompositions of the number N into k integers at most, for which the function (9) achieves its maximum. All the summands m_i of these three decompositions fulfil the inequalities

$$2m - 1 \leq m_i \leq 2m + 1, \quad \text{where} \quad m = \left[\frac{N+k}{2k}\right].$$

It is relatively complicated to calculate the number of individual addends and I shall omit it here. See /12, theorem 3/. I shall therefore only give a simple corollary of this theorem here:

$$P(N,k) \leq H_o^+(2m+1, 2m+1, \ldots 2m+1);$$

Ultimately if $N = k(2m-1)$, then

$$P(N,k) \leq H_o^+(2m-1, 2m-1, \ldots 2m-1).$$

We can easily obtain the upper bound for the number $p(N,k)$. We have

$$p(N,k) \leq H_o^+(N-k+1, 1, 1, \ldots, 1).$$

B. TORUS

1.4. The complete graph K_n. The problem of crossing numbers was originally proposed only for a plane. The first time these questions were discussed for a different topological surface was in the case of a complete graph by R. K. Guy, T. Jenkyns and J. Schaer for the torus. In thier paper /14/ they proved for $n \leq 16$:

TABLE 6.

n	≤ 7	8	9	10	11	12	13	14	15	16
$c_1^+(K_n)$	0	4	9	23	≤ 42 ≥ 37	≤ 70 ≥ 56	≤ 105 ≥ 81	≤ 154 ≥ 114	≤ 226 ≥ 156	≤ 326 ≥ 208

For the numbers $n \geq 15$ these estimates are proved:

(10) $$c_1^+(K_n) \geq D_1^+(n) = \frac{1}{210} n(n-1((n-2)(n-3),$$

(11) $$c_1^+(K_n) \leq H_1^+(n) = \frac{59}{5184} (n-1)(n-2)(n-3)(n-4).$$

The next two results are an easy consequence of the given inequalities (10) and (11).

In the paper /14/ J. W. Moon's method for studying geodetic crossing numbers of the graph K_n were used for a torus. The result is

$$g_1^+(K_n) \leq \frac{5}{432} n(n-1)(n-2)(n-3).$$

<u>1.5. The regular bicomplete graph</u> $K_{m,n}$. The crossing number $c_1^+(K_{m,n})$ was investigated by R. K. Guy and T. Jenkyns. For small numbers m, n we can find in /15/, besides others, the following results:

$c_1^+(K_{m,n})$	4	5	6	7	8	9	10
4	0	2	4	6	8	≤ 12 ≥ 11	≤ 16 ≥ 14
5	2	5	8	12	≤ 18 ≥ 16	≤ 24 ≥ 21	≤ 30 ≥ 27
6	4	8	12	≤ 22 ≥ 18	≤ 32 ≥ 24	≤ 42 ≥ 31	≤ 56 ≥ 41
7	6	12	≤ 22 ≥ 18	≤ 35 ≥ 26	≤ 48 ≥ 35	≤ 64 ≥ 45	≤ 84 ≥ 58

The crossing number $c_1^+(K_{m,n})$ is known if min (m,n) = 3. E.g.

$$c_1^+(K_{3,n}) = \left[\frac{(n-3)^2}{12} \right].$$

The main result of /15/ is the theorem:

$$\frac{1}{15} \binom{m}{2} \binom{n}{2} \leq c_1^+(K_{m,n}) \leq \frac{1}{6} \binom{m-1}{2} \binom{n-1}{2}$$

provided in the lower bound m and n are at least equal to one of the (unordered) pairs (7,45), (8,44), (10,43), (14,42), (19,41).

The crossing number $c_1^+(K_{5,n})$ was also investigated by D. Kleitman /16/.

C. Orientable surfaces of genus $g \geq 0$

1.6. The graph G(n,m,t). We denote by G(n,m,t) a graph without loops and parallel edges of n vertices, m edges having the girth t (minimum number of edges in a cycle). Because trees are planar graphs, it can be assumed that $t \geq 3$.

A lower bound for crossing numbers of graphs G(n,m,t) for orientable surfaces has been found recently by P. C. Kainen /18/:

For all orientable surfaces of genus $g \geq 0$ and all graphs G = = G(n,n,t), where t = 3, we have

$$c_g^+(G) \geq \delta_g(G) = m - \frac{t}{t-2} \left[n - 2(1 - g) \right].$$

Simultaneously examples are shown of graphs G = G(n,m,t), for which

$$c_g^+(G) > \delta_g(G).$$

However P. C. Kainen gives the hypothesis:

$$c_{g(G)}^+(G) = \delta_{g(G)}(G),$$

where G is either a complete graph or a regular bicomplete graph and g(G) is the greatest integer for which $\delta_{g(G)} \geq 0$.

He also shows, that the hypothesis holds for K_5, K_6, K_7, $K_{3,3}$, $K_{3,4}$, $K_{4,4}$. From the results of R. K. Guy /14/ and /16/ follows the hypothesis for K_8 and $K_{5,4}$.

In /18/ the crossing numbers of multigraphs are also studied. Among others the following theorem is proved:

If G is a multigraph with at most k parallel edges joining any pair of vertices and UG the induced simple graph, then

$$c_g^+(G) = k^2 .\ c_g^+(UG).$$

2.-A. P R O J E C T I V E   P L A N E

2.1. The complete graph K_n. First of all, we shall investigate the complete graphs K_n for $n \leqq 15$. For its crossing numbers the values in the table 7 have been found.

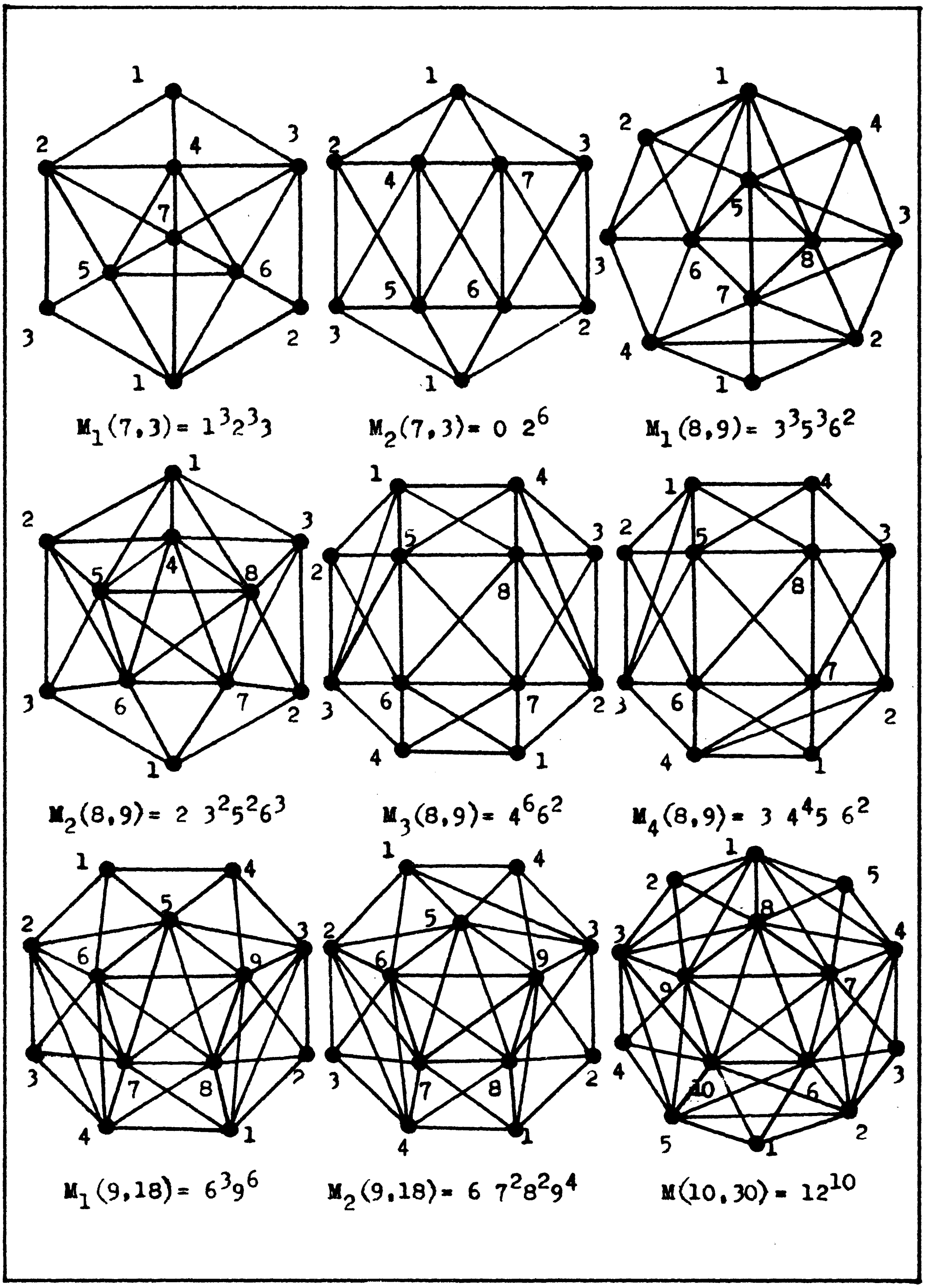
TABLE 7.

n	$\leqq 6$	7	8	9	10	11	12	13	14	15
$c_1(K_r)$	0	3	9	18	30	$\leqq 57$ $\geqq 49$	$\leqq 92$ $\geqq 74$	$\leqq 137$ $\geqq 107$	$\leqq 203$ $\geqq 150$	$\leqq 287$ $\geqq 205$

These results we shall complete by the table 8, which gives for $n \leqq 10$ the numbers $d_1(K_n)$ of topologically different models of graphs K_n in the projective plane.

TABLE 8.

n	1	2	3	4	5	6	7	8	9	10
$d_1(K_n)$	1	1	2	3	2	1	2	4	2	1



$M_1(7,3) = 1^3 2^3 3$
$M_2(7,3) = 0\ 2^6$
$M_1(8,9) = 3^3 5^3 6^2$
$M_2(8,9) = 2\ 3^2 5^2 6^3$
$M_3(8,9) = 4^6 6^2$
$M_4(8,9) = 3\ 4^4 5\ 6^2$
$M_1(9,18) = 6^3 9^6$
$M_2(9,18) = 6\ 7^2 8^2 9^4$
$M(10,30) = 12^{10}$

Fig. 1

The survey of all models M(n) of the complete graphs K_n for n = = 7,8,9,10 gives figure 1. (The projective plane and models M(n) in it, arise if the opposite sides and nodes of the corresponding polygons are identified in the usual way.) There we use e.g. the denotation $M(7,3) = 1^3 2^3 3$ for the model of graph K_7 which has three crossings distributed on its edges in the following way:

a) One crossing lies on edges which coincide with each of certain three vertices;

b) On edges which coincide with each of other three vertices there are two crossings;

c) Three crossings lie on the edges issuing from the last vertex.

From Fig. 1 the upper bounds follow

$$c_1(K_7) \leqq 3, \quad c_1(K_8) \leqq 9, \quad c_1(K_9) \leqq 18, \quad c_1(K_{10}) \leqq 30.$$

We shall prove that in all these cases the equalities take place. We shall use the inequality

$$(12) \qquad r_n \geqq \frac{n}{n-4} r_{n-1} \qquad (n > 4),$$

where r_n denotes the number of crossings of an arbitrary model of graph K_n and r_{n-1} the minimum of the numbers of intersections in any of its submodel generated by n-1 vertices. A corollary of the inequality (12) is a well-known relation between the crossing numbers $c_1(K_n)$ and $c_1(K_{n-1})$:

$$(13) \qquad c_1(K_n) \geqq \frac{n}{n-4} c_1(K_{n-1}), \qquad (n > 4).$$

We shall prove

$$(14) \qquad c_1(K_7) = 3, \quad d_1(K_7) = 2.$$

We know that $c_1(K_7) \leqq 3$. If we suppose that $c_1(K_7) < 3$, then from the inequality (12) follows that a minimal model M(7) must exist which contains a submodel M(6,0). In the projective plane it forms a map the regions of which are triangles only. Therefore on edges with endvertex $u \in M(7) - M(6,0)$ there are at least three crossings, but this is a contradiction. Therefore $c_1(K_7) = 3$ is true. From here and from Fig. 1 follows $d_1(K_7) \geqq 2$. The inequality (12) implies that each model M(7,3) has as a submodel, one of the models M(6,0), M(6,1). It is easy to see that there are only two models of this type: $M(6,0) = 0^6$, $M(6,1) = 0^2 1^4$.

These can be extended into models $M(7,3) = 1^3 2^3 3$ and $M(7,3) = 02^6$ only. This completes the proof of the equality $d_1(K_7) = 2$.

In the same way we can prove

(15) $$c_1(K_8) = 9, \qquad d_1(K_8) = 4.$$

In the proof it is necessary to investigate all extentions of four models of types $M(7,3)$ and $M(7,4)$.

Using the same method we can obtain the equalities

(16) $$c_1(K_9) = 18, \qquad d_1(K_9) = 2.$$

To verify $c_1(K_9) = 18$ it is sufficient to investigate all possible extensions of four models $M(8,9)$ which are models of the graph K_9. The proof of the equality $d_1(K_9) = 2$ is much longer. In this case it is necessary to investigate 9 models $M(6,x)$, where $x \leq 2$, 18 models $M(7,y)$, where $y \leq 5$ and finally 12 models $M(8,10)$.

On the other hand from (16) the equalities

(17) $$c_1(K_{10}) = 30, \qquad d_1(K_{10}) = 1$$

can easily be derived. E.g. the inequality $c_1(K_{10}) = 30$ follows immediately from (13) and (16). The equality $d_1(K_{10}) = 1$ can be verified if we investigate all extensions of two models $M(9)$ to models of graph K_{10}.

We obtain the lower bounds of crossing numbers $c_1(K_n)$ for $11 \leq n \leq 15$ given in the table 7 from the equality $c_1(K_{10}) = 30$ using repeatedly the inequality (13). The upper estimates follow from the existence of models $M(11,57)$, $M(12,92)$, $M(13,139)$, $M(14,203)$, $M(15,287)$.

E.g. the models

$$M(11,57) = 18^3 19^3 21^2 24^2 27,$$

$$M(12,92) = 25^2 27\ 30^3 31\ 32\ 34^2 35^2$$

are generated by the vertices $1,2,\dots,i$ (where $i = 11,12$) in the model $M(13,139) = 38\ 39\ 40\ 41\ 42\ 43^2 44^2 45^3 47$. (Fig. 2; the symbol $\langle 7 \rangle$ denotes that in the regular polygon 7 8 9 10 11 12 13 all diagonals have been drawn.)

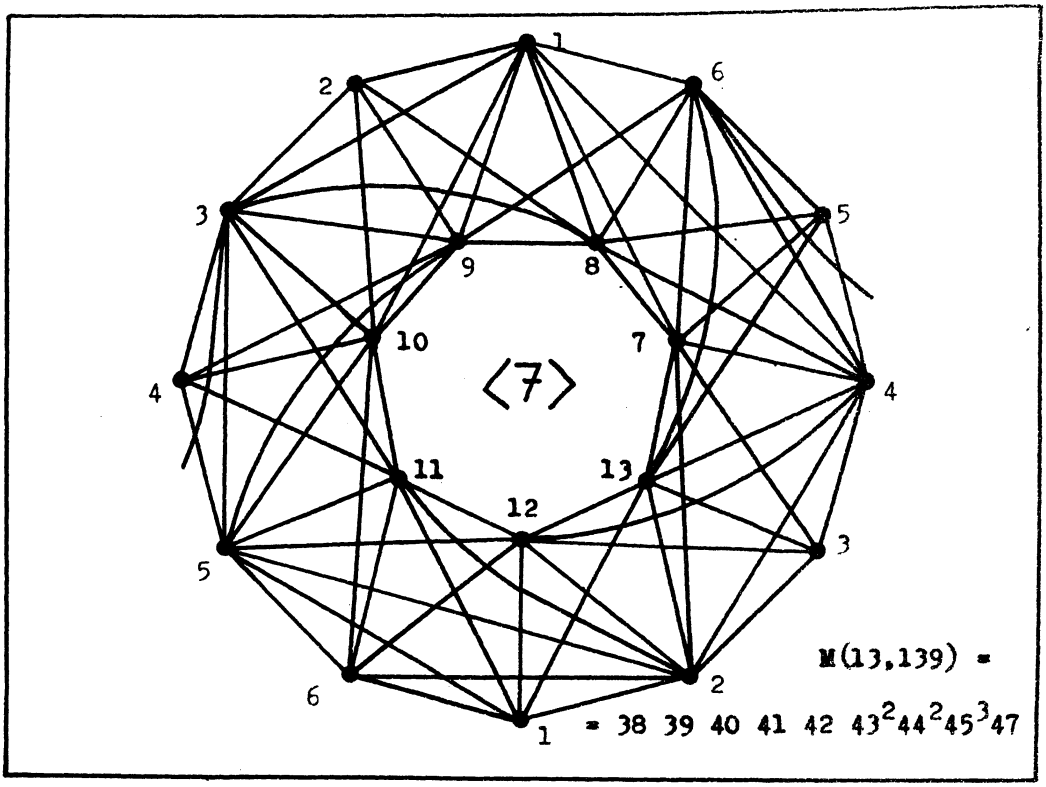


Fig. 2

The model $M(14,203) = 55^7 61^7$ is for $n = 14$ the model $A(n)$, which will be constructed later. Finally the model

$$M(15,287) = 72\ 73\ 74^2 75\ 76^3 77^2 78\ 79\ 80^2 81$$

is presented by Figure 3.

It is interesting to give the Hasse diagram (fig. 4) of the binary relation "to be a submodel" for the models of graphs K_n with minimal number of vertices ($6 \leq n \leq 10$) and the models $M(11,57)$ to $M(15,287)$ presented above.

An immediate consequence of inequalities (13) and $c_1(K_{15}) \geq 205$ is the following theorem, which gives a lower bound for the crossing numbers $c_1(K_n)$, where $n > 15$.

<u>T H E O R E M 1.</u> For all natural numbers n 15 we have

(18) $$c_1(K_n) \geq D_1(n) = \frac{41}{6552} n(n-1)(n-2)(n-3).$$

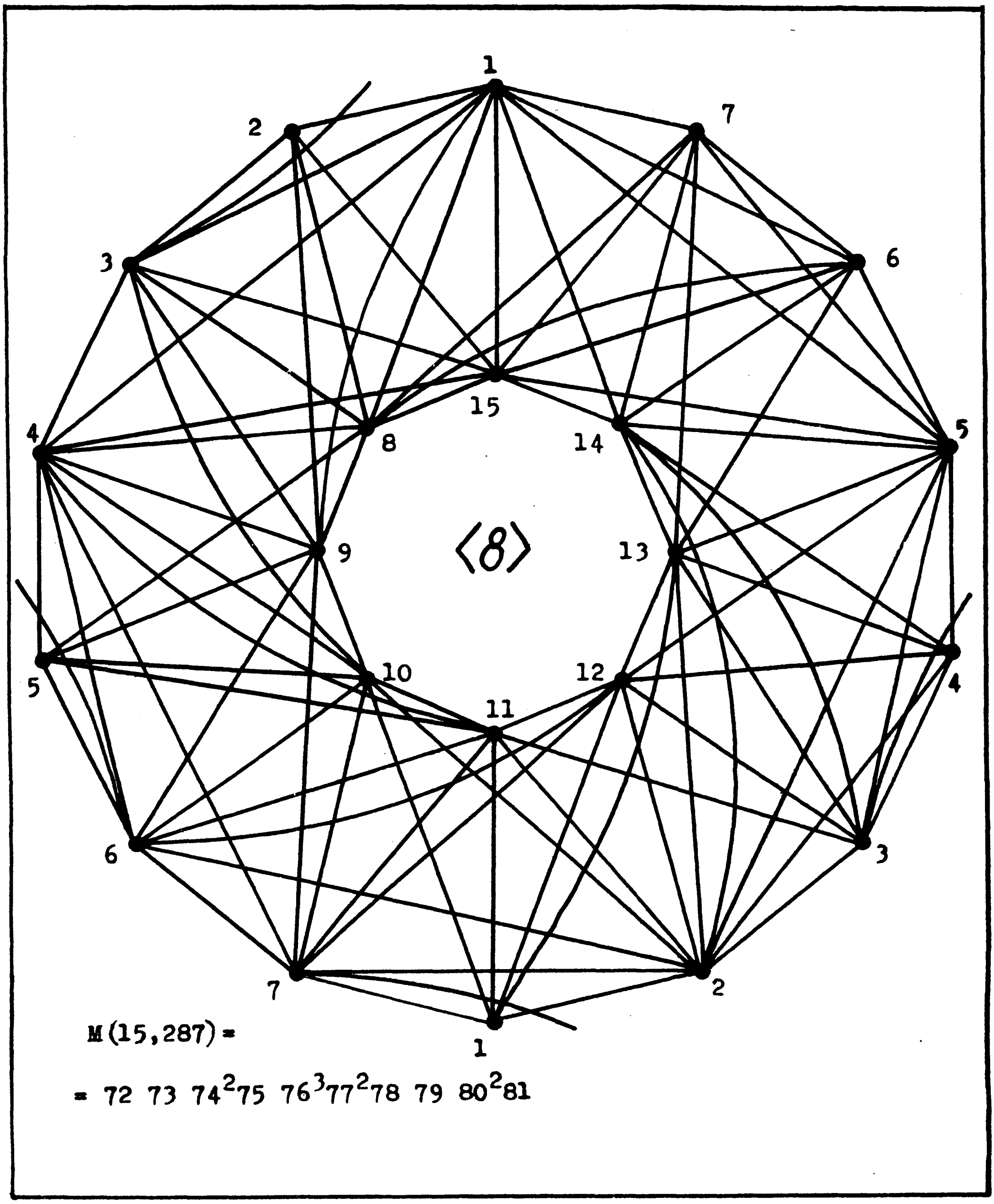


Fig. 3

We shall obtain the upper bounds for crossing numbers $c_1(K_n)$, where $n \geq 8$, $n \neq 11$ by constructing the models of graphs K_n.

<u>The auxiliary construction 1.</u> In the Euclidean plane we shall construct an auxiliary model M. Let $n \equiv \pm 2 \pmod 8$ be given. Denote

(19) $m = \frac{n}{2}$, $r = \left[\frac{n}{8}\right]$, $s = \left[\frac{m}{2}\right]$.

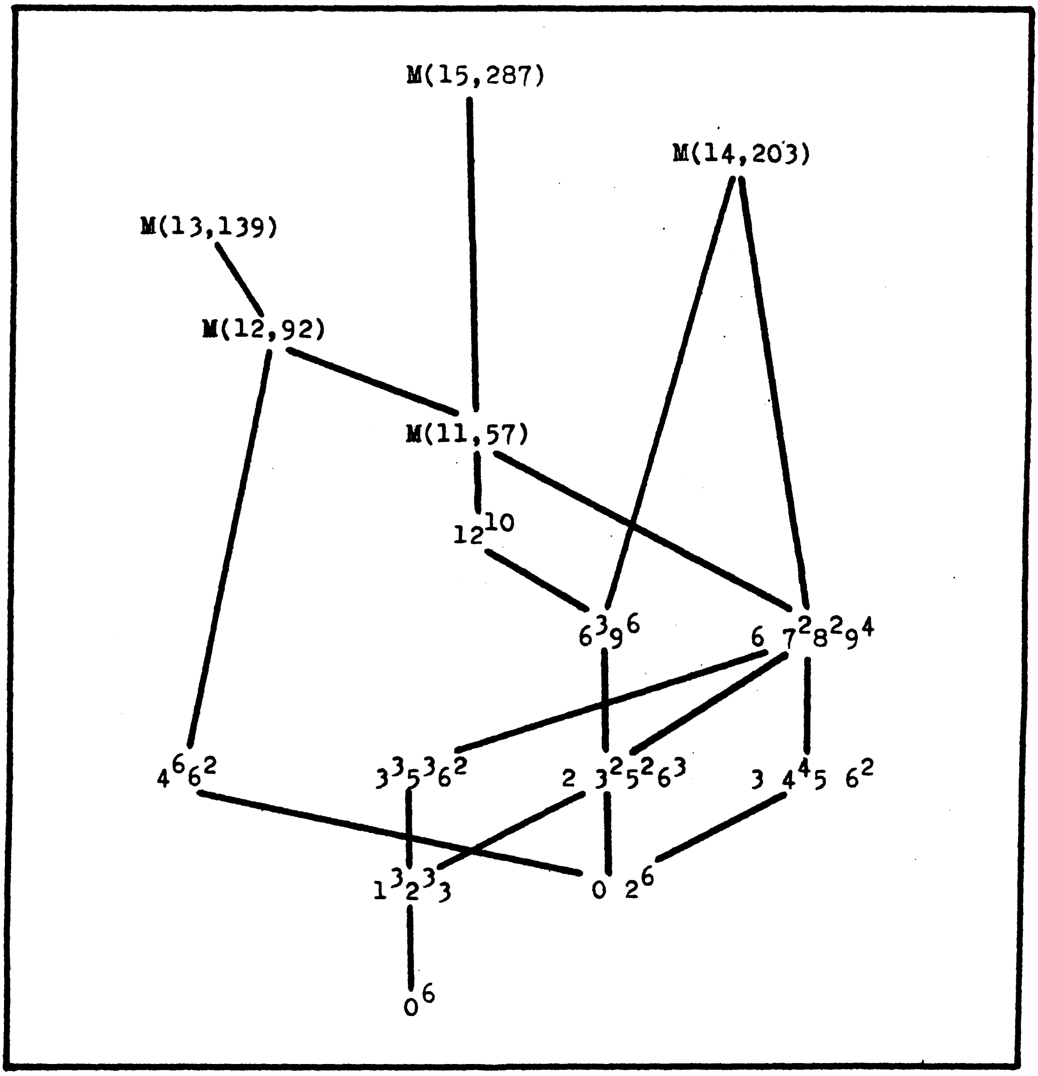


Fig. 4

We construct three regular polygons with girdles

$$W = w_1 w_2 \ldots w_m, \qquad V = v_1 v_2 \ldots v_m, \qquad U = u_1 u_2 \ldots u_m,$$

with a commón centre, for the mapping

$$H_1 : W \to V,\ w_i \to v_i \qquad (i = 1, 2, \ldots, m)$$

to be a homothety with the coefficient $h_1 = -\frac{1}{2}$ and the mapping

$$H_2 : V \to U,\ v_i \to u_i \qquad (i = 1, 2, \ldots, m)$$

to be a homothety with the coefficient

$$h_2 = \frac{1}{2}, \quad \text{if } m \equiv 1 \pmod 4, \quad h_2 = -\frac{1}{2}, \quad \text{if } m \equiv -1 \pmod 4.$$

The points w_i, v_i, u_i $(i = 1,2,\ldots,m)$ are the vertices of the auxiliary model M.

Denote by C_U, C_{VU}, C_{WL}, C_L, C_{LU} the bouned closed regions with boundaries

$$U, \quad V \cup U, \quad W \cup L, \quad L = w_1 v_{1+s} w_2 v_{2+s} w_3 \ldots w_m v_s, \quad L\ U$$

respectively. Now we shall construct the edges of models M, which we shall draw as curves homeomorphic with straight segments even as straight segments if possible. We shall simultaneously take care that no unnecessary crossing take place. We shall divide the edges of model M into five classes.

For all $i, j = 1,2,\ldots,m$ we construct

1) a-edges $a_{ij} = v_i w_j$ in the region C_{WV}, if

(20) $$j-i \not\equiv 0, \pm 1, \pm 2, \ldots, \pm r \pmod m.$$

2) b-edges $b_{ij} = v_i v_j$ in the region C_{VU}, if

(21) $$j-i \equiv \pm 1, \pm 2, \ldots, \pm r \pmod m.$$

3) c-edges $c_{ij} = v_i u_j$, if

(22) $$j-i \equiv 0, \pm 1, \pm 2, \ldots, \pm r \pmod m$$

and simultaneously $m \equiv 1 \pmod 4$ or if (20) and $m \equiv -1 \pmod 4$ hold. We construct all c-edges so that they are parts of the region C_{VU}.

4) d-edges $d_{ij} = u_i w_j$, if (20) and $m \equiv 1 \pmod 4$ is fulfilled or if (22) and $m \equiv -1 \pmod 4$ is fulfilled. We draw all d-edges in the region C_{LU}.

5) e-edges $e_{ij} = u_i u_j$ for all $i \neq j$ in the region C_U.

We conctruct the model so that the parts of the model M belonging to "opposite" triangles

$w_j v_i w_k$, $v_j w_i v_k$, where $j-i \equiv i-k \equiv s \pmod m$

are homeomorphic drawings.

<u>Constructions of models A(n) of the graphs</u> K_n ($8 \leq n \neq 11$).

a) Let $n \equiv \pm 2 \pmod 8$. We shall construct an auxiliary model M. From the region C_L and the corresponding part of the model M we obtain a model of the projective plane and in it a model A(n) of a complete graph K_n by identifying all pairs of "opposite" vertices w_i, v_i ($i=1,2, \ldots, m$), "opposite" a-edges.

a_{ij}, a_{ji}, where $j-i \equiv s \pmod m$

and corresponding crossings on these edges. The arising model A(n) has the set of vertices

$$A_o(n) = \{v_1=w_1, v_2=w_2, \ldots, v_m=w_m, u_1, u_2, \ldots, u_m\}.$$

b) Let $n + 1 \equiv \pm 2 \pmod 8$. Then the model A(n) is generated in the model A(n+1) by the set of vertices

$$A_o(n) = A_o(n+1) - \{v_1\}.$$

c) Let $n + 2 \equiv \pm 2 \pmod 8$. Then we construct the model A(n) as a submodel of A(n+2) generated by the set of vertices

$$A_o(n) = A_o(n+2) - \{v_1, v_{r+1}\}.$$

d) If $n + 3 \equiv \pm 2 \pmod 8$, we construct the model A(n) as a part of the model A(n+3) generated by the set of vertices

$$A_o(n) = A_o(n+3) - \{v_1, v_r, v_{2r-1}\}.$$

In this way for all $n \geq 8$, $n \neq 11$ the model A(n) is constructed for each graph K_n in the projective plane. The number of crossings of these models give the upper estimate for the number $c_1(K_n)$.

During the enumeration we use the following denotation: Similarly as for the auxiliary model M, we shal use the terms a-edge, b-edge, etc for the model A(n). The crossing of a y-edge and z-edge ($y,z = a,b,c,d,e$) will be called a yz-crossing. The number of all yz-crossings lying on the edges with endvertex $x \in A(n)$ we denote $p_{yz}(x)$. The number of all crossings lying on the edges issuing from the vertex x will be denoted p(x). Similarly $p_{yz}(x_1,x_2)$ means the mumber of yz-crossings which lie on at least one edge coinciding with the vertex x_1 and simultaneously on at least one edge with the endvertex x_2. The number of all crossing points of the model A(n) will be denoted a(n).

For the model A(n), where $n \equiv \pm 2 \pmod 8$ we can easily find the data given in the following table 9; there

$$t = \left[\frac{m+1}{4}\right] \text{ and } i = 1,2,\dots,m.$$

TABLE 9.

	v_i	u_i
p_{aa}	$2\binom{2t}{3}$	-
p_{ad}	$6(t+r)\binom{t}{2}$	$2(t+r)\binom{t}{2}$
p_{bb}	$\binom{2r}{3}$	-
p_{bc}	$3(t+r+1)\binom{r}{2}$	$(t+r+1)\binom{r}{2}$
p_{bd}	$3(t+r)\binom{r+1}{2}$	-
p_{cc}	$2\binom{t+r+1}{3}$	$2\binom{t+r+1}{3}$
p_{cd}	$4\binom{t+r+1}{3}$	$4\binom{t+r+1}{3}$
p_{dd}	$2\binom{t+r}{3}$	$2\binom{t+r}{3}$
p_{ee}	-	$\binom{2t+2r}{3}$

The values $p_{yz}(x)$, which are not given in Table 9, are equal to zero.

From the data in the table 9 $p(v_i)$, $p(u_i)$ $(i=1,2,\dots,m)$ can be calculated for $n \equiv \pm 2 \pmod 8$ and from them, using the equality

$$a(n) = \frac{m}{4}[p(v_1) + p(u_1)]$$

the number $a(n)$:

For $n \equiv 2 \pmod 8$ we obtain

(23) $p(v_i) = \frac{1}{6}r(160r^2 - 87r - 1)$, $p(u_i) = \frac{1}{6}r(152r^2 - 81r + 1)$,

(24) $a(n) = r^2(4r+1)(13r-7) = 2^{-10}n(n-2)^2(13n-82)$.

For $n \equiv -2 \pmod 8$ we obtain

(25) $p(v_i) = \frac{1}{6}r(160r^2 + 165r + 41)$, $p(u_i) = \frac{1}{6}r(152r^2 + 147r + 31)$,

(26) $a(n) = r(4r+3)(13r^2+13r+3) = 2^{-10}n(n-6)(13n^2-52n+36)$.

Because the differences

$$p(v_i) - p(u_i) = \frac{1}{3}r(r-1)(4r+1) \quad \text{if } n \equiv 2 \pmod 8,$$

$$p(v_i) - p(u_i) = \frac{1}{3}r(r+1)(4r+5) \quad \text{if } n \equiv -2 \pmod 8$$

are non-negative for all $n \equiv \pm 2 \pmod 8$, we obtain from (23) to (26):

The model $A(n)$, where $n + 1 \equiv \pm 2 \pmod 8$ is a model of the graph K_n which, of all the submodels in $A(n+1)$, has the smallest number of crossings; while for $n + 1 \equiv 2 \pmod 8$

(27) $a(n) = \frac{1}{6}r(2r-1)(156r^2-47r-1) = 3^{-1}2^{-10}(n-1)(n-5)(39n^2-172n+$ $+117)$ is true, and for $n + 1 \equiv -2 \pmod 8$

(28) $a(n) = \frac{1}{6}r(2r+1)(156r^2+115r+13) = 3^{-1}2^{-10}(n-1)(n-5)(39n^2-160n+33)$ holds.

For further calculations we use Table 10 which is a result of the construction of the model $A(n)$ for $n \equiv \pm 2 \pmod 8$. The values $p_{yz}(v_j,v_k)$ which are not given in the table 10 are all equal to zero.

TABLE 10.

	$j-k \equiv \pm i \pmod m$ $0 < i \leq r$	$j-k \equiv \pm(t+r-i) \pmod m$ $0 \leq i < t$
$p_{aa}(v_j,v_k)$	$\binom{2t-i}{2}$	$(3t-1)(t-1) - \binom{i}{2}$
$p_{ad}(v_j,v_k)$	$2(t+r)(t-i)$	$2(t+r)(t-1)$
$p_{bb}(v_j,v_k)$	$r(2r-3) - \frac{1}{2}(i+1)(i-2)$	$\binom{i+1}{2} - (t-r)i$
$p_{bc}(v_j,v_k)$	$(t+r+1)(2r-i-1)$	-
$p_{bd}(v_j,v_k)$	$(t+r)i$	$2(t+r)(r-i)$
$p_{cc}(v_j,v_k)$	$\binom{t+r+1-i}{2}$	$\binom{i+1}{2}$
$p_{cd}(v_j,v_k)$	$2\binom{i}{2}$	$2\binom{t+r-i}{2}$
$p_{dd}(v_j,v_k)$	$\binom{t+r-i}{2}$	$\binom{i}{2}$

We shall now prove: For n large enough, where $n+2 \equiv \pm 2 \pmod 8$ the model A(n) is a model of the graph K_n with the smallest number of crossings, which is a part of the model A(n+2). Meanwhile for $n + 2 \equiv 2 \pmod 8$

(29) $a(n) = \frac{1}{6}r(2r-1)(156r^2-127r+25)=3^{-1}2^{-10}n(n-4)(39n^2-254n+400)$

is true. For $n + 2 \equiv -2 \pmod 8$

(30) $a(n) = \frac{1}{3}(156r^4+113r^3+12r^2+r+9)=3^{-1}2^{-10}(39n^4-398n^3+1224n^2-$
$-544n+7296)$

holds.

The number of crossings of a model of the graph K_n, where $n + 2 \equiv \pm 2 \pmod 8$, which is a part of the model A(n+1) while it has the minimal number of crossings, is equal to

(31) $$\min_{\substack{x,y\in A(n+2)\\ x\neq y}} [a(n+2) - p(x) - p(y) + p(x,y)] .$$

Because $p(u_i)$, $p(v_i)$, $i = 1,2,\dots,m$ are polynomials of the 3^{rd} degree and $p(u_i) \leq p(v_i)$ holds; because to the function $p(x,y)$ where $x,y \in A(n+2)$ there exists a majorant polynomial of the 2^{nd} degree, e.g. $(2m-1)^2$, it is evident that for an n large enough the function

(32) $$f(x,y) = a(n+2) - p(x) - p(y) + p(x,y)$$

reaches its minimum (30) for some pair of vertices v_j, v_k. For these v_j, v_k the function $p(x,y)$ must also reach its minimum. Using Table 10 we can calculate that

$$\min_{1\leq j<k\leq m} p(v_j,\ v_k) = \tfrac{1}{2}r(15r-9), \quad \text{if } n \equiv 2 \pmod 8,$$

$$\min_{1\leq j<k\leq m} p(v_j,\ v_k) = 8r^2+5r+3, \quad \text{if } n \equiv -2 \pmod 8.$$

In both cases the minimum is achieved for such j, k that $j-k \equiv \pm r \pmod 8$. For such j,k the value of the function (32) is equal to the minimum (31) as well as to the value of the function $a(n)$ given by the formulae (29), (30) respectively. So we conclude the proof.

Finally we shall calculate the values $a(n)$ for $n + 3 \equiv \pm 2 \pmod 8$, $n \geq 15$. We obtain:

(33) $$a(15) = 312,\ a(19) = 946$$

and for $n + 3 \equiv 2 \pmod 8$, $n \geq 23$

(34) $$a(n) = \tfrac{1}{2}(104r^4-190r^3+133r^2-35r+6) =$$
$$= 2^{-10}(13n^4-138n^3+572n^2-630n+1719)$$

and for $n + 3 \equiv -2 \pmod 8$, $n \geq 27$

(35) $$a(n) = \tfrac{1}{2}(104r^4+22r^3-r^2-11r+60) =$$
$$= 2^{-10}(13n^4-134n^3+496n^2-1466n+33219).$$

Formulae (33) to (35) can be deduced from the equality

(36) $$a(n) = a(n+3) - 3p(v_1) + 2p(v_1,v_r) + p(v_1,v_{2r-1}) -$$
$$- p(v_1,v_r,v_{2r-1}).$$

While in the equality (36) for n = 15, 19

$$p(v_1, v_r, v_{2r-1}) = 2$$

and for $n > 19$

$$p(v_1, v_r, v_{2r-1}) = 0$$

is true.

Now it is easy to prove:

T H E O R E M 2. For the crossing numbers $c_1(K_n)$ we have

(37) $$c_1(K_n) \leq a(n) \qquad n \geq 7,$$

where the function $a(n)$ is defined by the equations (24), (26) to (30), (34) and (35) if $n \equiv 2, -2, 1, -3, 0, -4, -1, 3 \pmod 8$ respectively.

P r o o f : For $n \leq 15$ the inequality (37) follows from Table 7. For $16 \leq n \neq 19$ the theorem follows from the constructions of models A(n). For n = 19 the theorem stems from the existence of the model M(19, 919) of the graph K_{19}, which is generated in the model A(22) by the set of vertices $A_0(22) = \{v_1, v_3, v_5\}$. For n =19 - because of this fact - the inequality (37) gives the estimate 937.

Now we shall state a few simple consequences of theorems 1 and 2.

T H E O R E M 3. a) For all $n \geq 6$ the crossing numbers $c_1(K_n)$ fulfil the inequality

(38) $$c_1(K_n) < H_1(n) = \frac{13}{2^{10}} n(n-1)(n-3)(n-6) - \frac{1}{3.2^{10}} h(n) n^2 (4n-135),$$

where the function h(n) is defined by Table 11.

b) For all $n \geq 20$ the difference

$$o(n^2) = H_1(n) - a(n)$$

is a polynomial of the second degree at the most.

TABLE 11.

n (mod 8)	-1	0	1	2	3	4	5	6
h(n)	6	5	4	3	3	2	1	0

The proof for $n \geq 9$, $n \neq 19$ follows from Table 12, where the differences $H_1(n) - a(n)$ are noted. For $n = 19$ we obtain $H_1(19) = 923{,}98\ldots$. From the inequality $c_1(K_{19}) \leq 919$, for $n = 19$ the inequality (38) is proved. For $n = 6, 7, 8$ the proof is obvious.

TABLE 12.

n (mod 8)	$H_1(n) - a(n)$
-1	$207(n-3)^2 + 2016(n-3) + 270$
0	$362\, n^2 + 898n$
1	$289(n-1)^2 + 1438(n-1) + 564$
2	$348n^2 + 282n$
3	$3696(n-27) + 135$
4	$119(n-9)^2 + 1984(n-9) + 921$
5	$10(n-1)^2 + 316(n-1) + 141$
6	$9(n-6)^2 + 54(n-6)$

T H E O R E M 4. a) For all natural numbers $n>5$

(39) $$c_1(K_n) < \frac{13}{16} H_o^+(n) = 0{,}8125.H_o^+(n),$$

holds, where $H_o^+(n)$ is given by (4).

b) For all natural numbers $n > 16$

(40) $$c_1(K_n) < \frac{13.3^4}{59.24} H_1^+(n) < 1{,}1155.H_1^+(n)$$

holds, where $H_1^+(n)$ is defined by (11).

The proof of (39) for $n \geqq 33$ and $n = 22, 30$ follows from the inequalities

$$H_1(n) \leqq \frac{13}{2^{10}} n(n-1)(n-3)(n-6),$$

$$n(n-1)(n-3)(n-6) < (n-1)(n-2)(n-3)(n-4),$$

$$\frac{1}{16} n(n-1)(n-3)(n-6) < \left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right] \cdot \left[\frac{n-2}{2}\right] \cdot \left[\frac{n-3}{2}\right] .$$

The proof of (39) for $11 \leqq n \leqq 33$ follows from inequalities

$$H_1(n) \leqq \frac{13}{2^{10}} n(n-1)(n-3)(n-6) + \frac{1}{2^9} n^2(135-4n) < \frac{13}{16} H_0^+(n).$$

For all other n the inequality (39) follows from the table 7.

The inequality (40) for $23 \leqq n < 33$ follows from these inequalities

$$H_1(n) \leqq \frac{13}{2^{10}} n(n-1)(n-3)(n-6) + \frac{1}{2^9} n^2(135-4n) < \frac{13.3^4}{59.2^4} H_1^+(n).$$

For $17 \leqq n \leqq 22$ we obtain the estimate (40) from the inequalities

$$H_1(n) \leqq \frac{13}{2^{10}} n(n-1)(n-3)(n-6) + \frac{5}{3.2^{10}} n^2(135-4n) < \frac{13.3^4}{59.2^4} H_1^+(n).$$

Thus theorem 4 is proved.

I shall conclude with the limiting property of crossing numbers.

<u>T H E O R E M 5.</u> For crossing numbers $c_1(K_n)$ the inequalities

(41) $$\limsup c_1(K_n).n^{-4} \leqq \frac{13}{2^{10}} \doteq 0,0127,$$

(42) $$\liminf c_1(K_n).n^{-4} \geqq \frac{41}{6552} \doteq 0,0063$$

hold.

The proof follows immediately from the inequalities (38) and (18).

B. K L E I N ' S B O T T L E

2.2. The complete graph K_n. Similarly as in other preceding cases, first of all I give the results concerning the crossing numbers of the graphs K_n for $n \leq 15$. They are given in the table 13.

TABLE 13.

n	= 6	7	8	9	10	11	12	13	14	15
$c_2(K_n)$	0	1	4	9	≤ 24 ≥ 22	≤ 44 ≥ 35	≤ 72 ≥ 53	≤ 109 ≥ 77	≤ 161 ≥ 108	≤ 239 ≥ 148

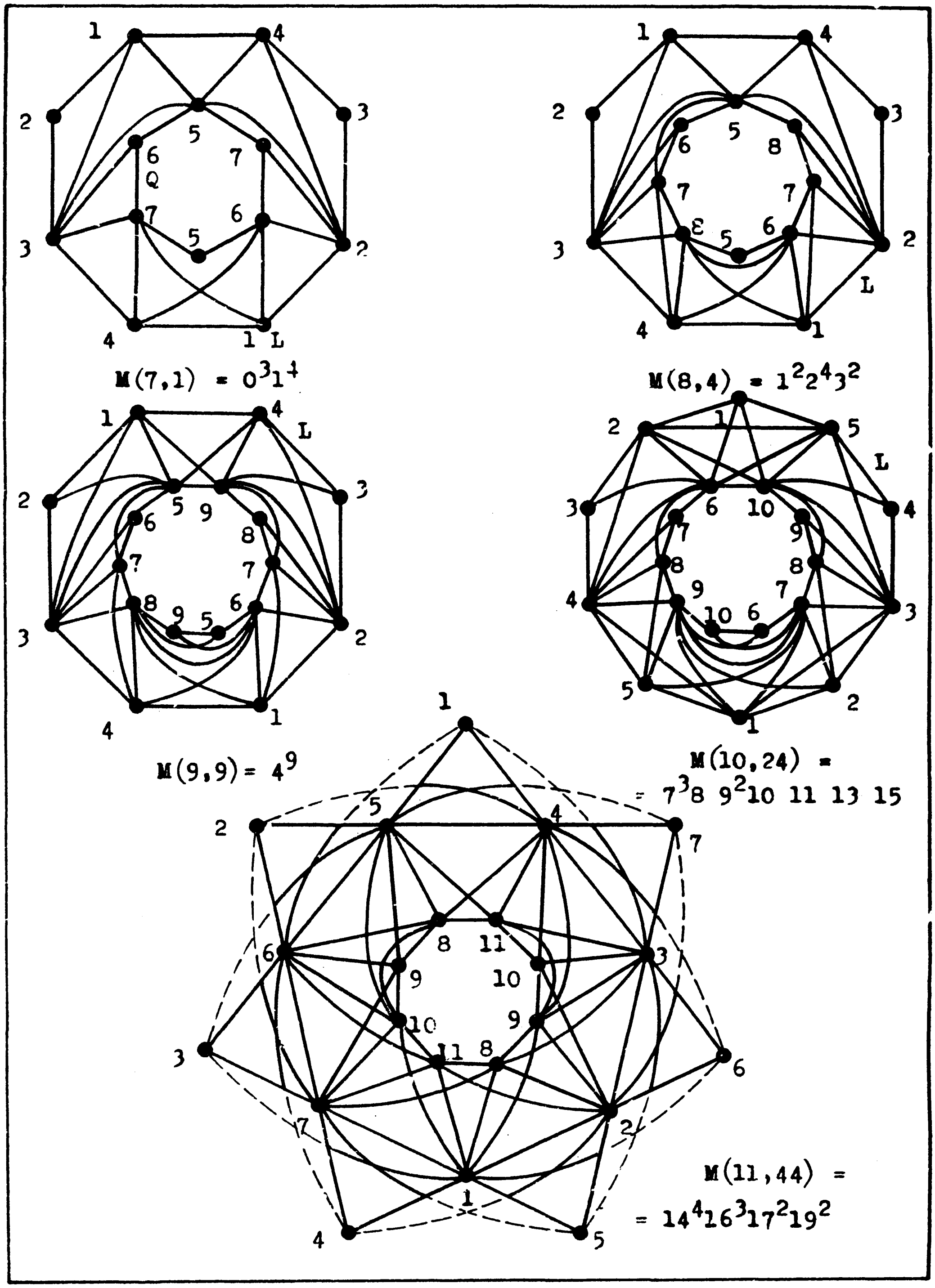
The upper bounds follow from the existence of corresponding models M(n) of the graphs $K_n (n = 7,8,\dots,15)$ in Klein's bottle. The figure 5 shows the models M(n) for n = 7,8,...,11. The Klein's bottle and on it the models M(n) arise if we identify the opposite points on each of two polygonal boundaries of presented regions. (The regions are homeomorphic to an annulus.)

The model $M(12,72) = 19\ 20\ 23^3 24\ 25^2 26^2 27^2$ is generated by the vertices 1,2,...,5,7,8,...,13 in the model M(13,109) = $= 31^2 32^2 33^3 34^3 35\ 37^2$. We can construct this model in this way: We draw two auxiliary models M_1 and ⟪9⟫ (fig. 6a,b) in two projective planes with polygonal holes 1,2,...,9. After identifying the corresponding vertices 1,2,...,9 lying on these two surfaces we obtain Klein's bottle and the model M(13,109) on it.

The model $M(14,161) = 40^2 44^2 45^2 46\ 47^2 48^3 51^2$ is generated by the vertices 1,2,...,14 in the model

$$M(15,239) = 53\ 55\ 57^2 59\ 60\ 61\ 62^2 64\ 66\ 71\ 73\ 78^2.$$

We draw it similarly to the model M(13,139). We construct two axiliary models M_2 (fig. 7) and ⟪9⟫ (fig. 6b) in two projective planes with polygonal holes 1,2,...,9. After identifying the corresponding points 1,2,...,9 lying in these two surfaces the Klein's bottle arises with the model M(15,239).



$M(7,1) = 0^3 1^4$
$M(8,4) = 1^2 2^4 3^2$
$M(9,9) = 4^9$
$M(10,24) = 7^3 8\ 9^2 10\ 11\ 13\ 15$
$M(11,44) = 14^4 16^3 17^2 19^2$

Fig. 5

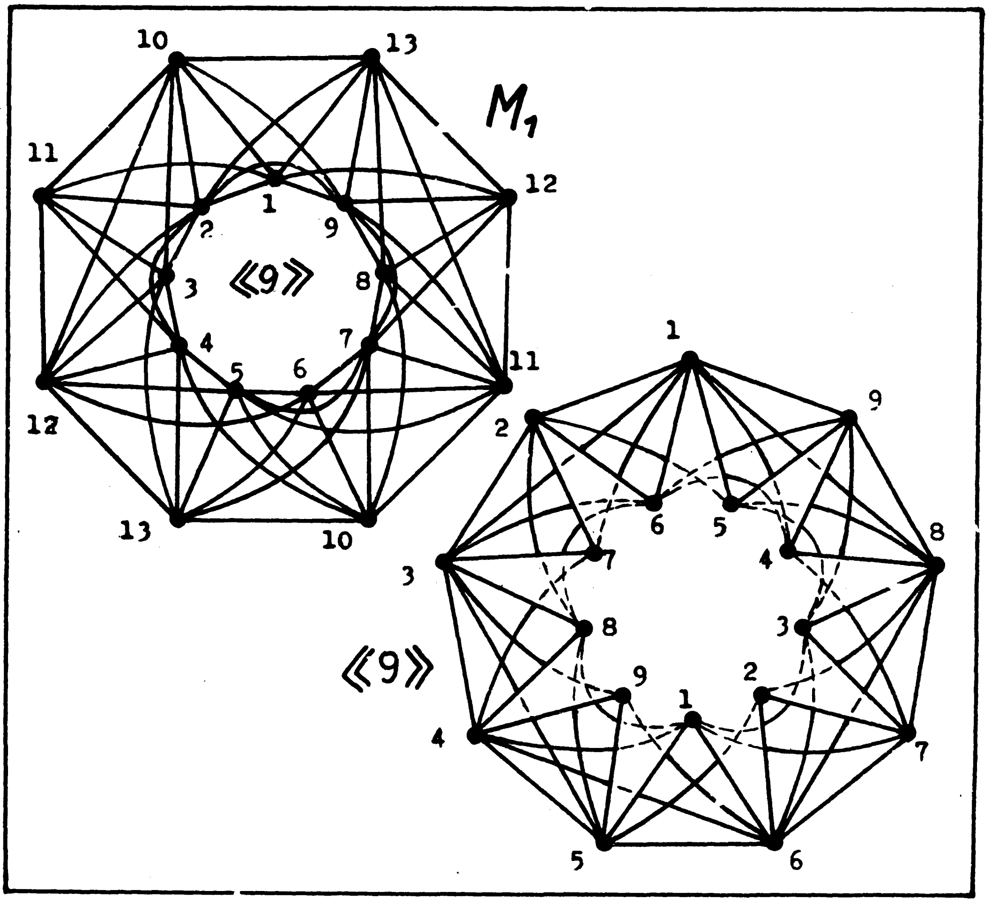


Fig. 6a, b

Now we shall prove the lower estimates from the table 13. It is known that $c_2(K_7) > 0$ (see e.g. G. Ringel /15/). This inequality together with $c_2(K_7) \leq 1$ gives

(43) $$c_2(K_7) = 1.$$

We shall prove

(44) $$c_2(K_8) = 4.$$

Suppose, that $c_2(K_8) = x < 4$. Then any model $M(8,x)$ has at least one submodel $M(7,1)$. Each model $M(7,1)$ forms in Klein's bottle a map P, which has one qudrangle and 14 triangles. The map has namely $a_0 = 7+1 = 8$ vertices, $a_1 = 21+2 = 23$ edges and from Euler's polyhedral formula $a_2 = 23-8 = 15$ regions. The number

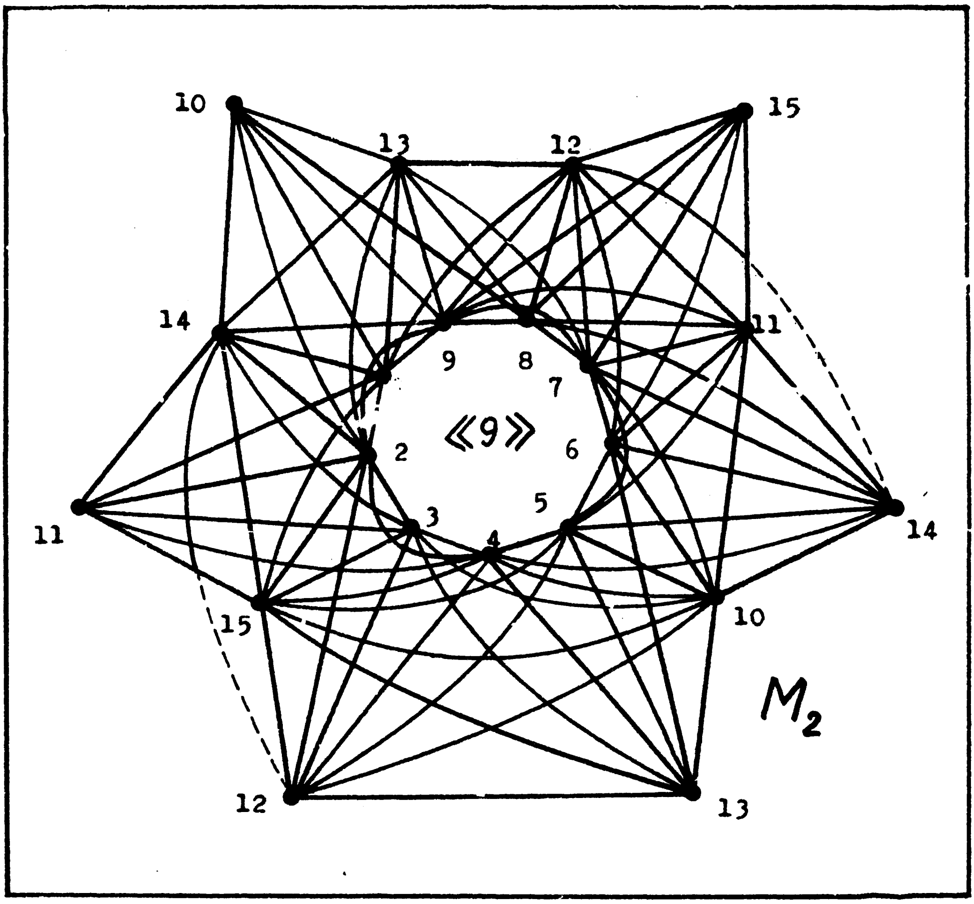


Fig. 7

of edges can be expressed also from the known equality

(45) $$2a_1 = 3a_2 + t_4 + 2t_5 + 3t_6 + \dots ,$$

where t_k ($k = 4,5,\dots$) denotes the number of k-sided regions. Hence we obtain $t_4 = 1$, $t_5 = t_6 = \dots = 0$. The number of triangles follows from the equality

(46) $$2a_1 = 3t_3 + 4t_4 + 5t_5 + \dots .$$

The point $u \in M(8,x) - M(7,1)$ belongs to a region of the map P. It can be easily verified that on the edges issuing from the vertex u, there are at most three crossings, that means $x \geq 1+3 = 4$, but this is a contradiction. Therefore it is true that $c_2(K_8) \geq 4$. Hence together with the inequality $c_2(K_8) \leq 4$ we obtain $c_2(K_8) = 4$.

Similarly we shall prove

(47) $$c_2(K_9) = 9.$$

Suppose that $c_2(K_9) = y < 9$. Then any model $M(9,y)$ must contain at least one submodel $M(8,4)$. The corresponding map P has, according to (12), triangular regions only. Therefore on the edges which are incident to the vertex $u \in M(9,y) - M(8,4)$ lie at least five crossings and hence we obtain $y \geq 4+5 = 9$, but this is a contradiction. Thus $c_2(K_9) \geq 9$. Now from the existence of the model $M(9,9) = 4^9$ follows $c_2(K_9) = 9$.

For the next crossing number we shall only prove

(48) $$22 \leq c_2(K_{10}) \leq 24.$$

According to (13) the inequality $c_2(K_{10}) \geq 15$ holds. Le us suppose that $c_2(K_{10}) = 15$. Then every model $M(10,15)$ must contain a submodel $M(9,9)$. The corresponding map P has, according to (45), 36 faces in all, all of them triangular. Therefore on every edge there is, at the most, one intersection point (otherwise all the faces could not be triangular). All triangles of the map P have two vertices and one crossing of the model $M(9,9)$ as thier vertices. From this it is easy to find out that every model $M(10)$ which contains the submodel $M(9,9)$ has at least 24 crossings. That is a contradiction to the supposition $c_2(K_{10}) = 15$. So $c_2(K_{10}) \geq 16$.

Let $c_2(K_{10}) = y < 19$. Then there exists a model $M(10,y)$, every submodel $M(9)$ of which has at least 10 crossings, of which at least one has exactly 10 crossings. The respective map P has 37 regions one of which is a quadrangle and the others are triangles. It is easy to find out that the quadrangle has, as vertices, 3 crossings and one vertex of the model $M(9)$. Of all the triangles of the map P only one has for vertices only vertices of the model $M(9,10)$. If the vertex $u \in M(10,y) - M(9,10)$ belongs to the quadrangle, then at least 15 crossings lie on the edges outgoing from it. If the vertex u belongs to a triangle, then on the edges incident to it there lie at least 12 intersection points. Therefore $y \geq 10+12 = 22$ holds and that is a contradiction to the supposition $c_2(K_{10}) < 19$.

Let $c_2(K_{10}) = 19$. Then there exists a model $M(10,19)$, which contains a submodel $M(9,11)$ and does not contain any submodel

M(9,y), where $y < 11$. Every model M(9,11) forms a map P, which has 38 regions. Either 36 of them are triangles and two are quadrangles or 37 are triangles and one is a pentagon. In all cases there are at most four triangles all the vertices of which are vertices of the model M(9,11). From this it is easy to find out that on the edges outgoing from the vertex $u \in M(10,19) - M(9,11)$ there lie at least 12 crossings; that is a contradiction. Therefore $c_2(K_{10}) \geq 20$.

Let $c_2(K_{10}) = y$, where $20 \leq y \leq 21$. Then there must exist a model M(10,y) which contains the submodel M(9,12) but no submodel M(9,z), where $z < 12$. An arbitrary model M(9,12) forms a map P which has 39 regions, there are either 3 quadrangles or one quadrangle and one pentagon or one hexagon. In all cases there are at most 6 triangles the vertices of which are all vertices of the model M(9,12). Therefore at least 10 crossings lie on the edges issuing from the vertex $u \in M(10,y) - M(9,12)$; that is a contradiction to the supposition $c_2(K_{10}) \leq 21$.

So at last the inequality $c_2(K_{10}) \geq 22$ is proved. The inequality $c_2(K_{10}) \leq 24$ follows from the existence of the model M(10,24).

Lower estimates

(49) $$c_2(K_{11}) \geq 35, \quad c_2(K_{12}) \geq 53, \quad c_2(K_{13}) \geq 77,$$
$$c_2(K_{14}) \geq 108, \quad c_2(K_{15}) \geq 148$$

are gained by using repeatedly the inequalities (13).

As direct consequence of the inequality (13) and the inequality $c_2(K_{15}) \geq 148$ we obtain the lower estimate for an arbitrary crossing number $c_2(K_n)$, where $n > 15$.

T H E O R E M 6. For all natural numbers $n > 15$

(50) $$c_2(K_n) \geq D_2(n) = \frac{37}{8190}\, n(n-1)(n-2)(n-3).$$

We obtain the upper estimate for the number $c_2(K_n)$, where $n \geq 8$, $n \neq 11$ using the construction of the models of the graphs K_n.

Auxiliary construction 2. In the Euclidean plane we shall construct an auxiliary model M. Let a number $n \equiv \pm 2 \pmod 8$ be given. The numbers m, r, s will be defined by (19). We construct two regular concentric polygons with girdles

$$U = u_1 u_2 \dots u_m, \qquad T = t_1 t_2 \dots t_m,$$

so that the mapping

$$H : U \rightarrow T, \ u_i \rightarrow t_i \qquad (i = 1,2,\dots,m)$$

is a homothety with the coefficient $h = -\frac{1}{2}$.

The points u_i, t_i $(i = 1,2,\dots,m)$ are vertices of the model $\bar{M}$. Let us denote by C_{UT}, C_{UQ} the bounded closed sets with boundaries $U \cup T$, $U \cup Q$ respectively, where

$$Q = u_1 t_{1+s} u_2 t_{2+s} u_3 \dots u_m t_s.$$

Now we shall construct the edges of the model $\bar{M}$, which we shall draw as curves homeomorphic with a straight segment, or where possible, as straight segments. We shall at the same time take care, that no unnecessary crossings form. We shall divide the edges of the model $\bar{M}$ into two groups. For all $i,j = 1,2,\dots m$ we shall construct inside the region C_{UT}

1) f-edges $f_{ij} = u_i t_j$, if (20) holds;
2) g-edges $g'_{ij} = t_i t_j$, $g''_{ij} = u_i u_j$, if $i \neq j$ and (22) hold.

During the construction we take care to obtain homeomorphic drawings as parts of the model $\bar{M}$ belonging to the "opposite" triangles

$$u_j t_i u_k, \quad t_j u_i t_k, \quad \text{where } j-i \equiv i-k \equiv s \pmod m.$$

Now from the set C_{UQ} and the appropriate part of the model $\bar{M}$ we obtain a model of the Möbius strip and a model G(m) ot the graph K_m by identifying all pairs of "opposite" vertices u_i, t_i $(i=1,2,\dots,m)$, "opposite" edges f_{ij}, f_{ji}, where $j-i \equiv s \pmod m$ and the corresponding crossings on these edges. Simultaneously the appropriate parts of the edges g'_{ij} and g''_{ij} after identifying form a single edge g_{ij}. The model G(m) has the set of vertices

$$G_0(m) = \{u_1 = t_1, \ u_2 = t_2, \dots, \ u_m = t_m\}.$$

Constructions of models B(n) of graphs K_n for $n \geq 8$, $n \neq 11$. In the projective plane we construct the model A(n). From the projective plane we cut out a region homeomorphic with a circle described by the edges e_{12}, e_{23}, ..., e_{m1}. The hole thus acquired we "join" with the Möbius strip containing the model G(m) by identifying all vertices which are denoted by the same letter in the models A(n) and G(m). So a model B(n) is formed.

Now we shall find out the number of crossings of the model B(n). We shall use a similar denotation as during calculations concerning the model A(n). Only instead of "p" we shall always write "q". E.g. q_{ab} means the number of ab-crossings of the model B(n). It is easy to find out that for $x, y \in \{a, b, c, d\}$, $x \neq y$ and for $n \equiv \pm 2 \pmod 8$

$$p_{xy}(u_i) = q_{xy}(u_i), \qquad p_{xy}(v_i) = q_{xy}(v_i)$$

is true. Further

$$q_{xf}(u_i) = q_{xg}(u_i) = 0$$

holds and finally

$$q_{ff}(u_i) = p_{aa}(v_i) = 2\binom{2t}{3},$$

$$q_{gg}(u_i) = p_{bb}(v_i) = \binom{2r}{3},$$

$$q_{fg}(u_i) = 8t\binom{r}{2}.$$

From here we easily obtain the theorem:

THE E O R E M 7. For the crossing numbers $c_2(K_n)$, where $n \geq 7$ the upper estimate is

$$c_2(K_n) \leq b(n) = a(n) - v(n), \tag{51}$$

where a(n) has the same meaning as in Theorem 2 and v(n) is defined in this way:

(52)

$$v(n) = \frac{1}{6}(4r+1)(4r-1)(r+1)r, \qquad \text{if } 8r-1 \leq n \leq 8r+2,$$

$$v(n) = \frac{1}{6}(4r+5)(4r+3)(r+1)r, \qquad \text{if } 8r+3 \leq n \leq 8r+6.$$

Similarly to Theorem 3 is

T H E O R E M 8. a) For all $n \geq 6$

$$c_2(K_n) \leq H_2(n) = H_1(n) - v(n) \tag{53}$$

holds, where $H_1(n)$ is defined by (38) and v(n) by (52).

b) For all $n \geq 20$ the difference $o(n^2) = H_2(n) - b(n)$ is a polynomial of the second degree at the most.

From Theorem 4 it is easy to derive other estimates for the crossing numbers $c_2(K_n)$.

T H E O R E M 9. a) For all natural numbers $n > 5$

(54) $$c_2(K_n) < \frac{37}{48} \cdot H_o^+(n) < 0{,}7709 \cdot H_o^+(n)$$

is true, where $H_o^+(n)$ is given by (4).

b) For all natural numbers $n > 16$

(55) $$c_2(K_n) < \frac{37 \cdot 3^3}{59 \cdot 2^4} H_1^+(n) < 1{,}0583 \cdot H_1^+(n)$$

holds, where $H_1^+(n)$ is given by (11).

The proof follows from Theorem 4 and from the inequalities

$$v(n) > \frac{1}{24} \cdot H_o^+(n), \qquad v(n) > \frac{27}{59 \cdot 2^3} \cdot H_1^+(n)$$

where $n > 16$.

Finally I shall give two limit properties of the numbers $c_2(K_n)$, which are an immediate consequence of the above mentioned theorems.

T H E O R E M 10. For the crossing numbers $c_2(K_n)$ we have

$$\limsup c_2(K_n) \cdot n^{-4} \leq \frac{37}{3 \cdot 2^{10}} \doteq 0{,}0121,$$

$$\liminf c_2(K_n) \cdot n^{-4} \geq \frac{37}{8190} \doteq 0{,}0045.$$

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