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Quasitrivial Groupoids and Balanced Identities

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Quasitrivial groupoids satisfying a balanced identity are described.

V článku jsou popsány kvazitriviální grupoidy splňující nějakou balancovanou identitu.

В статье опысываются квазитривиальные группоиды выполняющие некоторое сбалансированое тождество.

The aim of this paper is to describe quasitrivial groupoids satisfying a non-trivial balanced identity. To this purpose, balanced identities are divided into five types and the corresponding quasitrivial groupoids are determined in each of these five cases.

1. A groupoid G is said to be

- commutative if $ab = ba$ for all $a, b \in G$,
- idempotent if $aa = a$ for every $a \in G$,
- medial if $ab \,.\, cd = ac \,.\, bd$ for all $a, b, c, d \in G$,
- quasitrivial if $ab \in \{a, b\}$ for all $a, b \in G$,
- a semigroup if $a \,.\, bc = ab \,.\, c$ for all $a, b, c \in G$,
- an L-semigroup if $ab = a$ for all $a, b \in G$,
- an R-semigroup if $ab = b$ for all $a, b \in G$,
- a semilattice if it is a commutative idempotent semigroup.

Obviously, a groupoid G is quasitrivial iff every non-empty subset of G is a subgroupoid.

For a groupoid G, define a relation ϱ_G by $(a, b) \in \varrho$ iff $a, b \in G$ and either $a = b$ or $ab \neq ba$. Further, let σ_G designate the least congruence of G such that the corresponding factor is commutative. The groupoid G is called anticommutative (contracommutative) if $\varrho = G \times G$ ($\sigma = G \times G$).

1.1 Lemma. Let G be a quasitrivial gropoid. Then:

(i) $(a, b) \in \varrho_G$ iff $a, b \in G$ and $\{a, b\} = \{ab, ba\}$.

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(ii) $\varrho_G \subseteq \sigma_G$.
(iii) $\varrho_G = \sigma_G$ iff ϱ_G is a congruence of G.

Proof. (i) This assertion is clear.
(ii) Let f be the natural mapping of G onto G/σ. If $ab \neq ba$ for some $a, b \in G$ then $f(ab) = f(ba)$ implies $f(a) = f(b)$.
(iii) Let $a, b \in G$. If $ab = ba$ then $(ab, ba) \in \varrho$. If $ab \neq ba$ then $\{ab, ba\} = \{a, b\}$, $ab \cdot ba \neq ba \cdot ab$ and $(ab, ba) \in \varrho$.

1.2 Corollary. Every anticommutative quasitrivial groupoid is contracommutative.

Let H be a quasitrivial groupoid and G_i, $i \in H$, pair-wise disjoint groupoids. Define a groupoid $K = \mathrm{U}(G_i, i \in H)$ as follows: $K = \bigcup G_i$; the groupoids G_i are subgroupoids of K; $g_i g_j = g_{ij}$ for all $i, j \in H$, $i \neq j$, $g_i \in G_i$ and $g_j \in G_j$.

1.3 Lemma. Let H be a quasitrivial groupoid and G_i, $i \in H$, pair-wise disjoint groupoids. Then $\mathrm{U}(G_i, i \in H)$ is quasitrivial iff each G_i is.

Proof. Obvious.

1.4 Proposition. Let G be a quasitrivial groupoid. Then:

(i) G/σ is a commutative quasitrivial groupoid.
(ii) Every block of σ is a contracommutative quasitrivial groupoid.
(iii) $G = \mathrm{U}(i, i \in G/\sigma)$.

Proof. See [1, Proposition 2.11].

1.5 Proposition. Let G be a quasitrivial semigroup. Then $\varrho = \sigma$. Moreover, if G is contracommutative then G is either an L-semigroup or an R-semigroup.

Proof. The result is easy and well known (see e.g. [1, Lemmas 3.1, 3.5]).

1.6 Corollary. A groupoid G is a quasitrivial semigroup iff there exist a quasitrivial semilattice H and pair-wise disjoint groupoids G_i, $i \in H$, such that $G = \mathrm{U}(G_i, i \in H)$ and each G_i is either an L-semigroup or an R-semigroup.

Let G, H be two groupoids with $G \cap H = \emptyset$. Define a groupoid $K = G : H$ as follows: $K = G \cup H$; both G and H are subgroupoids of K; $gh = h = hg$ for all $g \in G$ and $h \in H$. Clearly, K is quasitrivial iff G and H are so.

Let G be a groupoid. An element $e \in G$ is said to be a left (right) unit if $ea = a$ ($ae = a$) for every $a \in G$. An element $z \in G$ is said to be a left (right) zero if $za = z$ ($az = z$) for every $a \in G$. Further, for every $a \in G$, we define two transformations L_a and R_a of G by $L_a(b) = ab$ and $R_a(b) = ba$.

Let G be a groupoid. The opposite groupoid $G^{\mathrm{op}} = G(\circ)$ is defined by $a \circ b = ba$ for all $a, b \in G$.

2. Consider the following twelve groupoids defined on a three-element set $S = \{u, v, w\}$.

S_1 : $uu = uv = uw = u,\ vu = vv = vw = v,\ wu = wv = wv = ww = w;$
S_2 : $uu = uv = u,\ vu = vv = v,\ uw = vw = wu = wv = ww = w;$
S_3 : $uu = uv = uw = wu = u,\ vu = vv = vw = wv = v,\ ww = w;$
S_4 : $uu = u,\ uv = vu = vv = v,\ uw = vw = wu = wv = ww = w;$
S_5 : $uu = uv = u,\ vu = vv = vw = v,\ uw = wu = wv = ww = w;$
S_6 : $uu = uv = u,\ vu = vv = vw = wv = v,\ uw = wu = ww = w;$
S_7 : $uu = uv = wu = u,\ vu = vv = vw = v,\ uw = wv = ww = w;$
S_8 : $uu = uv = wu = u,\ vu = vv = vw = wv = v,\ uw = ww = w;$
S_9 : $uu = uv = vu = u,\ vv = vw = wv = v,\ uw = wu = ww = w;$
S_{10} : $uu = uv = vu = wu = u,\ vv = vw = v,\ uw = wv = ww = w;$
S_{11} : $uu = uv = vu = u,\ vv = vw = v,\ uw = wu = wv = ww = w;$
S_{12} : $uu = uv = vu = wu = u,\ vv = vw = wv = v,\ uw = ww = w.$

2.1 Proposition. The groupoids $S_1, S_2, S_3, S_4, S_1^{\mathrm{op}}, S_2^{\mathrm{op}}, S_3^{\mathrm{op}}$ are pair-wise non-isomorphic three-lement quasitrivial semigroups. Every three-element quasitrivial semigroup is isomorphic to one of these seven groupoids.

Proof. The assertion is an easy consequence of 1.6.

2.2 Lemma. Let G be a quasitrivial groupoid and $a, b, c \in G$. Then $a \,.\, bc \neq ab \,.\, c$ iff at least one of the following two conditions is satisfied:

(i) $a \neq b \neq c,\ a \neq c$ and $ab = a,\ bc = b,\ ac = c$.
(ii) $a \neq b \neq c,\ a \neq c$ and $ab = b,\ bc = c,\ ac = a$.

Proof. Easy.

2.3 Lemma. Let G be a quasitrivial groupoid and let $a, b, c \in G$ be such that $a \,.\, bc \neq ab \,.\, c$. Put $H = \{a, b, c\}$. Then H is a subgroupoid of G and H is isomorphic to at least one of the groupoids $S_5, \ldots, S_{12}, S_5^{\mathrm{op}}, \ldots, S_{12}^{\mathrm{op}}$.

Proof. Use 2.2.

2.4 Lemma. (i) The groupoids $S_5, \ldots, S_9, S_5^{\mathrm{op}}, \ldots, S_8^{\mathrm{op}}$ are quasitrivial, non-associative and pair-wise non-isomorphic.

(ii) S_6 is isomorphic to S_{11} and S_{12}^{op}, S_8 is isomorphic to S_{10}^{op} and $S_9 = S_9^{\mathrm{op}}$.

Proof. Easy.

2.5 Proposition. The groupoids $S_1, \ldots, S_9, S_1^{\mathrm{op}}, S_2^{\mathrm{op}}, S_3^{\mathrm{op}}, S_5^{\mathrm{op}}, \ldots, S_8^{\mathrm{op}}$ are pair-wise non-isomorphic three-element quasitrivial groupoids. Every three-element quasitrivial groupoid is isomorphic to one of these sixteen groupoids.

Proof. Apply 2.1, 2.3 and 2.4.

2.6 Corollary. The groupoids S_4 and S_9 are up to isomorphism the only three-element commutative quasitrivial groupoids.

2.7 Proposition. A quasitrivial groupoid G is a semigroup iff no subgroupoid of G is isomorphic to one of the groupoids $S_5, \ldots, S_9, S_5^{op}, \ldots, S_8^{op}$.

Proof. Apply 2.3 and 2.4.

3. Let $X = \{x_1, x_2, \ldots\}$ be an infinite countable set of variables and W the absolutely free groupoid of terms over X. For every $t \in W$, define a positive integer $\mathrm{l}(t)$ and a non-empty set $\mathrm{var}(t)$ by $\mathrm{l}(x) = 1$, $\mathrm{var}(x) = \{x\}$ for every $x \in X$ and $\mathrm{l}(rs) = \mathrm{l}(r) + \mathrm{l}(s)$, $\mathrm{var}(rs) = \mathrm{var}(r) \cup \mathrm{var}(s)$ for all $r, s \in W$. Further, for all $x \in X$ and $t \in W$, define a non-negative integer $\mathrm{i}(t, x)$ by $\mathrm{i}(x, x) = 1$, $\mathrm{i}(y, x) = 0$ for $x \neq y \in X$ and $\mathrm{i}(rs, x) = \mathrm{i}(r, x) + \mathrm{i}(s, x)$ for all $r, s \in W$. Finally, put $\mathrm{o}(x) = x = (x)\mathrm{o}$ and $\mathrm{o}(rs) = \mathrm{o}(r)$, $(rs)\mathrm{o} = (s)\mathrm{o}$.

Let $t \in W$ and $\mathrm{n} = \mathrm{l}(t)$. We define an ordered n-tuple $\mathrm{v}(t)$ as follows: If $\mathrm{n} = 1$ then $t = x_\mathrm{i}$ for some $1 \leqq \mathrm{i}$ and we put $\mathrm{v}(t) = (\mathrm{i})$; if $2 \leqq \mathrm{n}$ then $t = rs$, $r, s \in W$, $\mathrm{l}(r) = \mathrm{m}$, $\mathrm{l}(s) = \mathrm{k}$, $\mathrm{n} = \mathrm{m} + \mathrm{k}$, $\mathrm{v}(r) = (\mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{m})$, $\mathrm{v}(s) = (\mathrm{j}_1, \ldots, \mathrm{j}_\mathrm{k})$ and we put $\mathrm{v}(t) = (\mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{m}, \mathrm{j}_1, \ldots, \mathrm{j}_\mathrm{k})$.

A term t is said to be balanced if $\mathrm{i}(t, x) \leqq 1$ for every $x \in X$.

An identity is an ordered pair of terms. Let (r, s) be an identity and G a groupoid. We say that G satisfies this identity if $f(r) = f(s)$ for every homomorphism f of W into G.

An identity (r, s) is called non-trivial if $r \neq s$.

An identity (r, s) is called balanced if $\mathrm{var}(r) = \mathrm{var}(s)$ and both r and s are balanced.

Let $\mathscr{R}$ ($\mathscr{S}$, $\mathscr{T}_\mathrm{l}$, $\mathscr{T}_\mathrm{r}$, resp.) denote the fully invariant congruence of W generated by the pair (x_1x_2, x_2x_1) ($(x_1 . x_2x_3, x_1x_2 . x_3)$, $(x_1 . x_2x_3, x_2 . x_1x_3)$, $(x_1x_2 . x_3, x_1x_3 . x_2)$, resp.).

Let (r, s) be a balanced identity. We shall say that (r, s) is

- of type 1 if $(r, s) \in \mathscr{R}$ and $r \neq s$;
- of type 2 if $(r, s) \in \mathscr{S}$ and $r \neq s$;
- of type 3 if $(r, s) \in \mathscr{T}_\mathrm{r}$ and $r \neq s$;
- of type 4 if $(r, s) \in \mathscr{T}_\mathrm{l}$ and $r \neq s$;
- of type 5 if $(r, s) \notin \mathscr{R} \cup \mathscr{S} \cup \mathscr{T}_\mathrm{r} \cup \mathscr{T}_\mathrm{l}$;
- strong if $(r, s) \notin \mathscr{R}$.

Let t be a balanced term. We denote by $\mathrm{var}_\mathrm{l}(t)$ the set of all variables x such that xr is a subterm of t for some $r \in W$. Further, we put $\mathrm{var}_\mathrm{r}(t) = \mathrm{var}(t) \setminus \mathrm{var}_\mathrm{l}(t)$.

Let t be a balanced term and $x \in X$ be such that $t \neq x$. Define a balanced term $\mathrm{u}(t, x)$ as follows: If $x \notin \mathrm{var}(t)$ then $\mathrm{u}(t, x) = t$; if $t = px$ for some $p \in W$ then $\mathrm{u}(t, x) = p$; if $t = xq$ for some $q \in W$ then $\mathrm{u}(t, x) = q$; if $t = rs$ for some $r, s \in W$, $r \neq x \neq s$, then $\mathrm{u}(t, x) = \mathrm{u}(r, x) . \mathrm{u}(s, x)$.

A balanced identity (r, s) is said to be irreducible if $t \in X$ whenever t is a subterm of both the terms r and s.

3.1 Lemma. Let $r, s \in W$. Then $(r, s) \in \mathscr{R}$ iff every commutative groupoid satisfies (r, s).

Proof. Easy.

3.2 Lemma. The following conditions are equivalent for a balanced identity (r, s):

(i) $(r, s) \in \mathscr{R}$.

(ii) If p is a subterm of r then there exists a subterm q of s such that $\mathrm{var}(p) = \mathrm{var}(q)$.

(iii) If q is a subterm of s then there exists a subterm p of r such that $\mathrm{var}(q) = \mathrm{var}(p)$.

Proof. (i) implies (ii) and (iii). Define a relation $\mathscr{U}$ on W by $(p, q) \in \mathscr{U}$ iff for every subterm t of p there is a subterm w of q such that $\mathrm{var}(t) = \mathrm{var}(w)$ and $\mathrm{l}(t) = \mathrm{l}(w)$. Put $(p, q) \in \mathscr{V}$ iff (p, q) and (q, p) belong to $\mathscr{U}$. Then $\mathscr{V}$ is a congruence of W and $W/\mathscr{V}$ is a commutative groupoid. Hence $\mathscr{R} \subseteq \mathscr{V}$.

(ii) implies (i). We shall proceed by induction on $\mathrm{l}(r)$. Let $r = r_1 r_2$, $s = s_1 s_2$ and let f be a homomorphism of W into a commutative groupoid G. Then $f(r) = f(r_1) f(r_2) = f(r_2) f(r_1)$, $f(s) = f(s_1) f(s_2) = f(s_2) f(s_1)$ and either $\mathrm{var}(r_1) = \mathrm{var}(s_1)$ or $\mathrm{var}(r_1) = \mathrm{var}(s_2)$. The rest of the proof is clear.

3.3 Lemma. The following conditions are equivalent for an identity (r, s):

(i) $(r, s) \in \mathscr{S}$.

(ii) Every semigroup satisfies (r, s).

(iii) $\mathrm{v}(r) = \mathrm{v}(s)$.

Proof. Obvious.

3.4 Lemma. Let $(r, s) \in \mathscr{T}_r$. Then $\mathrm{o}(r) = \mathrm{o}(s)$, $\mathrm{l}(r) = \mathrm{l}(s)$, $\mathrm{var}(r) = \mathrm{var}(s)$ and $\mathrm{i}(r, x) = \mathrm{i}(s, x)$ for every $x \in X$.

Proof. Easy.

3.5 Lemma. Let $0 \leqq \mathrm{n}, \mathrm{m}$, $r_1, \ldots, r_\mathrm{n}$, $s_1, \ldots, s_\mathrm{m} \in W$, $x \in X$ and $r = (((x r_1) r_2) \ldots) r_\mathrm{n}$, $s = (((x s_1) s_2) \ldots) s_\mathrm{m}$. Then $(r, s) \in \mathscr{T}_r$ iff $\mathrm{n} = \mathrm{m}$ and there exists a permutation π such that $(r_\mathrm{i}, s_{\pi(\mathrm{i})}) \in \mathscr{T}_r$ for every $1 \leqq \mathrm{i} \leqq \mathrm{n}$.

Proof. Define a relation $\mathscr{V}$ on W by $(p, q) \in \mathscr{V}$ iff there are $0 \leqq \mathrm{k}$, $p_1, \ldots, p_\mathrm{k}$, $q_1, \ldots, q_\mathrm{k} \in W$, $y \in X$ and a permutation σ such that $p = ((y p_1) \ldots) p_\mathrm{k}$, $q = (y q_1) \ldots) q_\mathrm{k}$ and $(p_\mathrm{i}, q_{\sigma(\mathrm{i})}) \in \mathscr{T}_r$ for every $1 \leqq \mathrm{i} \leqq \mathrm{k}$. It is easy to check that $\mathscr{V}$ is a congruence of W, $\mathscr{V} \subseteq \mathscr{T}_r$ and $W/\mathscr{V}$ satisfies the identity $(x_1 x_2 \cdot x_3, x_1 x_3 \cdot x_2)$. Hence $\mathscr{V} = \mathscr{T}_r$.

3.6 Lemma. Let r be a balanced term and $y \in \mathrm{var}_r(r)$. Then there exists a balanced term $s \in W$ such that $y = (s)o$ and $(r, s) \in \mathscr{T}_r$.

Proof. By induction on $l(r)$. If $l(r) = 1$ then $r = y$ and $y = (r)o$. Let $l(r) \geqq 2$. There are $n \geqq 1$, $x \in X$ and $r_1, \ldots, r_n \in W$ such that $r = ((xr_1) \ldots) r_n$. Since $y \in \mathrm{var}_r(r)$, $y \neq x$ and we can assume that $y \in \mathrm{var}(r_n)$. If $l(r_n) = 1$ then $r_n = y$, $(r)o = y$ and we put $s = r$. If $l(r_n) \geqq 2$ then $y \in \mathrm{var}_r(r_n)$ and $(p)o = y$ for some $p \in W$ such that $(r_n, p) \in \mathscr{T}_r$ and we put $s = (((xr_1) \ldots) r_{n-1})\, p$.

3.7 Proposition. Every non-trivial balanced identity is of exactly one of the types 1, 2, 3, 4 and 5.

Proof. Apply 3.2, 3.3, 3.4 and 3.5.

3.8 Lemma. Let Y, Z be subsets of X and t a balanced term such that $Y \cap Z = \emptyset$, $\mathrm{var}(t) \subseteq Y \cup Z$ and $\mathrm{var}(t) \cap Y \neq \emptyset \neq \mathrm{var}(t) \cap Z$. Then there exist $r, s \in W$ such that rs is a subterm of t and either $\mathrm{var}(r) \subseteq Y$, $\mathrm{var}(s) \subseteq Z$ or $\mathrm{var}(r) \subseteq Z$, $\mathrm{var}(s) \subseteq Y$.

Proof. By induction on $l(t)$.

3.9 Lemma. Let (r, s) be a balanced identity such that $(r, s) \in \mathscr{R}$ ($(r, s) \in \mathscr{S}$) and $x \in X$ such that $r \neq x$. Then $(\mathrm{u}(r, x), \mathrm{u}(s, x)) \in \mathscr{R}$ ($\in \mathscr{S}$).

Proof. By induction on $l(t)$.

4.

4.1 Lemma. Let G be a groupoid containing at least two left zeros and satisfying an identity (r, s). Then $o(r) = o(s)$.

Proof. Let $o(r) = x \neq y = o(s)$. Define a homomorphism f of W into G by $f(x) = a$ and $f(z) = b$ for every $x \neq z \in X$, $a \neq b$ being left zeros of G. Then $f(r) = a \neq b = f(s)$, a contradiction.

4.2 Lemma. Let G be a groupoid containing a left zero a such that $ba = b = bb$ for some $a \neq b \in G$. Suppose that G satisfies an identity (r, s). Then $o(r) = o(s)$.

Proof. Similar to that of 4.1.

4.3 Lemma. Let G be a groupoid satisfying a balanced identity (r, s) such that $2 \leqq l(r)$. Suppose that G contains a right unit and $x \in \mathrm{var}_r(r) \cap \mathrm{var}_r(s)$. Then G satisfies the identity $(\mathrm{u}(r, x), \mathrm{u}(s, x))$.

Proof. Let f be a homomorphism of W into G and let $e \in G$ be a right unit. There is a homomorphism g such that $g(y) = f(y)$ for every $x \neq y \in X$ and $g(x) = e$. It is easy to show by induction on $l(t)$ that $f(\mathrm{u}(t, x)) = g(t)$ for every balanced term t such that $2 \leqq l(t)$ and $x \in \mathrm{var}_r(t)$.

4.4 Lemma. Let G be a groupoid with $G = \{ab \mid a, b \in G\}$ and (r, s) a balanced identity such that G satisfies (r, s). Suppose that a term t is a subterm of both r and s, $x \in \mathrm{var}(t)$ and define an endomorphism f of W by $f(y) = y$ for every variable $y \neq x$ and $f(x) = t$. Then:

(i) There exist uniquely determined $r', s' \in W$ such that $f(r') = r$, $f(s') = s$, (r', s') is a balanced identity and G satisfies (r', s').

(ii) If (r, s) is of type 2 then (r', s') is of this type.

(iii) If $r \neq s$ then $r' \neq s'$.

Proof. Easy.

4.5 Lemma. Let G be a groupoid containing a right unit and satisfying a balanced identity (r, s) of type 2. Suppose that $\mathrm{n} = \mathrm{l}(r) \leqq \mathrm{l}(r')$ whenever (r', s') is a balanced identity of type 2 such that G satisfies (r', s'). Then $3 \leqq \mathrm{n}$ and there exists $1 \leqq \mathrm{m} \leqq \mathrm{n} - 2$ such that G satisfies (p, q), where $p = x_1(x_2(\ldots(x_{n-1}x_n)))$ and $q = x_1(\ldots(x_{m-1}((x_m(\ldots(x_{n-2}x_{n-1})))\,x_n)))$.

Proof. We can assume that $\mathrm{v}(r) = (1, 2, \ldots, \mathrm{n})$. Since $r \neq s$, $3 \leqq \mathrm{n}$. Further, $(\mathrm{u}(r, x_n), \mathrm{u}(s, x_n)) \in \mathscr{S}$ and G satisfies this identity. Consequently, $\mathrm{u}(r, x_n) = \mathrm{u}(s, x_n)$. On the other hand, there are $1 \leqq \mathrm{i}, \mathrm{j}$ and $r_1, \ldots, r_i, s_1, \ldots, s_j \in W$ such that $r = r_1(\ldots(r_i x_n))$ and $s = s_1(\ldots(s_j x_n))$. We must distinguish the following two cases:

(i) $2 \leqq \mathrm{i} \leqq \mathrm{j}$. Then $r_1(\ldots(r_{i-1}r_i)) = \mathrm{u}(r, x_n) = \mathrm{u}(s, x_n) = s_1(\ldots(s_{j-1}s_j))$, $r_1 = s_1, \ldots, r_{i-1} = s_{i-1}$, $r_i = s_i(\ldots(s_{j-1}s_j))$, $s_1, \ldots, s_j \in X$ by 4.4, $\mathrm{j} = \mathrm{n} - 1$, $s_1 = x_1, \ldots, s_{n-1} = x_{n-1}$, $s = x_1(\ldots(x_{n-1}x_n))$ and
$r = x_1(\ldots(x_{i-1}((x_i(\ldots(x_{n-2}x_{n-1})))\,x_n)))$. Since $r \neq s$, $\mathrm{i} \leqq \mathrm{n} - 2$.

(ii) $1 = \mathrm{i} \leqq \mathrm{j}$. Then $r_1 = s_1(\ldots(s_{j-1}s_j))$, $\mathrm{j} = \mathrm{n} - 1$, $s_1 = x_1, \ldots, s_{n-1} = x_{n-1}$, $r = (x_1(\ldots(x_{n-2}x_{n-1})))\,x_n$ and $s = x_1(\ldots(x_{n-1}x_n))$.

5.

5.1 Lemma. The groupoids S_5 and S_5^{op} satisfy no balanced identity of type 2.

Proof. Let (r, s) be a balanced identity of type 2 such that S_5 satisfies (r, s). Put $\mathrm{n} = \mathrm{l}(r)$ and suppose that $r = x_1(\ldots(x_{n-1}x_n))$ and
$s = x_1(\ldots(x_{m-1}((x_m(\ldots(x_{n-2}x_{n-1})))\,x_n)))$ where $1 \leqq \mathrm{m} \leqq \mathrm{n} - 2$ (see 4.5). Define a homomorphism f of W into S_5 by $f(x_1) = f(x_2) = \ldots = f(x_{n-2}) = u$, $f(x_{n-1}) = v$ and $f(x_n) = f(x_{n+1}) = \ldots = w$. Then $f(r) = u \neq w = f(s)$, a contradiction.

5.2 Lemma. The groupoids S_6, S_7, S_6^{op}, S_7^{op} satisfy no balanced identity of type 2.

Proof. Similar to that of 5.1.

5.3 Lemma. The groupoids S_8 and S_8^{op} satisfy no balanced identity of type 2.

Proof. Suppose that S_8^{op} satisfies (r, s), $1 \leqq \mathrm{m} \leqq \mathrm{n} - 2$, $r = x_1(\ldots(x_{n-1}x_n))$ and $s = x_1(\ldots((x_m(\ldots(x_{n-2}x_{n-1})))\,x_n)))$ (see 4.5). Define a homomorphism f

of W into S_8^{op} by $f(x_1) = \ldots = f(x_{n-2}) = v$, $f(x_{n-1}) = w$ and $f(x_n) = \ldots = u$. Then $f(r) = v \neq u = f(s)$, a contradiction.

5.4 Lemma. The groupoid S_9 satisfies no balanced identity of type 2.

Proof. We are going to show by induction on $l(r)$ that for every balanced identity (r, s) of type 2 there exists a homomorphism f of W into S_9 such that $f(r) \neq f(s)$. We can assume without loss of generality that $4 \leqq n = l(r)$, $v(r) = (1, 2, \ldots, n)$, $r = r_1 r_2$ and $s = s_1 s_2$.

(i) Let $l(r_1) = m < k = l(s_1)$. Define a homomorphism f of W into S_9 by $f(x_1) = \ldots = f(x_m) = u$, $f(x_{m+1}) = \ldots = f(x_k) = v$ and $f(x_{k+1}) = \ldots = w$. Then $f(r) = f(r_1) f(r_2) = uv = u \neq w = uw = f(s_1) f(s_2) = f(s)$.

(ii) Let $l(r_1) = m = l(s_1)$. Then $l(r_2) = l(s_2)$. Assume first $r_1 \neq s_1$. Then (r_1, s_1) is a balanced identity of type 2 and there is a homomorphism g of W into S_9 with $g(r_1) \neq g(s_1)$. We have $\{g(r_1), g(s_1), z\} = \{u, v, w\} = S$ for some $z \in S$. Define f by $f(x_i) = g(x_i)$ for $1 \leqq i \leqq m$ and $f(x_j) = z$ for $m + 1 \leqq j$. Then $f(r) = g(r_1) z \neq g(s_1) z = f(s)$. If $r_1 = s_1$ then $r_2 \neq s_2$ and we can proceed similarly.

5.5 Theorem. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity of type 2.
(ii) G is a semigroup.
(iii) G satisfies every balanced identity of type 2.

Proof. Apply 2.7, 5.1, 5.2, 5.3 and 5.4.

6.

6.1 Lemma. The groupoid S_5 satisfies the identity $(x_1 x_2 \,.\, x_3, x_1 x_3 \,.\, x_2)$.

Proof. Easy.

6.2 Lemma. Let (r, s) be a balanced identity such that S_5 satisfies (r, s). Then $(r, s) \in \mathcal{T}_r$.

Proof. The proof will be divided into eight parts.

(i) By 4.1, $o(r) = o(s)$. Suppose $o(r) = x_1$.

(ii) Let $x \in \text{var}_l(r)$. We are going to show that $x \in \text{var}_l(s)$. Suppose, on the contrary, that xp is a subterm of r and qx of s for some $p, q \in W$. Obviously, $x \neq x_1$ and we can assume $x = x_2$ and $\text{var}(p) = \{x_3, \ldots, x_m\}$, $3 \leqq m$. Define a homomorphism f of W into S_5 by $f(x_1) = f(x_{m+1}) = f(x_{m+2}) = \ldots = u$, $f(x_2) = v$ and $f(x_3) = f(x_4) = \ldots = f(x_m) = w$. Then $f(xp) = v$, $f(qx) = f(q) \in \{u, w\}$ and $f(r) = u \neq w = f(s)$, a contradiction.

(iii) By (ii), $\text{var}_l(r) = \text{var}_l(s)$ and $\text{var}_r(r) = \text{var}_r(s)$.

(iv) Now, we are going to prove by induction on $l(r) = n$ that $(r, s) \in \mathscr{T}_r$. With regard to (i), (iii), 3.4, 3.6 and 6.1, we can assume that $3 \leqq n$ and $(r)o = x_n = (s)o$. Put $r' = u(r, x_n)$ and $s' = u(s, x_n)$. By 4.3, S_5 satisfies the identity (r', s'). Hence $(r', s') \in \mathscr{T}_r$. On the other hand (see 3.5), there are $1 \leqq k$, $r_1, \dots, r_k$, $s_1, \dots, s_k \in W$ and a permutation π such that $r' = ((x_1 r_1) \dots) r_k$, $s' = ((x_1 s_1) \dots) s_k$ and $(r_i, s_{\pi(i)}) \in \mathscr{T}_r$ for every $1 \leqq i \leqq k$.

(v) Let $r = r' x_n$ and $s = s' x_n$. Then $(r, s) \in \mathscr{T}_r$ trivially.

(vi) Let $r = (((x_1 r_1) \dots) r_{k-1}) p$, $p \in W$, and $s = s' x_n$. Put $q = (((x_1 s_{\pi(1)}) \dots) . s_{\pi(k)}) x_n$ and define a homomorphism f of W into S_5 by $f(x_1) = f(x) = u$ for every $x \in \mathrm{var}(r_1) \cup \dots \cup \mathrm{var}(r_{k-1})$, $f(y) = v$ for every $y \in \mathrm{var}(r_k)$ and $f(z) = w$ for any other variable z. Then S_5 satisfies (r, q) and we have $f(r) = uv = u \neq w = uw = uv \,.\, w = f(q)$, a contradiction.

(vii) Let $r = (((x_1 r_1) \dots) r_{k-1}) p$, $s = (((x_1 s_1) \dots) s_{k-1}) q$ and $\pi(k) = i < k$. Then $\pi(j) = k$ for some $1 \leqq j < k$ and we put $p' = ((((((x_1 r_1) \dots) r_{j-1}) r_{j+1}) \dots) . r_{k-1}) r_j) p$, $q' = ((((((x_1 s_{\pi(1)}) \dots) s_{\pi(j-1)}) s_{\pi(j+1)}) \dots) s_{\pi(k-1)}) s_i) q$ if $j \neq k - 1$ and $p' = r$, $q' = ((((x_1 s_{\pi(1)}) \dots) s_{\pi(k-2)}) s_i) q$ if $j = k - 1$. Then S_5 satisfies the identity (p', q'). Define f by $f(x_n) = w$, $f(y) = v$ for every $y \in \mathrm{var}(r_j)$ and $f(z) = u$ for $z \in X$, $z \neq x_n$, $z \notin \mathrm{var}(r_j)$. Then $f(p') = uv \,.\, w = w \neq u = uv = f(q')$, a contradiction.

(viii) Let $r = (((x_1 r_1) \dots) r_{k-1}) p$, $s = (((x_1 s_1) \dots) s_{k-1}) q$ and $\pi(k) = k$. Assume first that S_5 satisfies (p, q). Then $(p, q) \in \mathscr{T}_r$, and hence $(r, s) \in \mathscr{T}_r$. Now, let S_5 do not satisfy (p, q). Since $u(p, x_n) = r_k$, $u(q, x_n) = s_k$ and $(r_k, s_k) \in \mathscr{T}_r$, we have $o(p) = o(q)$. From this, $\{f(p), f(q)\} = \{u, w\}$ for every homomorphism f of W into S_5 such that $f(p) \neq f(q)$. However, such a homomorphism f exists and we define g by $g(x) = u$ for every $x \in \mathrm{var}(x_1 r_1) \cup \mathrm{var}(r_2) \cup \dots \cup \mathrm{var}(r_{k-1})$, $g(y) = y$ for the remaining variables $y \in X$. Then $\{g(r), g(s)\} = \{uu, uw\} = \{u, w\}$, $g(r) \neq g(s)$, a contradiction.

6.3 Corollary. The groupoid S_5 (S_8^{op}) satisfies a non-trivial balanced identity (r, s) iff (r, s) is of type 3 (4).

7.

7.1 Lemma. Let (r, s) be a balanced identity such that the groupoid S_6 satisfies (r, s). Then $r = s$.

Proof. The proof will be divided into eight parts.

(i) By 4.2, $o(r) = o(s)$. Suppose $o(r) = x_1$.

(ii) Let $x \in \mathrm{var}_l(r)$. We are going to show that $x \in \mathrm{var}_l(s)$. Let, on the contrary, xp be a subterm of r and qx of s. Then we can assume $x = x_2$ and $\mathrm{var}(p) = \{x_3, \dots, x_m\}$ for some $3 \leqq m$. Define a homomorphism f of W into S_6 by $f(x_2) = u$, $f(x_3) = f(x_4) = \dots = f(x_m) = v$ and $f(x_1) = f(x_{m+1}) = \dots = w$. Then $f(xp) = u$, $f(qx) = f(q) \in \{v, w\}$ and $f(r) = w \neq v = f(s)$, a contradiction.

(iii) By (ii), $\mathrm{var_l}(r) = \mathrm{var_l}(s)$ and $\mathrm{var_r}(r) = \mathrm{var_r}(s)$.

(iv) Let $r = pq_1$ and $s = pq_2$, $p, q_1, q_2 \in W$, and let f be a homomorphism of W into S_6 with $f(q_1) \neq f(q_2)$. Taking into account that the inequalities $wu \neq wv$, $uu \neq uw$ and $uv \neq uw$ hold in S_6, it is easy to check that there exists a homomorphism g such that $g(x) = f(x)$ for every $x \in \mathrm{var}(q_1q_2)$ and $g(pq_1) \neq g(pq_2)$, a contradiction. We have proved that S_6 satisfies (q_1, q_2).

(v) Assume that $r \neq s$ and $r' = s'$ whenever (r', s') is balanced, S_6 satisfies (r', s') and $\mathrm{l}(r') < \mathrm{n} = \mathrm{l}(r)$. Then $3 \leqq \mathrm{n}$ and (r, s) is irreducible by 4.4. Further, let $\mathrm{var}(r) = \{x_1, \ldots, x_n\}$ and $(r)\mathrm{o} = x_\mathrm{n}$. Then $x_\mathrm{n} \in \mathrm{var_r}(r) = \mathrm{var_r}(s)$ and S_6 satisfies the identity $(\mathrm{u}(r, x_\mathrm{n}), \mathrm{u}(s, x_\mathrm{n}))$. Consequently, $\mathrm{u}(r, x_\mathrm{n}) = \mathrm{u}(s, x_\mathrm{n})$ and the following three cases can arise:

(vi) $r = pq$, $s = p'q'$ and $x_\mathrm{n} \in \mathrm{var}(q')$. First, let $q \neq x_\mathrm{n} \neq q'$. Then $p\mathrm{u}(q, x_\mathrm{n}) = \mathrm{u}(r, x_\mathrm{n}) = \mathrm{u}(s, x_\mathrm{n}) = p'\mathrm{u}(q', x_\mathrm{n})$, $p = p'$ and S_6 satisfies (q, q') by (iv). Thus $q = q'$ and $r = s$, a contradiction. Further, let $q = x_\mathrm{n} \neq q'$. Then $p = p'\mathrm{u}(q', x_\mathrm{n})$, $p' \in X$, $p' = x_1$, $r = x_1\mathrm{u}(q', x_\mathrm{n}) \,.\, x_\mathrm{n}$, $s = x_1q'$ and $f(r) \neq f(s)$ where $f(x_1) = u$, $f(x_\mathrm{n}) = w$ and $f(x) = v$ for $x \in X$, $x \neq x_1$, $x \neq x_\mathrm{n}$, a contradiction. Similarly if $q \neq x_\mathrm{n} = q'$. Finally, if $q = x_\mathrm{n} = q'$ then $p = p'$ and $r = s$, a contradiction.

(vii) $r = pq$, $s = p'q'$, $2 \leqq \mathrm{l}(q)$ and $x_\mathrm{n} \in \mathrm{var}(p')$. Then $p = \mathrm{u}(p', x_\mathrm{n})$, $q' = \mathrm{u}(q, x_\mathrm{n})$ and there are $1 \leqq \mathrm{k}, \mathrm{m}$, $T_1, \ldots, T_\mathrm{k} \in \{L, R\}$ and $r_1, \ldots, r_\mathrm{k}, s_1, \ldots, s_\mathrm{m} \in W$ such that $p' = T_{1,r_1} \ldots T_{\mathrm{k},r_\mathrm{k}}(x_\mathrm{n})$, $T_\mathrm{k} = L$ and $q = s_1(\ldots(s_\mathrm{m}x_\mathrm{n}))$. Then $p = T_{1,r_1} \ldots T_{\mathrm{k}-1,r_{\mathrm{k}-1}}(r_\mathrm{k})$, $q' = s_1(\ldots(s_{\mathrm{m}-1}s_\mathrm{m}))$ and $r_1, \ldots, r_\mathrm{k}$, $s_1, \ldots, s_\mathrm{m} \in X$, since (r, s) is irreducible. We have $1 \leqq \mathrm{m}$, $s_\mathrm{m} \in \mathrm{var_l}(r)$ and $s_\mathrm{m} \in \mathrm{var_r}(s)$, a contradiction.

(viii) $r = px_\mathrm{n}$, $s = p'q'$ and $x_\mathrm{n} \in \mathrm{var}(p')$. Then $p = \mathrm{u}(p', x_\mathrm{n})\, q'$ and $q' \in X$. There are $1 \leqq \mathrm{k}$, $T_1, \ldots, T_\mathrm{k} \in \{L, R\}$ and $r_1, \ldots, r_\mathrm{k} \in W$ such that $p' = T_{1,r_1} \ldots T_{\mathrm{k},r_\mathrm{k}}(x_\mathrm{n})$, $T_\mathrm{k} = L$. Then $p = T_{1,r_1} \ldots T_{\mathrm{k}-1,r_{\mathrm{k}-1}}(r_\mathrm{k}) \,.\, q'$, $r_1, \ldots, r_\mathrm{k} \in X$. Assume $\mathrm{k} \geqq 2$. Since $r_\mathrm{k} \in \mathrm{var_l}(s)$, we have $r_\mathrm{k} \in \mathrm{var_l}(r)$, $T_{\mathrm{k}-1} = R$, S_6 satisfies $(\mathrm{u}(r, r_{\mathrm{k}-1}), \mathrm{u}(s, r_{\mathrm{k}-1}))$, $\mathrm{u}(r, r_{\mathrm{k}-1}) = \mathrm{u}(s, r_{\mathrm{k}-1})$ and $x_\mathrm{n} = q'$, a contradiction. Thus $\mathrm{k} = 1$, $\mathrm{n} = 3$, $r = x_1x_2 \,.\, x_3$, $s = x_1x_3 \,.\, x_2$. But $uw \,.\, v \neq uv \,.\, w$ is true in S_6, a contradiction.

7.2 Corollary. The groupoids S_6 and S_6^{op} satisfy no non-trivial balanced identity.

8.

8.1 Lemma. Let (r, s) be a balanced identity such that the groupoid S_7 satisfies (r, s). Then $r = s$.

Proof. By 4.2, $\mathrm{o}(r) = \mathrm{o}(s)$. Moreover, the subgroupoid $\{u, w\}$ of S_7 is an **R**-semigroup, and therefore $(r)\mathrm{o} = (s)\mathrm{o}$. Since S_7 has a right unit, S_7 satisfies the identity $(\mathrm{u}(r, x), \mathrm{u}(s, x))$, $x = (r)\mathrm{o}$. Hence $(\mathrm{u}(r, x))\mathrm{o} = (\mathrm{u}(s, x))\mathrm{o}$, etc., and we have $\mathrm{v}(r) = \mathrm{v}(s)$ and $(r, s) \in \mathscr{S}$. By 5.2, $r = s$.

8.2 Lemma. Let (r, s) be a balanced identity such that the groupoid S_8 satisfies (r, s). Then $r = s$.

Proof. We have $o(r) = o(s)$ by 4.2, S_8 has a left unit, $v(r) = v(s)$ and $r = s$ by 5.3.

8.3 Corollary. The groupids $S_7, S_8, S_7^{op}, S_8^{op}$ satisfy no non-trivial balanced identity.

9.

9.1 Lemma. Let $r, s, r', s' \in W$ be such that $\mathrm{var}(r) \cap \mathrm{var}(r') \neq \emptyset \neq \mathrm{var}(s) \cap \cap \mathrm{var}(r')$ and the pair $(rs, r's')$ is a strong balanced identity. Then the groupoid S_9 does not satisfy this identity.

Proof. Let $Y = \mathrm{var}(r) = \{x_1, \ldots, x_m\}$ and $Z = \mathrm{var}(s) = \{x_{m+1}, \ldots, x_n\}$, $1 \leqq \leqq m < n$. By 3.8, there are $p, q \in W$ such that pq is a subterm of r' and either $\mathrm{var}(p) \subseteq Y$ and $\mathrm{var}(q) \subseteq Z$ or $\mathrm{var}(p) \subseteq Z$ and $\mathrm{var}(q) \subseteq Y$. Suppose that $\mathrm{var}(p) \subseteq Y$ and $\mathrm{var}(q) \subseteq Z$, the other case being similar. If $\mathrm{var}(q) \neq Z$ then $x_k \in \mathrm{var}(q)$ for some $m + 1 \leqq k \leqq n$ and we define a homomorphism f of W into S_9 by $f(x_1) = \ldots \ldots = f(x_m) = u$, $f(x_k) = v$ and $f(x) = w$ for the remaining variables $x \in X$. Then $f(rs) = uv = u$ and $f(r's') = w$. If $\mathrm{var}(q) = Z$ then we can assume $\mathrm{var}(s') = = \{x_k, x_{k+1}, \ldots, x_m\}$ for some $2 \leqq k \leqq m$ and we define f by $f(x_1) = \ldots \ldots = f(x_{k-1}) = u$, $f(x_k) = \ldots = f(x_m) = w$ and $f(x_{m+1}) = \ldots = v$. Then $f(rs) = = wv = v$ and $f(r's') = uw = w$.

9.2 Lemma. The groupoid S_9 satisfies no strong balanced identity.

Proof. Let (r, s) be a strong balanced identity. We shall prove by induction on $l(r)$ that $f(r) \neq f(s)$ for a homomorphism f of W into S_9. We have $3 \leqq l(r)$, $r = r_1 r_2$ and $s = s_1 s_2$. With respect to 9.1 and the fact that S_9 is commutative, we may assume that $\mathrm{var}(r_1) = \mathrm{var}(s_1)$ and $\mathrm{var}(r_2) = \mathrm{var}(s_2)$. First, let (r_2, s_2) be a strong balanced identity. Then $g(r_2) \neq g(s_2)$ for a homomorphism g and we define f by $f(x) = g(x)$ for $x \in \mathrm{var}(r_2)$ and $f(y) = z$ for $y \in X$, $y \notin \mathrm{var}(r_2)$, where $z \in S$ is such that $\{g(r_2), g(s_2), z\} = \{u, v, w\} = S$. Then $f(r) = z\, g(r_2) \neq z\, g(s_2) = f(s)$. Finally, let $(r_2, s_2) \in \mathscr{R}$. Then $(r_1, s_1) \notin \mathscr{R}$, (r_1, s_1) is a strong balanced identity and we can proceed similarly.

10.

10.1 Proposition. Every quasitrivial groupoid satisfying a balanced identity of type 5 is a semigroup.

Proof. Apply 2.7, 6.3, 7.2, 8.3 and 9.2.

10.2 Proposition. A groupoid G is a medial quasitrivial groupoid iff at least one of the following five assertions is true:

(i) G is a quasitrivial semilattice.

(ii) G is an L-semigroup.

(iii) G is an R-semigroup.

(iv) There exist an L-semigroup H and a quasitrivial semilattice K such that $H \cap K = \emptyset$ and $G = H : K$.

(v) There exist an R-semigroup H and a quasitrivial semilattice K such that $H \cap K = \emptyset$ and $G = H : K$.

Proof. See [1, Theorem 5.5].

10.3 Corollary. Every medial quasitrivial groupoid is a semigroup.

10.4 Theorem. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity of type 5.

(ii) G is medial.

(iii) G satisfies every balanced identity (r, s) such that $o(r) = o(s)$ and $(r)o = (s)o$.

Proof. (i) implies (ii). By 10.1, G is a semigroup. Let $A, B \in G/\sigma$ be such that $A \neq B$ and $AB = B$. Suppose that $2 \leqq \text{card } B$ and $v(r) = (1, 2, \dots, n)$. Since $(r, s) \notin \mathscr{S}$, there are $1 \leqq i < j \leqq n$ such that $v(s) = (\dots, j, \dots, i, \dots)$. Now, take $a \in A$, $b, c \in B$, $b \neq c$, and define a homomorphism f of W into G by $f(x_i) = b$, $f(x_j) = c$ and $f(x) = a$ for $x_i \neq x \neq x_j$. Then, by 1.4 and 1.5, $f(r) = bc \neq cb = f(s)$, a contradiction. have proved that card $B = 1$ and the rest is now clear from 10.2.

(ii) implies (iii). Apply 10.2.

(iii) implies (i). This is obvious.

10.5 Corollary. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity (r, s) of type 5 such that $(r)o \neq (s)o$.

(ii) G is either a semilattice or an L-semigroup or $G = H : K$ where H is an L-semigroup and K a quasitrivial semillatice with $H \cap K = \emptyset$.

(iii) G satisfies the identity $(x_1x_2 \,.\, x_3, x_1 \,.\, x_3x_2)$.

(iv) G satisfies every balanced identity (r, s) such that $o(r) = o(s)$.

10.6 Corollary. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G is commutative and satisfies a strong balanced identity.

(ii) G is a semilattice.

(iii) G satisfies every balanced identity.

(iv) G satisfies the identity $(x_1x_2 \,.\, x_3, x_3x_2 \,.\, x_1)$.

(v) G satisfies a balanced identity (r, s) of type 5 such that $o(r) \neq o(s)$ and $(r)o \neq (s)o$.

11.

11.1 Proposition. The following conditions are equivalent for a quasitrivial groupoid G:

(i) $\varrho = \sigma$.

(ii) G contains no subgroupid isomorphic to one of the groupoids S_5, S_6, S_8, S_5^{op}, S_6^{op}, S_8^{op}.

Proof. Easy (use 1.1(iii) and 2.5).

11.2 Lemma. Let G be a contracommutative quasitrivial groupoid satisfying a balanced identity of type 1. Then G is either an L-semigroup or an R-semigroup.

Proof. By 6.3, 7.2, 8.3 and 11.1, $\varrho = G \times G$ and G is anticommutative. Hence S_9 is not isomorphic to a subgroupoid of G and G is a semigroup. The result follows now from 1.5.

We shall say that a quasitrivial groupoid G is semicommutative if at least one of the following five assertions is true:

(i) G is commutative.

(ii) G is an L-semigroup.

(iii) G is an R-semigroup.

(iv) There exist an L-semigroup H and a commutative quasitrivial groupoid K such that $H \cap K = \emptyset$ and $G = H : K$.

(v) There exist an R-semigroup H and a commutative quasitrivial groupoid K such that $H \cap K = \emptyset$ and $G = H : K$.

11.3 Theorem. The following conditions are equivalent for a quasitrivial groupoid G.

(i) G satisfies a balanced identity of type 1.

(ii) G is semicommutative.

(iii) G satisfies every balanced identity (r, s) of type 1 such that $o(r) = o(s)$ and $(r)o = (s)o$.

(iv) G satisfies the identity $((x_1 \cdot x_2x_3)\, x_4,\ (x_1 \cdot x_3x_2)\, x_4)$.

Proof. (i) implies (ii). Let $A, B \in G/\sigma$ be such that $AB = B$ and $A \neq B$. Suppose $2 \leqq \text{card}\, B$ and $v(r) = (1, 2, \ldots, n)$ where (r, s) is balanced identity of type 1 such that G satisfies (r, s). Since $(r, s) \in \mathscr{R}$ and $r \neq s$, $(r, s) \notin \mathscr{S}$ and there are $1 \leqq i < j \leqq n$ such that $v(s) = (\ldots, j, \ldots, i, \ldots)$. Take $a \in A$, $b, c \in B$, $b \neq c$, and define a homomorphism f of W into G by $f(x_i) = b$, $f(x_j) = c$ and $f(x) = a$ for every $x \in X$, $x_i \neq x \neq x_j$. Then $f(r) = bc$ and $f(s) = cb$. However, $bc \neq cb$ by 11.1, a contradiction. We have proved that card $B = 1$ and the rest is clear from 1.4 and 11.2.

(ii) implies (iii). Assume that $G = H : K$ for an L-semigroup H and a commutative quasitrivial groupoid K such that $H \cap K = \emptyset$. Obviously, $\sigma_G = (H \times H) \cup$

$\cup \mathrm{id}_G$. Denote by g the natural homomorphism of G onto G/σ. Let (r, s) be a balanced identity of type 1 such that $\mathrm{o}(r) = \mathrm{o}(s)$. We have $g f(r) = g f(s)$ and $(f(r), f(s)) \in \sigma$ for every homomorphism f of W into G. Hence either $f(r) = f(s)$ or $f(r), f(s) \in H$. If $f(r) \in H$ then $f(\mathrm{var}(r)) \subseteq H$ and $f(r) = f(s)$. The rest is similar.

(iii) implies (iv) and (iv) implies (i). These implications are clear.

11.4 Corollary. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity (r, s) of type 1 such that $(r)\mathrm{o} \neq (s)\mathrm{o}$.

(ii) G is either commutative or an L-semigroup or $G = H : K$ for an L-semigroup H and a commutative quasitrivial groupoid K with $H \cap K = \emptyset$.

(iii) G satisfies every balanced identity (r, s) of type 1 such that $\mathrm{o}(r) = \mathrm{o}(s)$.

(iv) G satisfies the identity $(x_1 . x_2x_3, x_1 . x_3x_2)$.

11.5 Corollary. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity (r, s) of type 1 such that $\mathrm{o}(r) \neq \mathrm{o}(s)$ and $(r)\mathrm{o} \neq (s)\mathrm{o}$.

(ii) G is commutative.

(iii) G satisfies every balanced identity of type 1.

12.

12.1 Lemma. Let G be a quasitrivial semigroup satisfying a balanced identity of type 3. Then G is medial.

Proof. Similar to that of 10.4.

12.2 Proposition. Let G be a quasitrivial groupoid satisfying the identity $(x_1x_2 . x_3, x_1x_3 . x_2)$. Define a relation π by $(a, b) \in \pi$ iff $a, b \in G$ and $ab = b$. Then:

(i) π is an ordering.

(ii) If $a, b, c \in G$ and $(a, b), (a, c) \in \pi$ then either $(b, c) \in \pi$ or $(c, b) \in \pi$.

Proof. Easy.

12.3 Proposition. Let π be an ordering on a non-empty set G such that either $(b, c) \in \pi$ or $(c, b) \in \pi$ whenever $a, b, c \in G$ and $(a, b), (a, c) \in \pi$. Define a multiplication on G by $ab = b$ if $(a, b) \in \pi$ and $ab = a$ in the opposite case. Then G is a quasitrivial groupoid satisfying $(x_1x_2 . x_3, x_1x_3 . x_2)$.

Proof. Easy.

12.4 Theorem. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity (r, s) of type 3 such that $(r)\mathrm{o} \neq (s)\mathrm{o}$.

(ii) G satisfies the identity $(x_1x_2 . x_3, x_1x_3 . x_2)$.
(iii) G satisfies every balanced identity of type 3.

Proof. (i) implies (ii). Let H be a subgroupoid of G containing at most three elements. If H is a semigroup then H is medial by 12.1 and it is easy to check that H satisfies $(x_1x_2 . x_3, x_1x_3 . x_2)$ (use 10.2). Assume that H is not associative. According to 2.7, 6.3, 7.2, 8.3 and 9.2, H is isomorphic to S_5 and the result follows from 6.1.

12.5 Theorem. The following conditions are equivalent for a quasitrivial groupoid G:

(i) G satisfies a balanced identity of type 3.
(ii) G satisfies the identity $((x_1x_2 . x_3) x_4, (x_1x_3 . x_2) x_4)$.
(iii) At least one of the following assertions is true:
(iii1) G satisfies the identity $(x_1x_2 . x_3, x_1x_3 . x_2)$.
(iii2) G is an R-semigroup.
(iii3) $G = H : K$ for an R-semigroup H and a quasitrivial semilattice K with $H \cap K = \emptyset$.
(iv) G satisfies every balanced identity (r, s) of type 3 such that $(r)\mathrm{o} = (s)\mathrm{o}$.

Proof. (i) implies (ii). Let H be a subgroupid of G containing at most four elements. We are going to show that H satisfies the identity $((x_1x_2 . x_3) x_4, (x_1x_3 . . x_2) x_4)$. The groupoid H satisfies a balanced identity of type 3, say (r, s). We can assume that $\mathrm{l}(r) \leqq \mathrm{l}(r')$ whenever (r', s') is a balanced identity of type 3 such H satisfies (r', s'). If $(r)\mathrm{o} \neq (s)\mathrm{o}$ then 12.4 may be applied. Suppose $\mathrm{o}(r) = x = \mathrm{o}(s)$, $(r)\mathrm{o} = y = (s)\mathrm{o}$ $(x \neq y$, since $4 \leqq \mathrm{l}(r))$ and put $r_1 = \mathrm{u}(r, x)$, $r_2 = \mathrm{u}(r, y)$, $s_1 = = \mathrm{u}(s, x)$, $s_2 = \mathrm{u}(s, y)$. We must distinguish the following cases:

(1) H is a semigroup. By 12.1, H is medial and the result follows easily from 10.2.

(2) H is not associative and H contains a left unit. Then H satisfies (r_1, s_1). If $r_1 = s_1$ then $\mathrm{v}(r) = \mathrm{v}(s)$, $(r, s) \in \mathscr{S}$, a contradiction. Therefore (r_1, s_1) is a balanced identity of type $T \in \{1, 2, 4, 5\}$. If $T = 1$ $(T = 2,\ T = 5)$ then H is associative by 11.3 and 10.6 (5.5, 10.1), a contradiction. Hence $T = 4$ and, since H is not associative, H contains a subgroupoid isomorphic to S_5^{op} (use 2.7, 6.3, 7.2, 8.3, 9.2), a contradiction with 6.3.

(3) H is not associative and H contains a right unit. In this case, we can proceed similarly as in (2).

(4) H is not associative and contains no left unit and no right unit. We can assume without loss of generality that S_5 is a subgroupoid of H. If $H = S_5$ then we have a contradiction with 6.1. Hence H contains just four elements, $H = \{u, v, w, z\}$. Since u and v are not right units of H and z is not a left unit, $zu = u$, $zv = v$, $zw = z$. The subgroupoid $K = \{u, w, z\}$ is not associative, since $z . uw = z \neq w = zu . w$. Consequently, K is isomorphic to S_5, a contradiction with the fact that K contains at most one left zero.

(ii) implies (iii). Suppose that $d = ab \,.\, c \neq ac \,.\, b = e$ for some $a, b, c \in G$. We have $dg = eg$ for every $g \in G$, and so $dg = g = eg$. From this, $gh \,.\, k = (dg \,.\, h)\, k = (dh \,.\, g)\, k = hg \,.\, k$ for all $g, h, k \in G$ and the rest is clear from 11.3 and 10.6.

(iii) implies (iv) and (iv) implies (i). Easy.

Reference

[1] JEŽEK J. and KEPKA T.: Quasitrivial and nearly quasitrivial distributive groupoids and semigroups, Acta Univ. Carolinae Math. Phys. *19/2* (1978), 25—44.