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On the Diagonalisation of von Neumann Regular Matrices

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In the paper, there are studied rings over which von Neumann regular matrices are diagonisable.

V článku se studují okruhy, nad nimiž jsou diagonizovatelné matice regulární ve smyslu von Neumannově.

В статье изучаются кольца, над которыми диагонизируемые матрицы регулярные в смысле фон Неймaнна.

1. Introduction

Let R be any ring, associative with unit element, A an m-by-n matrix over R then A is said to be von Neumann regular if there exists an m-by-n matrix X over R such that $AXA = A$.

If $\mathcal{M}(R)$ denotes the set of all matrices over R then an involution * on $\mathcal{M}(R)$ is a mapping from $\mathcal{M}(R)$ to $\mathcal{M}(R)$ such that for all A, B: $(A^*)^* = A$; $(AB)^* = B^*A^*$.

The m-by-n matrix A is said to have a Moore-Penrose inverse with respect to the involution * iff there exists an n-by-m matrix such that $AXA = A$; $XAX = X$; $(AX)^* = AX$; $(XA)^* = XA$. The solution, if it exists, is unique and denoted by $A^\dagger$.

Several authors considered the problem of characterizing those matrices for which an MP-inverse exists, cf. (1), (4), (5). These results for matrices over the integers, over pricipal ideal domains, over polynomial rings in several variables over a field all follow from a general result of R. Puystjens and D. W. Robinson. They proved that if an m-by-n matrix A over a ring is of the form:

$$A = (P_1P_2)\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

with I_r the r-by-r identity matrix, $P = (P_1P_2)$ and $Q = (Q_1Q_2)^T$ invertible matrices, then A has an MP-inverse with respect to an involution * iff $P_1^*P_1$ and $Q_1Q_1^*$ are

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invertible. The MP-inverse of A is then given by the formula:

$$A^{\dagger} = Q_1^*(Q_1 Q_1^*)^{-1}(P_1^* P_1)^{-1} P_1^* .$$

The rings studied so far in this context are rings over which all idempotent matrices are diagonisable, so called ID-rings. One question we consider is whether ID-rings form exactly that class of rings for which the above characterisation for the existence of MP-inverses hold.

This lead to the following; clearly in order that all von Neumann regular matrices are diagonisable it is necessary that the ring is ID. Is this also sufficient?

We do not solve the latter problem in its full generality, but bring together some evidence for its truth.

2. ID-rings

Definition 1. R is said to be an *ID-ring* provided that for every $E = E^2$ in $\mathcal{M}(R)$, there exist invertible matrices P and Q such that PEQ is a diagonal matrix.

Lemma 2. If R is an integral domain then R is an ID-ring if and only if R is projective free.

Proof. This is a very easy known result e.g. cf. (2).

Remark. In case R is either a domain or a commutative ring, it is equivalent to consider conjugacy with a diagonal matrix, i.e. if $E = E^2$ then there exists an invertible matrix P such that PEP^{-1} is a diagonal matrix. cf. (2) and (6).

As a consequence one has:

Corollary 3. Let R be an ID-domain then every von Neumann regular matrix is diagonisable.

Proof. Let A be von Neumann regular, say $AXA = A$. Then $ImA = ImAX$, since clearly $ImAX$ is part of ImA and $ImAXA$ is contained in $ImAX$. But AX is an idempotent matrix so $ImAX$ is free. We obtain $ImA \oplus F = R^n$ with both ImA and F free modules. Choosing appropriate bases in both modules yields A equivalent with a diagonal matrix.

In case R has zero divisors the above proof does not hold any more. For commutative ID-rings one can prove that every projective module is stable free. We don't know whether this hold in general.

But the corollary still holds in some cases.

The fact that $AXA = A$ implies $ImA = ImAX$ still gives that the module ImA is projective and the isomorphism property then yields coker $A \simeq$ coker AX.

For a large class of rings the latter isomorphism implies that A is equivalent with AX. This has been investigated by Steinitz, Levy, Guralnick, Robson and

Warfield. A good survey for which rings coker $A \simeq$ coker B implies A equivalent with B, may be found in (7). Remark that it holds for rings having 1 in the stable range, among which are:

- Rings with $R/J(R)$ artinian.
- Module finite algebras over commutative rings R with $R/J(R)$ von Neumann regular.
- Module finite algebras over local rings.

We conjecture:

If R is an ID-ring and A is a von Neumann regular matrix then AX is equivalent with A, for every v.N. regular inverse X of A.

In general coker $A \simeq$ coker B does not imply equivalence of A and B. Not even when the cokernels are projective or when A is von Neumann regular, the truth of which would imply the conjecture. We illustrate this in section 4 with matrices over the Weyl algebra.

3. MP-inverses over ID-rings

From what we said in the previous section it follows that the Puystjens-Robinson characterisation for the existence of MP-inverses of matrices holds in any ID-domain.

It can be shown that the characterisation can also be extended to matrices of the form:

$$A = P \left(\begin{array}{ccc|c} e_1 & & & \\ & \ddots & & 0 \\ & & e_r & \\ \hline & 0 & & 0 \end{array}\right) Q$$

with P, Q invertible matrices and $e_1, \ldots, e_n$ idempotent elements in R symmetric with respect to the involution * (to appear). So if the conjecture holds the criterium for MP-inverses can be used over *any ID-ring*.

Without having the conjecture one still can apply the criterium over commutative ID-rings and when * is the transposition of matrices. This is done by using the following trick. Let A be a von Neumann regular m-by-n matrix, say $AXA = A$. Then:

$$(A\ 0)\begin{pmatrix} X & 1 - XA \\ 1 & -A \end{pmatrix} = (AX\ 0)$$

note that:

$$\begin{pmatrix} X & 1 - XA \\ 1 & -A \end{pmatrix}^{-1} = \begin{pmatrix} A & 1 - AX \\ 1 & -X \end{pmatrix}$$

so the matrix $(A\ 0)$ is equivalent with $(AX\ 0)$. Now AX is idempotent, over an ID-ring we have:

$$PAXQ = \operatorname{diag}(e_1, \ldots, e_r, 0, \ldots, 0)$$

with P, Q invertible matrices and the e_i's idempotents in R. Therefore

$$P(AX\ 0)\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = (PAXQ\ 0)$$

$$= \left(\begin{array}{ccc|c|c} e_1 & & & & \\ & \ddots & & 0 & \\ & & e_r & & 0 \\ \hline & 0 & & 0 & \end{array}\right) \qquad (*).$$

and this implies that also the matrix $(A\ 0)$ can be brought in this form (*).

Applying the criterium gives a formula for the MP-inverse of $(A\ 0)$, say $(A\ 0)^{\dagger} = (Y\ Y')^T$ with Y an n-by-m matrix and Y' an m-by-m matrix. So:

$$(A\ 0)\,(Y\ Y')^T\ (A\ 0) = (AYA\ 0) = (A\ 0)$$

implying: $A = AYA$.

$$(Y\ Y')^T\ (A\ 0)\ (Y\ Y')^T =$$

$$\begin{pmatrix} YA & 0 \\ Y'A & 0 \end{pmatrix}\begin{pmatrix} Y \\ Y' \end{pmatrix} = (YAY\ Y'AY')^T = (Y\ Y')^T$$

implying: $YAY = Y$.

$$((A\ 0)\ (Y\ Y')^T)^T = (Y^T\ Y'^T)\ (A^T\ 0) = Y^T A^T = AY.$$

implying: $AY = (AY)^T$

$$((Y\ Y')^T (A\ 0))^T = (A^T\ 0)\,(Y^T\ Y'^T) = \begin{pmatrix} YA & 0 \\ Y'A & 0 \end{pmatrix}$$

implying: $(YA)^T = YA$.

Therefore Y is the MP-inverse for the matrix A.

4. An example over the Weyl algebra

We give an example of a 2-by-2 matrix over the Weyl algebra, which is von Neumann regular but for which A is not equivalent to AX and this for any von Neumann regular inverse X. This shows that the condition "R is an ID-ring" is essential in the conjecture of section 2.

Let F be a field of characteristic 0.

$W = F[x, y, \delta]$, x, y variables and δ the derivation defined by $xy - yx = 1$.

W is a Noetherian simple domain. Every element of W has a unique representation as $\sum a_{ij}x^i y^j$ with a_{ij} in F or as $\sum b_{ij}y^i x^j$ with b_{ij} in F. cf. (2).

Lemma 4. Let f, g, h, k be elements of W. Suppose $fh + gk$ is an element in F not equal to 0.

If $(f\ g)$ is extendable to a 2-by-2 invertible matrix then $(h\ k)^T$ is also extendable to a 2-by-2 invertible matrix.

Prof. Let $fh + gk = a \neq 0$ in F and let

$$\begin{pmatrix} f & g \\ r & s \end{pmatrix}\begin{pmatrix} f' & g' \\ r' & s' \end{pmatrix} = I_2$$

then:

$$\begin{pmatrix} f & g \\ r & s \end{pmatrix}\begin{pmatrix} h & g' \\ k & s' \end{pmatrix} = \begin{pmatrix} a & 0 \\ * & 1 \end{pmatrix}$$

The latter matrix is invertible. Therefore also the second one of the first part of the equation is invertible.

Consider the matrix $A = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$, A is von Neumann regular since $X = \begin{pmatrix} -y & x \\ 0 & 0 \end{pmatrix}$ is a von Neumann regular inverse, i.e. $AXA = A$.

Now let Y be any von Neumann regular element. And suppose there are P, Q invertible 2-by-2 matrices over W, such that $PAQ = AY$. It follows that $PAQA = A$. So if $P = (p_{ij})$ and $Q = (q_{ij})$ then the following equations hold:

$$(p_{11}x + p_{12}y)(q_{11}x + q_{12}y) = x$$

$$(p_{21}x + p_{22}y)(q_{11}x + q_{12}y) = y$$

Comparing degrees in the monomials $x^i y^j$, one finds that either $p_{11}x + p_{12}y \neq 0$ in F and $p_{21}x + p_{22}y \neq 0$ in F or $q_{11}x + q_{12}y \neq 0$ in F.

The first case is impossible since it would yield $q_{11}x + q_{12}y = ax = by$ with a, b in F, which contradicts the unique representation of elements of W.

So $q_{11}x + q_{12}y$ is a nonzero element of F. But Q is invertible so lemma 4 implies that $(x\ y)^T$ is extendable to an invertible 2-by-2 matrix.

This is only possible if $Wx \cap Wy$ is a principal ideal, cf. (2). We show that the latter is not true.

Suppose $Wx \cap Wy = W(\sum a_{ij}x^i y^j)\,x$. Since $x^2 y = x(1 + yx) = (1 + xy)\,x$ is in the intersection of Wx and Wy, we must have $x^2 y = (\sum a_{ij}x^i y^j)\,x$, yielding $1 + xy = \sum a_{ij}x^i y^j$. So $a_{00} = 1$, $a_{11} = 1$ and all the other a_{ij}'s are zero.

But also $y^2 x = y(yx) = y(xy - 1) = (yx - 1)\,y$ is in the intersection of Wx and Wy, so we have $y^2 x = (\sum a_{ij}x^i y^j)\,x$. This implies $a_{02} = 1$ and all the other a_{ij}'s zero. Both conditions are not compatible so the intersection cannot be principal.

This proves that the matrix A is not equivalent with AY for any von Neumann regular inverse Y.

Remark. W is not an ID-ring. Not every projective module over the Weyl algebra is free. Since W is a domain, corollary 3 together with the above example yields another proof for this fact.

One can still ask whether the matrix A is diagonisable over W. (This is sufficient to study the existence of MP-inverses). However since A is not invertible the only possible diagonalisation would be diag (1, 0). But

$$XA = \begin{pmatrix} -y & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Now, an analogous calculation as above shows that A is not equivalent with XA.

For suppose $PAQ = XA$, with P, Q invertible matrices then $APAQ = AXA = A$.

This leads to the equations:

$$x(p_{11}xq_{11} + p_{12}yq_{11}) = x$$

$$x(p_{11}xq_{12} + p_{12}yq_{12}) = 0$$

Therefore:

$$(p_{11}x + p_{12}y)\, q_{11} \neq 0 \text{ and an element in } F,$$

and

$$(p_{11}x + p_{12}y)\, q_{12} = 0\,.$$

This yields $q_{12} = 0$ and $p_{11}x + p_{12}y$ a nonzero element of F. Since P is invertible lemma 4 would imply that $(x\ y)^T$ is extendable to an invertible matrix. This is not possible as we showed already.

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