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Graphs and quasitrivial groupoids

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In the paper, some questions concerning graphs and their connection with quasitrival groupoids are studied.

V článku se studují některé otázky týkající se grafů a jejich souvislosti s kvazitriviálními grupoidy.

В статье изучаются некоторые вопросы касающиеся графов и квазитривиальных группоидов.

1. Introduction

The solution of some questions from the theory of quasitrivial groupoids (e.g. estimations of number of associative triples e.t.c., see [1]) leads to the following problem: Let $L_1, L_2, \dots, L_{16}$ be all pair-wise non-isomorphic graphs with 3 vertices, i.e. $L_j = (V(L_j), E(L_j)), j = 1, \dots, 16$, where $V(L_j) = \{1, 2, 3\}$ for every $j = 1, \dots, 16$ and $E(L_1) = \{(1, 2), (1, 3), (2, 3)\}$, $E(L_2) = \{(1, 2), (1, 3), (2, 3), (3, 2)\}$, $E(L_3) = \{(1, 2), (1, 3)\}$, $E(L_4) = \{(1, 2), (1, 3), (2, 1), (2, 3)\}$, $E(L_5) = \{(1, 2), (3, 2)\}$, $E(L_6) = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $E(L_7) = \emptyset$, $E(L_8) = \{(1, 2), (2, 3), (3, 1)\}$, $E(L_9) = \{(1, 2), (2, 3)\}$, $E(L_{10}) = \{(1, 2), (2, 3), (3, 1), (1, 3)\}$, $E(L_{11}) = \{(1, 2), (2, 3), (3, 2)\}$, $E(L_{12}) = \{(1, 2), (1, 3), (3, 1)\}$, $E(L_{13}) = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$, $E(L_{14}) = \{(1, 3), (3, 1)\}$, $E(L_{15}) = \{(1, 2), (2, 1), (2, 3), (3, 1), (3, 2)\}$, $E(L_{16}) = \{(1, 3)\}$.

In the following text, the word graph is used for the simple antireflexive graph, i.e. a couple $K = (V(K), E(K))$ where $V(K) \neq \emptyset$ will be always a finite set and $E(K) \subseteq V(K) \times V(K)$ such that $E(K) \cap \{(a, a), a \in V(K)\} = \emptyset$.

We shall use the symbol N for the set of all natural numbers. The letters i, j, k, l, m, n will usually denote elements of N.

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Further, an isomorphism of graphs will be denoted by $\cong$. If K is an induced subgraph of a graph L (i.e. $V(K) \subseteq V(L)$ and $E(K) = E(L) \cap (V(K) \times V(K))$) then we shall write $K \subseteq \subseteq L$.

We shall say (for a graph K) that a couple $a, b \in V(K)$ is *a bolt* (denoted by $a \to b$) iff $(a, b) \in E(K)$ and $(b, a) \notin E(K)$, a couple $a, b \in V(K)$ is *a double-bolt* (denoted by $a \leftrightarrow b$) iff $(a, b) \in E(K)$ and $(b, a) \in E(K)$. If $(a, b) \notin E(K)$ and $(b, a) \notin E(K)$ then we shall write $a \not\leftrightarrow b$.

Now, for any graph G and any $1 \leqq i \leqq 16$, denote by $q(G, i)$ the number of all induced subgraphs of G which are isomorphic to L_i, i.e.

$$q(G, i) = \operatorname{card} \{H \subseteq \subseteq G,\ H \cong L_i\} .$$

Our task is to find which values can be achieved by these numbers, especially to determine

$$q_i(n) = \max \{q(G, i),\ \operatorname{card} G = n\}$$

for every natural number n and each $1 \leqq i \leqq 16$.

2. Preliminary results

2.1. Lemma. *The following conditions are satisfied for any* $1 \leqq i \leqq 16$:

(i) *if* $n \leqq 2$ *then* $q_i(n) = 0$,

(ii) *for every* $n \in N$, $0 \leqq q_i(n) \leqq \binom{n}{3}$.

Proof. Evident.

2.2. Lemma. *For any* $n \in N$,

(i) $q_2(n) = q_3(n) = q_4(n) = q_5(n)$,

(ii) $q_9(n) = q_{10}(n)$,

(iii) $q_{11}(n) = q_{12}(n)$,

(iv) $q_{13}(n) = q_{14}(n)$,

(v) $q_{15}(n) = q_{16}(n)$.

Proof. Let us define for a graph G a graph G^{op} (the opposite graph of G) and a graph G^{in} (the inverse graph of G) by $V(G^{op}) = V(G^{in}) = V(G)$ where $(a, b) \in E(G^{op})$ iff $(b, a) \in E(G)$ and the set $E(G^{in})$ is defined in this way: $a \to b$ in the graph G^{in} iff $a \to b$ in the graph G, $a \leftrightarrow b$ in the graph G^{in} iff $a \not\leftrightarrow b$ in the graph G. Clearly, the applications $G \to G^{op}$, $G \to G^{in}$, $G \to (G^{op})^{in}$ define mutually unique correspondences of graphs with the same vertices. Now, if we consider the equality $L_{16} = (L_{15})^{in}$ then we obtain (v).

The situation in (i)–(iv) is quite analogous.

2.3. Lemma. *For any $n \in N$, $n \geqq 3$, we have*

(i) $q_7(n) = \binom{n}{3}$,

(ii) $q_6(n) = \binom{n}{3}$,

(iii) $q_1(n) = \binom{n}{3}$.

Proof. We must find in all the three cases a graph G, card $V(G) = n$, such that

$$q(G, i) = \binom{n}{3}, \quad i = 1, 6, 7. \tag{1}$$

Then we shall have the following inequalities:

$$q_i(n) \geqq \binom{n}{3}, \quad i = 1, 6, 7.$$

The converse inequalities are satisfied by Lemma 2.1 (ii). We choose the following graphs: the empty graph $(M, \emptyset)$ for the case (i), the complete graph $(M, M^2 \setminus \{(a, a), a \in M\})$ for the case (ii) and the graph

$$(M, \bigcup_{i=1}^{n} \{(i, j), j = 1, \ldots, i - 1\})$$

for the case (iii), where $M = \{1, 2, \ldots, n\}$. Then the formula (1) is proved by induction.

Let G be a graph and $a \in V(G)$. Then we put

$$\begin{aligned} f(a) &= f_G(a) = \text{card}\,\{b \in V(G),\ a \to b\}\,, \\ g(a) &= g_G(a) = \text{card}\,\{b \in V(G),\ b \to a\}\,, \\ h(a) &= h_G(a) = \text{card}\,\{b \in V(G),\ a \leftrightarrow b\}\,, \\ k(a) &= k_G(a) = \text{card}\,\{b \in V(G),\ a \not\leftrightarrow b\}\,. \end{aligned}$$

The following results are straightforward.

2.4. Lemma. *Let G be a graph* card $V(G) = n$. *Then for any $a \in V(G)$, $f(a) + g(a) + h(a) + k(a) = n - 1$.*

2.5. Lemma. *Let G be a graph. Then*

$$\sum_{a \in V(G)} f(a) = \sum_{a \in V(G)} g(a).$$

2.6. Lemma. *Let G be a graph. Then*

(i) $\sum_{a \in V(G)} \binom{f(a)}{2} = q(G, 1) + q(G, 2) + q(G, 3)$,

(ii) $\sum_{a \in V(G)} \binom{g(a)}{2} = q(G, 1) + q(G, 4) + q(G, 5),$

(iii) $\sum_{a \in V(G)} \binom{h(a)}{2} = q(G, 13) + q(G, 15) + 3 \cdot q(G, 6),$

(iv) $\sum_{a \in V(G)} \binom{k(a)}{2} = q(G, 14) + q(G, 16) + 3 \cdot q(G, 7).$

2.7. Lemma. *Let G be a graph. Then*

(i) $\sum_{a \in V(G)} f(a)\, g(a) = q(G, 1) + q(G, 9) + q(G, 10) + 3q(G, 8),$

(ii) $\sum_{a \in V(G)} f(a)\, h(a) = q(G, 10) + q(G, 12) + q(G, 15) + 2\, q(G, 4),$

(iii) $\sum_{a \in V(G)} f(a)\, k(a) = q(G, 9) + q(G, 11) + q(G, 16) + 2q(G, 5),$

(iv) $\sum_{a \in V(G)} g(a)\, h(a) = q(G, 10) + q(G, 11) + q(G, 15) + 2q(G, 2),$

(v) $\sum_{a \in V(G)} g(a)\, k(a) = q(G, 9) + q(G, 12) + q(G, 16) + 2q(G, 3),$

(vi) $\sum_{a \in V(G)} h(a)\, k(a) = q(G, 11) + q(G, 12) + 2q(G, 13) + 2q(G, 14).$

3. $q_8(n)$

3.1. Lemma. *Let G be a graph,* card $V(G) = n$, *such that*

(2) $$h(a) = k(a) = 0$$

for every $a \in V(G)$. *Then*

(i) $q(G, 8) = \frac{1}{3} \cdot \sum_{a \in V(G)} f(a)\, g(a) - \frac{1}{3} \cdot \sum_{a \in V(G)} \binom{f(a)}{2},$

(ii) $f(a) + g(a) = n - 1$ *for every* $a \in V(G)$.

Proof. The equality (2) implies the fact that every couple of vertices of G is a bolt, especially, $q(G, 9) = q(G, 4) = q(G, 10) = q(G, 5) = 0$. Now, it suffices to consider Lemmas 2.6, 2.8 and 2.9.

3.2. Lemma. *Let G be a graph satisfying* (2) *such that* card $V(G) = n$. *Then*

$$\sum_{a \in V(G)} f(a)^2 = \sum_{a \in V(G)} g(a)^2 .$$

Proof. Let us write (see Lemma 3.1):

$$\sum_{a \in V(G)} f(a)^2 - \sum_{a \in V(G)} g(a)^2 = \sum_{a \in V(G)} (f(a)^2 - g(a)^2) =$$
$$= \sum_{a \in V(G)} (f(a) + g(a))(f(a) - g(a)) =$$

$$= \sum_{a \in V(G)} (n-1)(f(a) - g(a)) = (n-1)\Big(\sum_{a \in V(G)} f(a) - \sum_{a \in V(G)} g(a)\Big) = 0 .$$

3.3. Lemma. *Let G be a graph with the property* (2), card $V(G) = n$. *Then*

$$q(G, 8) = \frac{n(n-1)(n+1)}{24} - \frac{1}{8} \cdot \sum_{a \in V(G)} (f(a) - g(a))^2 .$$

Proof. First, we shall modify the equality of Lemma 3.1(i)

$$\begin{aligned} q(G, 8) &= \tfrac{1}{6}\Big(\sum_{a \in V(G)} 2 . f(a) . g(a) - \sum_{a \in V(G)} (f(a)^2 - g(a)^2)\Big) = \\ &= \tfrac{1}{6} . \sum_{a \in V(G)} (2 . f(a) . ((n-1) - f(a)) - f(a)^2 + f(a)) = \\ &= \tfrac{1}{6} . \sum_{a \in V(G)} ((2n-1) . f(a) - 3 . f(a)^2) . \end{aligned}$$

Now, we have by the preceding Lemma the following equality:

$$\begin{aligned} 4 . \sum_{a \in V(G)} f(a)^2 &= 2 . \Big(\sum_{a \in V(G)} f(a)^2 + \sum_{a \in V(G)} f(a)^2\Big) = \\ &= \sum_{a \in V(G)} ((f(a) + g(a))^2 + (f(a) - g(a))^2) \end{aligned}$$

which is to be substituted for the last term. We get in this way

$$\begin{aligned} q(G, 8) = \frac{1}{6} . \left(\frac{3n . (n-1)^2}{4} + (2n-1) . \sum_{a \in V(G)} f(a)\right) - \\ - \frac{1}{8} . \sum_{a \in V(G)} (f(a) - g(a))^2 . \end{aligned}$$

We have also to consider the equality

$$\sum_{a \in V(G)} f(a) = \frac{n . (n-1)}{2}$$

which follows from Lemma 2.7 and 3.1(ii).

3.4. Lemma. *For every $n \geqq 3$,*

$$q_8(n) \leqq \frac{n . (n-1) . (n+1)}{24} .$$

Proof. Let us choose any graph G with n vertices. We shall prove that

$$q(G, 8) \leqq \frac{n . (n-1) . (n+1)}{24} = v(n) . \tag{3}$$

First, let the graph G do not fulfill (2), i.e. $a \leftrightarrow b$ or $a \not\leftrightarrow b$ for some $a, b \in V(G)$. Then no induced subgraph $L \subseteq \subseteq G$ with $a, b \in V(L)$ is isomorphic to L_8. Hence, the graph G' defined by $V(G') = V(G)$, $E(G') = E(G) \cup (\{(a, b)\} \setminus \{(b, a)\})$, satisfies

$q(G, 8) \leqq q(G', 8)$. Proceeding similarly for every such $a, b \in V(G)$ we get (after finite number of steps) a graph H satisfying (2) such that

$$q(G, 8) \leqq q(H, 8) . \tag{4}$$

If the graph G fulfills (2) then we put $H = G$. Then Lemma 3.3 implies $q(H, 8) \leqq \leqq v(n)$ and (3) is a consequence of (4).

3.5. Theorem. *For every odd $n \geqq 3$,*

$$q_8(n) = \frac{n \,.\, (n-1) \,.\, (n+1)}{24} = v(n) .$$

Proof. Let us define the graph $G = (\{1, 2, \ldots, n\}, E)$ by

$$E = \bigcup_{i=1}^{n} \left\{(i, i + j \,[\text{mod } n]),\ j = 1, \ldots, \frac{n-1}{2}\right\} .$$

We shall prove by induction that $h(i) = k(i) = 0$ and $f(i) = g(i) = (n-1)/2$ for every $1 \leqq i \leqq n$. Then we get by Lemma 3.3: $q(G, 8) = v(n)$. Hence, $q_8(n) \geqq v(n)$. The converse inequality is satisfied by Lemma 3.4.

3.6. Lemma. *For every even number $n \geqq 4$,*

$$q_8(n) \leqq \frac{n \,.\, (n-2) \,.\, (n+2)}{24} .$$

Proof. Let G be a graph with n vertices. Similarly as in the proof of Lemma 3.4, we obtain a graph H, $V(H) = V(G)$, $q(G, 8) \leqq q(H, 8)$ which fulfills the assumptions of Lemma 3.3. Hence,

$$q(H, 8) = \frac{n \,.\, (n-1) \,.\, (n+1)}{24} - \frac{1}{8} \sum_{a \in V(H)} (f_H(a) - g_H(a))^2 .$$

Now, the number $f_H(a) + g_H(a) = n - 1$ is an odd number. Therefore,

$$(f_H(a) - g_H(a))^2 \geqq 1 .$$

This inequality implies:

$$q(G, 8) \leqq q(H, 8) \leqq \frac{n \,.\, (n-1) \,.\, (n+1)}{24} - \frac{1}{8} \sum_{a \in V(H)} 1 = \frac{n \,.\, (n-2) \,.\, (n+2)}{24} .$$

3.7. Theorem. *For every even number $n \geqq 4$,*

$$q_8(n) = \frac{n \,.\, (n-2) \,.\, (n+2)}{24} .$$

Proof. Let us define a graph $G = (\{1, 2, \ldots, n\}, E)$ by

$$E = \left(\bigcup_{i=1}^{n} \left\{ (i, i + j \,[\mathrm{mod}\ n]),\ j = 1, \ldots, \frac{n}{2} \right\} \right) \cup$$

$$\cup \left(\bigcup_{i=n/2+1}^{n} \left\{ (i, i + j \,[\mathrm{mod}\ n]),\ j = 1, \ldots, \frac{n}{2} - 1 \right\} \right).$$

By induction we can show that $h(i) = k(i) = 0$ and $|f(i) - g(i)| = 1$ for every $1 \leqq i \leqq n$. The rest is similar to the proof of Theorem 3.5.

4. $q_{13}(n)$ and $q_{14}(n)$

We have demonstrated in Lemma 2.2 that $q_{13}(n) = q_{14}(n)$ for every n. We shall find therefore only $q_{13}(n)$.

4.1. Lemma. *Let G be a graph which satisfies*

$$f(a) = g(a) = 0\,, \quad \textit{for every} \quad a \in V(G)\,. \tag{5}$$

Then

(i) $k(a) + h(a) = n - 1$ *for every* $a \in V(G)$,

(ii) $q(G, 13) + q(G, 14) = \dfrac{n\,.\,(n-1)^2}{8} - \dfrac{1}{8} \sum_{a \in V(G)} (h(a) - k(a))^2\,.$

Proof. The equality (i) is a consequence of (5) and of Lemma 2.6. Analogously, the following equality is a consequence of Lemma 2.9:

$$q(G, 13) + q(G, 14) = \frac{1}{2} \,.\, \sum_{a \in V(G)} k(a)\, h(a) =$$

$$= \frac{1}{8} \,.\, \sum_{a \in V(G)} (k(a) + h(a))^2 - \frac{1}{8} \,.\, \sum_{a \in V(G)} (h(a) - k(a))^2\,.$$

The equality (5) implies that the last term is equal to the expression in (ii).

4.2. Lemma. *For every even $n \geqq 4$,*

$$q_{13}(n) \leqq \frac{n^2(n-2)}{8}\,.$$

Proof. Let us choose a graph G with n vertices. Similarly as in Lemma 3.4 we shall find a graph H such that $V(H) = V(G)$, and $q(G, 13) \leqq q(H, 13)$.

Now, Lemma 4.1 says that

$$q(H, 13) \leqq \frac{n\,.\,(n-1)^2}{8} - \frac{1}{8} \sum_{a \in V(H)} (h_H(a) - k_H(a))^2\,.$$

The end of the proof is similar to the proof of Lemma 3.6.

4.3. Theorem. *For every even* $n \geqq 4$,

$$q_{13}(n) = \frac{n^2(n-2)}{8}.$$

Proof. Let us divide some set V, $\operatorname{card} V = n$, into two parts M and K, such that $\operatorname{card} K = \operatorname{card} M = n/2$, and define:

$a \nleftrightarrow b$ for every $a, b \in M$ (or $a, b \in K$);

$a \leftrightarrow b$ whenever $a \in M$, $b \in K$.

We have constructed a graph G with $V(G) = V$ which contains induced subgraphs isomorphic to L_{13} of just two types: in the first case, one vertice lies in M and two in K and in the second case one vertice lies in K and two in M. The number of the subgraphs of both these types is equal to $n/2 \,.\, \frac{1}{2} \,.\, n/2 \,.\, (n/2 - 1)$, since $\operatorname{card} M = \operatorname{card} K = n/2$.

From this:

$$q(G, 13) = \frac{n}{2} \cdot \frac{1}{2} \cdot \frac{n}{2} \cdot \left(\frac{n}{2} - 1\right) . 2 = \frac{n^2(n-2)}{8},$$

i.e.

$$q_{13}(n) \geqq \frac{n^2(n-2)}{8}.$$

The converse inequality is satisfied by Lemma 4.2.

4.4. Lemma. *Let* $n \geqq 3$ *be an odd number and* G *a graph,* $\operatorname{card} V(G) = n$, *which satisfies* (5). *Let* $a, b \in V(G)$ *be a couple of vertices such that*

$$a \leftrightarrow b \quad \text{and} \quad h(a) = h(b) = \frac{n-1}{2}. \tag{6}$$

Then there exists $c \in V(G)$ *such that the induced subgraph* $\{a, b, c\} \subseteq \subseteq G$ *is isomorphic to* L_{14}.

Proof. Put

$\{d \in V(G),\ d \neq b \text{ and } a \leftrightarrow d\} = \{a_1, \ldots, a_h\}$,

$\{d \in V(G),\ d \nleftrightarrow b\} = \{b_1, \ldots, b_k\}$ where the vertices $a, b \in V(G)$ fulfill the suppositions of the Lemma. The equalities (6) and Lemma 4.1(i) imply that

$$k(a) = k(b) = \frac{n-1}{2}.$$

Hence, $h = h(a) - 1 = k(b) - 1 < k$ and there is an i, $1 \leqq i \leqq k$, such that $b_i \notin \{a_1, \ldots, a_h\}$. Then $b_i \nleftrightarrow a$ and the induced subgraph $\{a, b, b_i\} \subseteq \subseteq G$ is isomorphic to L_{14}.

4.5. Lemma. *For every odd* $n \geqq 3$,

$$q_{13}(n) \leqq \frac{(n^2 - 1)(n - 2)}{8}.$$

Proof. Let G be a graph, card $V(G) = n$. There exists a graph H satisfying (5) such that $V(G) = V(H)$ and $q(G, 13) \leqq q(H, 13)$.

We have to prove that

$$q(H, 13) \leqq \frac{(n^2 - 1)(n - 2)}{8} = v(n). \tag{7}$$

First, we suppose that there does not exist a vertice $a \in V(H)$ with the following property:

$$h_H(a) = k_H(a) = \frac{n-1}{2}. \tag{8}$$

As $h_H(a) + k_H(a) = n - 1$ is an even number, we have for every $a \in V(H)$, $(h_H(a) - k_H(a))^2 \geqq 4$. Hence,

$$\sum_{a \in V(H)} (h_H(a) - k_H(a))^2 \geqq 4n.$$

We substitute this inequality in the equality from Lemma 4.1(ii), and so we get the inequality (7). Now, let $a \in V(H)$ be a vertice which fulfills (8). Denote $M = \{b \in K, h_H(b) \neq k_H(b)\}$ where $K = \{b \in V(H), b \leftrightarrow a\}$. If card $M = m$ then $q(H, 14) \geqq$ $\geqq$ card K $-$ card $M = (n - 1)/2 - m$, since for every $b \in K \setminus M$ we find (by Lemma 4.4) an induced subgraph of H which is isomorphic to L_{14} and these subgraphs are pair-wise different. Further, for every $b \in M$, $(h_H(b) - k_H(b))^2 \geqq 4$. Hence,

$$\tfrac{1}{8} \cdot \sum_{a \in V(H)} (h_H(a) - k_H(a))^2 \geqq \tfrac{1}{8} \cdot \sum_{a \in M} (h_H(a) - k_H(a))^2 \geqq$$
$$\geqq \tfrac{1}{8} \cdot 4 \cdot \text{card}\, M = \frac{m}{2}.$$

From this:

$$q(H, 14) + \tfrac{1}{8} \cdot \sum_{a \in V(H)} (h_H(a) - k_H(a))^2 \geqq \frac{n-1}{2} - \frac{m}{2} \geqq \frac{n-1}{4},$$

since card $M = m \leqq (n - 1)/2 =$ card K.

We substitute the last inequality into the equality from Lemma 4.1(ii), and so we get the inequality (7).

4.6. Theorem. *For every odd* $n \geqq 3$,

$$q_{13}(n) = \frac{(n^2 - 1)(n - 2)}{8}.$$

Proof. If we find a graph G such that card $V(G) = n$ and

$$q(G, 13) = \frac{(n^2 - 1)(n - 2)}{8} = v(n)$$

we are ready, since the rest follows by the preceding lemma. We shall construct such a graph G similarly as in the proof of Theorem 4.3 but we choose card $M =$ $= (n + 1)/2$, card $K = (n + 1)/2$.

5. $q_{15}(n)$ and $q_{16}(n)$

Analogously to the previous part we shall find only $q_{15}(n)$, because $q_{15}(n) =$ $= q_{16}(n)$, for every n.

5.1. Lemma. *Let G be a graph satisfying for every $a \in V(G)$:*

$$(9) \qquad k(a) = 0\,.$$

Then

(i) $h(a) + f(a) + g(a) = n - 1$, *for every* $a \in V(G)$,

(ii) $q(G, 15) + q(G, 4) + q(G, 2) + q(G, 10) = n\,.\,(n - 1)^2)/8 -$ $- \sum_{a \in V(G)} (h(a) - f(a) - g(a))^2$.

Proof. The equality (i) is a consequence of (9) and Lemma 2.6. Similarly, Lemma 2.9 implies:

$$q(G, 15) + q(G, 4) + q(G, 2) + q(G, 10) =$$
$$= \tfrac{1}{2} \,.\, \sum_{a \in V(G)} (h(a) f(a) + h(a)\, g(a)) =$$
$$= \tfrac{1}{2} \,.\, \sum_{a \in V(G)} ((h(a) + g(a) + f(a))^2 - (h(a) - g(a) - f(a))^2)\,.$$

This expression is equal to that in (ii), since the equality (i) is true.

5.2. Lemma. *For every even* $n \geqq 4$,

$$q_{15}(n) \leqq \frac{n^2(n - 2)}{8}\,.$$

Proof is analogous to that of Lemma 4.2.

5.3. Theorem. *For every even* $n \geqq 4$,

$$q_{15}(n) = \frac{n^2(n - 2)}{8}\,.$$

Proof. It is similar to that of Theorem 4.3 but the graph G is defined in this way: $a \leftrightarrow b$ whenever $a \in M$, $b \in K$ and if $a, b \in M$ (or $a, b \in K$) then the couple of vertices $a, b \in V$ is any bolt.

5.4. Lemma. *Let $n \geqq 3$ be an odd number and G a graph, card $V(G) = n$, which fulfills (9). Let $a, b \in V(G)$ be a couple of vertices such that $a \leftrightarrow b$ and*

$$h(a) = h(b) = \frac{n-1}{2}. \tag{10}$$

Then there exists $c \in V(G)$ such that the induced subgraph $\{a, b, c\} \subseteq \subseteq G$ is isomorphic to L_2 (or L_4, or L_{10}).

Proof. Let us denote

$$\begin{aligned}
\{d \in V(G),\ d \neq b \text{ and } a \leftrightarrow d\} &= \{a_1, \dots, a_h\}, \\
\{d \in V(G),\ b \to d\} &= \{b_1, \dots, b_f\}, \\
\{d \in V(G),\ d \to b\} &= \{b_{f+1}, \dots, b_{f+g}\}
\end{aligned}$$

where the vertices $a, b \in V(G)$ satisfy the suppositions of the lemma. The equalities (10) and Lemma 5.1 imply that

$$h = h(a) - 1 = f(b) + g(b) - 1 < f + g.$$

Hence, there exists i, $1 \leqq i \leqq f + g$, such that $b_i \notin \{a_1, \dots, a_h\}$. Then either $b_i \to a$ or $a \to b_i$. If $b_i \to a$ then $\{a, b, b_i\} \cong L_{10}$ (for $1 \leqq i \leqq f$) or $\{a, b, b_i\} \cong L_2$ (for $f + 1 \leqq i \leqq f + g$). If $a \to b_i$ then $\{a, b, b_i\} \cong L_4$ (for $1 \leqq i \leqq f$) or $\{a, b, b_i\} \cong \cong L_{10}$ (for $f + 1 \leqq i \leqq f + g$).

5.5. Lemma. *For every odd $n \geqq 3$,*

$$q_{15}(n) \leqq \frac{(n^2 - 1)(n - 2)}{8}.$$

Proof. Let G be a graph, card $V(G) = n$. We find a graph H satisfying (9) such that $V(H) = V(G)$ and $q(G, 15) \leqq q(H, 15)$.

We have to prove that

$$q(H, 15) \leqq \frac{(n^2 - 1)(n - 2)}{8} = v(n). \tag{11}$$

First, we suppose that there is no vertice $a \in V(H)$ with the following property:

$$h_H(a) = f_H(a) + g_H(a) = \frac{n-1}{2}. \tag{12}$$

As $h_H(a) + f_H(a) + g_H(a) = n - 1$ is an even number, we have for every $a \in V(H)$:

$$(h_H(a) - f_H(a) - g_H(a))^2 \geqq 4.$$

Hence,

$$\sum_{a\in V(H)} (h_H(a) - f_H(a) - g_H(a))^2 \geqq 4n .$$

We substitute this inequality in the equality from Lemma 5.1 (ii), and so we get the inequality (11).

Now, let $a \in V(H)$ be a vertice which fulfills (12). Put $M = \{b \in K,\ h_H(b) \neq (n-1)/2\}$ where $K = \{b \in V(H),\ a \leftrightarrow b\}$. Then (for card $M = m$): $q(H, 2) + q(H, 4) + q(H, 10) \geqq (n-1)/2 - m$, since for every $b \in K \setminus M$ we have (by Lemma 5.4) an induced subgraph of H which is isomorphic to L_2 (or L_4, or L_{10}) and these subgraphs are pair-wise different. Further, for every $b \in M$,

$$(h_H(b) - f_H(b) - g_H(b))^2 \geqq 4 .$$

Therefore,

$$\tfrac{1}{8} \cdot \sum_{a\in V(H)} (h_H(a) - f_H(a) - g_H(a))^2 \geqq \frac{m}{2} ,$$

and

$$\tfrac{1}{8} \cdot \sum_{a\in V(H)} (h_H(a) - f_H(a) - g_H(a))^2 + q(H, 2) +$$

$$+ q(H, 4) + q(H, 10) \geqq \frac{n-1}{2} - \frac{m}{2} \geqq \frac{n-1}{4} ,$$

(similarly as in Lemma 4.5).

We substitute this inequality in the equality from Lemma 5.1(ii) and we obtain (11).

5.6. Theorem. *For every odd $n \geqq 3$,*

$$q_{15}(n) = \frac{(n^2-1)(n-2)}{8} .$$

Proof. Analogously to the proof of Theorem 5.3 (or Theorem 4.3 or 4.6).

6. Some estimations

Now, we shall find an estimation of $q_2(n)$ (hence also of $q_3(n)$ and $q_4(n)$ and $q_5(n)$; see Lemma 2.2).

Let $\{P_i(n)\}_{n=1}^{\infty}$, $i = 0, 1, 2, \ldots$ be a sequence defined by

$$P_0(n) = n , \quad P_{i+1}(n) = [\tfrac{1}{3}(P_i(n) + 1)]$$

where $[x]$ denotes the integral part of x.

Further, we put for every n:

$$(13) \qquad Q(n) = \sum_{i\in I(n)} (\tfrac{1}{2} P_i(n)(P_{i-1}(n) - P_i(n))(P_{i-1}(n) - P_i(n) - 1))$$

where $I(n) = \{i \in N,\ P_i(n) \neq 0\}$.

6.1. Lemma. *Let* $n \geqq 3$. *Then*

$$Q(n) = Q(P_1(n)) + \tfrac{1}{2} P_1(n)(n - P_1(n))(n - P_1(n) - 1).$$

Proof. Obviously, for any $i \in N$, $P_i(n) = P_{i-1}(P_1(n))$. We substitute this equality into (13) and we have:

$$Q(n) = \sum_{\substack{i \in I(n) \\ i \neq 1}} (\tfrac{1}{2} P_{i-1}(P_1(n))(P_{i-2}(P_1(n)) - P_{i-1}(P_1(n))) .$$

$$. (P_{i-2}(P_1(n)) - P_{i-1}(P_1(n)) - 1)) + \tfrac{1}{2} P_1(n)(n - P_1(n))(n - P_1(n) - 1)).$$

Now, it suffices to consider the definition of $Q(n)$ and the fact that $i \in I(n)$ iff $i - 1 \in \in I(P_1(n))$.

6.2. Lemma. *For every* n, $q_2(n) \geqq Q(n)$.

Proof. We shall define by induction a graph G which satisfies

$$q(G, 2) = Q(n). \tag{14}$$

Let V be a set with n elements.

I. If $n = 1$ then $E = \emptyset$.

II. Divide the set V into two parts: $V = M \cup K$, $M \cap K = \emptyset$, such that card $M = m = [\frac{1}{3}(n + 1)]$. The graph G is defined on the set M, since card $M < n$, and we put $a \rightarrow b$ for $a \in M$, $b \in K$, and $a \leftrightarrow b$ for $a, b \in K$. Now, the equality (14) proved by induction (and Lemma 6.1).

6.3. Remark. We can show by induction (and Lemma 6.1) that for every n,

$$Q(n + 1) \geqq Q(n) + \frac{2n^2}{9}.$$

6.4. Corollary. *For every* $n \geqq 3$,

$$Q(n) \geqq \frac{2n^3 - 3n^2}{27}.$$

6.5. Theorem. *For every* $n \geqq 3$,

$$\frac{2n^3 - 3n^2}{27} \leqq q_2(n) \leqq \frac{n(n - 1)^2}{8}.$$

Proof. See Lemmas 5.1, 6.2 and 6.4.

References

[1] Kepka T. and Kratochvíl J.: Graphs and associative triples in quasitrivial groupoids (to appear).