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On Strong Continuity of Derivatives of Mappings

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Relations between strong continuity of derivatives and uniformity of differentiability of mappings in locally convex spaces are studied; results are formulated for families of mappings.

V práci je vyšetřována souvislost mezi zesílenou spojitostí derivací a stejnoměrnou diferencovatelností zobrazení v lokálně konvexních prostorech; výsledky jsou formulovány pro systémy zobrazení.

Изучаются соотношения между усиленной непрерывностью производных и равномерной дифференцируемостью отображений в локально выпуклых пространствах; утверждения формулированы для семейств отображений.

Consider a mapping f between two locally convex linear topological spaces and suppose there exists its derivative f'. In case of Banach spaces or more particularly if f is a real functional, various relations were obtained between properties of f and of f', namely as to compactness and continuity properties of f' due to their important role in applications (see [4], [6]–[9] and many others). Considerably less is known in case of general locally convex spaces, nevertheless there is for instance an interesting paper [5] concerning families of differentiable mappings in such spaces, or [2], [6] etc. Some results of [5] were completed or generalized later in [3] and the aim of our present paper is to give some further theorems in that direction. They are presented for families of mappings but nevertheless they provided new results even in case of single mappings, generalizing so some results of [6], [8], [9].

Basic definitions and notations

Throughout all the paper, the symbols X and Y will denote arbitrary locally convex topological linear spaces over the real field $\mathbf{R}$, $\mathscr{U}$ and $\mathscr{V}$ will denote the collections of all open convex balanced neighbourhoods of 0 in X and Y (in respective topologies), M will denote a subset of X and $\mathscr{B}$ and $\mathscr{B}_M$ the collections of all bounded

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subsets of X and M, respectively. We shall denote by $\mathscr{L}(X, Y)$ the space of all continuous linear mappings from X into Y with the topology of uniform convergence on bounded subsets of X, and by $\mathscr{Z}$ the base of neighbourhoods of 0 in $\mathscr{L}(X, Y)$ consisting of all sets of the form

$$(B, V) = \{u \in \mathscr{L}(X,Y) : u(B) \subset V\}$$

where $B \in \mathscr{B}$ and $V \in \mathscr{V}$.

Let $\mathscr{F}$ be a family of mappings from M into Y. This family is said to be weakly (strongly, respectively) equicontinuous (see also [5]) on M iff for each $x_0 \in M$ and each bounded net $(x_a, a \in A)$ in M, the weak convergence $x_a \rightharpoonup x_0$ $(a \in A)$ implies $f(x_a) \rightharpoonup f(x_0)$ (respectively $f(x_a) \to f(x_0)$) uniformly over $f \in \mathscr{F}$. The $\mathscr{F}$ is said to be uniformly weakly (resp. strongly) equicontinuous on a set $N \subset M$ iff for any bounded nets (x_a, A) and (x'_a, A) in N, the weak convergence $x_a - x'_a \rightharpoonup 0$ $(a \in A)$ implies $f(x_a) - f(x'_a) \rightharpoonup 0$ (resp. $f(x_a) - f(x'_a) \to 0$) uniformly over $f \in \mathscr{F}$.

The family $\mathscr{F}$ is said to be collectively precompact [5] in M iff for each $B \in \mathscr{B}_M$ the set $\{f(x) : x \in B, f \in \mathscr{F}\}$ is precompact in Y (let us recall that precompactness is equivalent to relative compactness in complete spaces). Note that the collective precompactness of the family $\mathscr{F}'$ of derivatives f' of mappings from $\mathscr{F}$ (see below) means – according to this definition – precompactness of the set $\{f'(x) : x \in B, f \in \mathscr{F}\}$ in the space $\mathscr{L}(X, Y)$ for each $B \in \mathscr{B}_M$.

We say that $\mathscr{F}$ is collectively locally precompact at a point $x_0 \in M$ (an analogy to the similar notion used in [9]) iff the validity of the condition above in the definition of collective precompactness is required only for such $B \in \mathscr{B}_M$ which are weakly to x_0 convergent nets. Obviously, both strong continuity and precompactness imply local compactness, but the converse does not hold.

We shall use the following concept of differentiability (see [1], [5], [6]): A mapping $f: M \to Y$ is said to be Gâteaux (Fréchet, respectively) differentiable at a point $x \in M$ iff there exists $u \in \mathscr{L}(X, Y)$ such that for each $h \in X$ ($B \in \mathscr{B}$, resp.) and $V \in \mathscr{V}$, there is a $\delta > 0$ such that

$$f(x + th) - f(x) - u(th) \in tV$$

whenever $|t| \leqq \delta$ (whenever $h \in B$ and $|t| \leqq \delta$, resp.); such mapping u is denoted by $f'(x)$ and called a derivative of f at x. Differentiability is called uniform on a set $N \subset M$ iff the δ above can be chosen independently of $x \in N$.

A family of mappings is said to be equidifferentiable (Gâteaux, Fréchet or uniformly) iff the mappings are differentiable in the respective sense and the δ in the respective definition can be chosen the same for all mappings from the family.

Throughout the paper, for a given family $\mathscr{F}$ of mappings, the following notations will be used for point sets and families of mappings induced by $\mathscr{F}$:

$$\mathscr{F}(x) = \{f(x) : f \in \mathscr{F}\}\,, \quad \mathscr{F}' = \{f' : f \in \mathscr{F}\}\,,$$
$$\mathscr{F}'(x) = \{f'(x) : f \in \mathscr{F}\}\,, \quad \mathscr{F}(N) = \{f(x) : x \in N,\, f \in \mathscr{F}\}\,,$$

and similar ones.

Pseudouniform differentiability

Let $f\colon M \to Y$ be a Fréchet or Gâteaux differentiable mapping with a derivative f'. In Fréchet case, the differentiability is called pseudouniform at a point $x_0 \in M$ iff an arbitrary $B \in \mathscr{B}$ and $V \in \mathscr{V}$ given, there exist $\delta > 0$ and $U \in \mathscr{U}$ such that

$$f(x + th) - f(x) - f'(x)\, th \in tV$$

for all $|t| \leqq \delta$, $h \in B$ and $x \in (x_0 + U) \cap M$; in case of Gâteaux differentiability, the definition of pseudouniformity is the same but only one-point sets B are considered in it.

Let us remark that such a point x_0, at which f is pseudouniformly differentiable, is sometimes called a point of uniform differentiability of f (see e.g. [2]) or the term locally uniform differentiability at x_0 instead of pseudouniform differentiability at x_0 is sometimes used.

Evidently, uniform differentiability of f on $N \subset M$ implies pseudouniform differentiability of f on N (i.e. at every point of N) but the converse does not hold.

In case the weak topology of X is considered in X, a slightly modified notion of pseudouniformity will be used: Fréchet differentiability is called weak-pseudouniform at $x_0 \in M$ iff for each $B \in \mathscr{B}$, $V \in \mathscr{V}$ and each bounded net (x_a, A) in M which weakly converges to x_0, there exist $a_0 \in A$ and $\delta > 0$ such that

$$f(x_a + th) - f(x_a) - f'(x_a)\, th \in tV$$

for all $|t| \leqq \delta$, $h \in B$ and $a \in A$, $a \succ a_0$; in case of Gâteaux differentiability, one-point sets B are considered only.

It is easy to see that both uniform differentiability (with respect to the original topology as well as to the weak topology in X) and pseudouniform differentiability with respect to the weak topology in X imply weak-pseudouniform differentiability of a mapping $f\colon M \to Y$.

A family $\mathscr{F}$ of mappings $f\colon M \to Y$ is said to be pseudouniformly (or weak-pseudouniformly) equidifferentiable at a point $x_0 \in M$ iff the mappings f are pseudouniformly (or weak-pseudouniformly, respectively) differentiable at x_0 and δ and U (or a_0, respectively) in the definitions above can be chosen the same for all $f \in \mathscr{F}$.

Main results

In our paper [3] several sufficient or necessary and sufficient conditions were derived for collective precompactness of a family $\mathscr{F}'$ of derivatives and among them also one or two ones for strong equicontinuity of $\mathscr{F}'$ — but in semireflexive spaces only. We shall now prove some other theorems on strong equicontinuity of $\mathscr{F}'$ without any space restrictions.

Let X, Y, M and $\mathscr{F}$ be as stated above and suppose that the mappings from $\mathscr{F}$ are Gâteaux differentiable in M (so that $\mathscr{F}'$ is defined on M). For the sake of simplicity suppose M to be open.

Theorem 1. Let $\mathscr{F}$ be strongly equicontinuous and Fréchet equidifferentiable, both uniformly on bounded subsets of M. Then $\mathscr{F}'$ is uniformly strongly equicontinuous on every set $N \subset M$ such that $\overline{N} \subset M$.

Proof. Let $N \subset M$ be an arbitrary set such that $\overline{N} \subset M$ and choose $U \in \mathscr{U}$ such that $\overline{N} + U \subset M$. Let (x_a, A) and (y_a, A) be two bounded nets in N and suppose the net $(x_a - y_a, A)$ is weakly convergent to 0. We are to prove that the corresponding net $(f'(x_a) - f'(y_a), A)$ converges to 0 (in the topology of the space $\mathscr{L}(X, Y)$ mentioned above) uniformly over $f \in \mathscr{F}$.

Let a neighbourhood (B, V) $(B \in \mathscr{B}, V \in \mathscr{V})$ of 0 in $\mathscr{L}(X, Y)$ be given and let us denote $W = \frac{1}{4}V$. It follows from the boundedness of B and the uniform Fréchet equidifferentiability of $\mathscr{F}$ on the bounded set

$$P = \{x_a\colon a \in A\} \cup \{y_a\colon a \in A\}$$

that a $\delta > 0$ can be found so that $\delta B \subset U$ and

$$r_f(x, th) = f(x + th) - f(x) - f'(x)\, th \in tW$$

for all $x \in P$, $h \in B$, $|t| \leqq \delta$ and $f \in \mathscr{F}$. Furthermore, in view of the uniform strong equicontinuity of $\mathscr{F}$ on the bounded set $P + (\delta B \cup \{0\})$, an $a_0 \in A$ can be chosen so that

$$f(u_{(a,h)}) - f(u'_{(a,h)}) \in \delta W$$

whenever $(a, h) \in A \times (B \cup \{0\})$, $a \succ a_0$ and $f \in \mathscr{F}$ where $u_{(a,h)} = x_a + \delta h$, $u'_{(a,h)} = y_a + \delta h$ and a partial order given by the relation

$$(a_1, h_1) \prec (a_2, h_2) \Leftrightarrow a_1 \prec a_2$$

is considered in $A \times (B \cup \{0\})$; in other words, it is

$$f(x_a) - f(y_a) \in \delta W \quad \text{and} \quad f(x_a + \delta h) - f(y_a + \delta h) \in \delta W$$

whenever $a \in A$, $a \succ a_0$, $h \in B$ and $f \in \mathscr{F}$. Hence it holds

$$\begin{aligned}(f'(x_a) - f'(y_a))\, h &= \frac{1}{\delta}[f(x_a + \delta h) - f(x_a) - r_f(x_a, \delta h)] - \\ &- \frac{1}{\delta}[f(y_a + \delta h) - f(y_a) - r_f(y_a, \delta h)] = \\ &= \frac{1}{\delta}[f(x_a + \delta h) - f(y_a + \delta h)] -\end{aligned}$$

$$- \frac{1}{\delta}[f(x_a) - f(y_a)] - \frac{1}{\delta} r_f(x_a, \delta h) +$$

$$+ \frac{1}{\delta} r_f(y_a, \delta h) \in W - W - W + W = V$$

for all $a \in A$, $a \succ a_0$, $h \in B$ and $f \in \mathscr{F}$ (we recall that both W and V are convex and balanced), which means

$$f'(x_a) - f'(y_a) \in (B, V)$$

for $a \succ a_0$ and $f \in \mathscr{F}$. It proves the theorem.

Let us remark that in general the assumption of uniformity on bounded subsets is weaker than usually used assumption of local uniformity (i.e. uniformity on on a neighbourhood of each point). Both concepts coincide in the case of normed spaces.

Corollary 1. Let $\mathscr{F}$ and $\mathscr{F}'$ be collectively precompact in M and suppose $\mathscr{F}$ is Fréchet equidifferentiable uniformly on bounded subsets of M. Then $\mathscr{F}'$ is uniformly strongly equicontinuous on every subset N of M such that $\bar{N} \subset M$.

Proof. According to [3, Theorem 2.1], it follows from collective precompactness of $\mathscr{F}'$ in M that $\mathscr{F}$ is uniformly weakly equicontinuous on bounded subsets of M and then, in view of collective precompactness of $\mathscr{F}$ in M and [3, Theorem 2.3], also uniformly strongly equicontinuous on bounded subsets of M. Our assertion follows now immediately from Theorem 1.

Corollary 2. Let $\mathscr{F}$ be Fréchet equidifferentiable uniformly on bounded subsets of M and let $\mathscr{F}(x_0)$ be precompact in Y for at least one $x_0 \in M$. Suppose that $\mathscr{F}'$ is collectively procompact in M and that each mapping $f'(x) \in \mathscr{L}(X, Y)$ $(x \in M)$ is precompact. Then $\mathscr{F}'$ is uniformly strongly equicontinuous on every subset N of M such that $\bar{N} \subset M$.

This corollary follows immediately from Corollary 1 and [3, Theorem 2.4]. Note that collective joint precompactness (see [3], [5]) of $\mathscr{F}'$ can be assumed instead of our conditions concerning $\mathscr{F}'$ and $f'(x)$.

A brief examination of the proof of Theorem 1 shows how the assumptions of Theorem 1 must modified to obtain an "original topology" analogue of it. Bearing into mind that the nets (x_a) and (y_a) considered in the proof are not necessarily bounded in that case, the analogue reads as follows:

Theorem 2. Let $\mathscr{F}$ be uniformly equicontinuous and uniformly Fréchet equidifferentiable on M. Then $\mathscr{F}'$ is uniformly equicontinuous on every subset N of M such that $\bar{N} \subset M$.

It is apparently the direct transcription of the well-known Vainberg's result [8], which concerns the case of a single mapping in Banach spaces, to our more general situation.

Theorem 1 concerns uniform strong equicontinuity of $\mathscr{F}'$. The following theorem shows that a localization of the differentiability assumptions, imposed on $\mathscr{F}$, yields the sufficient condition for simple strong equicontinuity.

From now on, we shall suppose M to be weakly open.

Theorem 3. Let $\mathscr{F}$ be strongly equicontinuous uniformly on bounded subsets of M and weak-pseudouniformly Fréchet equidifferentiable at a point $x_0 \in M$. Then $\mathscr{F}'$ is strongly equicontinuous at x_0.

Proof. The proof is similar to that of Theorem 1. Let (x_a, A) be a bounded net in M weakly convergent ot x_0, let a neighbourhood (B, V) $(B \in \mathscr{B}, V \in \mathscr{V})$ of 0 in $\mathscr{L}(X, Y)$ be given. Debote $W = \frac{1}{4}V$ and choose a weak neighbourhood U_0 of 0 in X so that $x_0 + U_0 + U_0 \subset M$. The set B being bounded and hence also weakly bounded, there is a $\delta_0 > 0$ so that $\delta_0 B \subset U_0$. Let $a_0 \in A$ be such that $x_a \in U_0$ whenever $a \in A$ and $a \succ a_0$.

It follows from weak-pseudouniform equidifferentiability of $\mathscr{F}$ at x_0 that an $a_1 \in A$, $a_1 \succ a_0$ and a $\delta > 0$, $\delta \leqq \delta_0$ can be found such that

$$r_f(x_a, th) = f(x_a + th) - f(x_a) - f'(x_a)\, th \in tW$$

for all $h \in B$, $|t| \leqq \delta$, $f \in \mathscr{F}$ and $a \in A$, $a \succ a_1$. In view of the continuity properties of $\mathscr{F}$, an $a_2 \in A$, $a_2 \succ a_1$ can be chosen so that

$$f(x_a) - f(x_0) \in \delta W \quad \text{and} \quad f(x_a + \delta h) - f(x_0 + \delta h) \in \delta W$$

for all $a \in A$, $a \succ a_2$, $h \in B$ and $f \in \mathscr{F}$. The same arguments as in the proof of Theorem 1 now show that

$$f'(x_a) - f'(x_0) \in (B, V)$$

for all $a \in A$, $a \succ a_2$ and $f \in \mathscr{F}$, which proves the theorem.

An analogical assertion to Corollary 1 (in local form) can be now stated but we shall prove that it holds even under slightly weaker assumptions. To that aim, two propositions are needed. The first one is a localization of [3, Theorem 2.1] and the second one is an improvement of the local part of [3, Theorem 2.3]. The proofs of both propositions are evident – compare their original proofs with the definition of local compactness – and then omitted.

Proposition 1. Let $\mathscr{F}$ be Gâteaux equidifferentiable in a weakly open set $U \subset X$ and suppose $\mathscr{F}'$ is collectively locally precompact in U. Then $\mathscr{F}$ is weakly equicontinuous in U.

Proposition 2. Let $\mathscr{F}$ be weakly equicontinuous and collectively locally precompact in a weakly open set $U \subset X$. Then it is strongly equicontinuous in U.

Before we formulate the promised assertion, the following theorem will be useful to state.

Theorem 4. Let $\mathscr{F}$ be weak-pseudouniformly Gâteaux equidifferentiable at a point $x_0 \in M$ and strongly equicontinuous in a weak neighbourhood of x_0. If $\mathscr{F}'$ is collectively locally precompact on a weak neighbourhood of x_0, then it is strongly equicontinuous at x_0.

Proof. Let x_0 be a given point of M and let U_0 be such a weak neighbourhood of 0 in X that $x_0 + U_0 + U_0 \subset M$, $\mathscr{F}'$ is collectively locally precompact on $x_0 + U_0$ and $\mathscr{F}$ is strongly equicontinuous on $x_0 + U_0$. Let $\mathscr{F}$ be weak-pseudouniformly Gâteaux equidifferentiable at x_0 and suppose to the contrary that $\mathscr{F}'$ is not strongly equicontinuous at x_0. In that case, we can find a bounded net (x_a, A) in M such that it converges weakly to x_0 and simultaneously, the corresponding net $(f'(x_a), A)$ does not converge to $f'(x_0)$ uniformly over $f \in \mathscr{F}$. Hence, there is a neighbourhood $(B, V) \in \mathscr{X}$ of 0 in $\mathscr{L}(X, Y)$, a cofinal directed subset A' of A and a mapping $f_a \in \mathscr{F}$ for each $a \in A'$, such that

$$f'_a(x_a) - f'_a(x_0) \notin (B, V) \tag{1}$$

holds for all $a \in A'$.

The subnet (x_a, A') of the original net (x_a, A) is weakly convergent to x_0, hence we can assume that A' is such that $x_a \in x_0 + U_0$ for all $a \in A'$. Double use of collective local precompactness of $\mathscr{F}'$ on $x_0 + U_0$ then implies that there is a cofinal directed subset A'' of A' such that the net $(f'_a(x_a) - f'_a(x_0), A'')$ is a Cauchy net in $\mathscr{L}(X, Y)$.

Denote by $\hat{Y}$ the completion of Y and let us consider $\mathscr{L}(X, Y)$ as embedded into its completion $\mathscr{L}(X, \hat{Y})$. The net $(f'_a(x_a) - f'_a(x_0), A'')$ has then its limit point in $\mathscr{L}(X, \hat{Y})$; denote it by z_0. Apparently, the formula (1) implies $z_0 \neq 0$, which means there is an $h_0 \in X$ such that $z_0(h_0) \neq 0$ in $\hat{Y}$ and hence, a neighbourhood W of 0 in $\hat{Y}$ can be found so that

$$z_0(h_0) \notin \overline{W}. \tag{2}$$

Eventually, find a neighbourhood $V_0 \in \mathscr{V}$ such that

$$4V_0 \subset W \cap Y. \tag{3}$$

(Evidently all the choice can be done in such way so that $h_0 \in B$ and $4V_0 \subset \frac{1}{2}V$, but it is of no importance for the proof.)

Now, choose $t_1 > 0$ so that $t_1 h_0 \subset U_0$. The point h_0 and the neighbourhood V_0 being given, the weak-pseudouniform Gâteaux equidifferentiability of $\mathscr{F}$ at x_0 implies that there is an $a_1 \in A''$ and a $t_2 > 0$ such that

$$t^{-1} r_f(x_a, th_0) \in V_0 \tag{4}$$

for all $a \in A''$, $a \succ a_1$, $|t| \leqq t_2$ and $f \in \mathscr{F}$ where

$$r_f(x, h) = f(x + h) - f(x) - f'(x)\, h\,.$$

Put $t_0 = \min(t_1, t_2)$ and find $a_2 \in A''$ so that $x_a \in x_0 + U_0$ whenever $a \in A''$, $a \succ a_2$. Furthermore, find $a_0 \in A''$ so that $a_0 \succ a_1$, $a_0 \succ a_2$ and both

$$f_a(x_a) - f_a(x_0) \in t_0 V_0 \tag{5}$$

and

$$f_a(x_a + t_0 h_0) - f_a(x_0 + t_0 h_0) \in t_0 V_0 \tag{6}$$

are valid for all $a \in A''$, $a \succ a_0$; it is possible to do so in view of the strong equicontinuity of $\mathscr{F}$ on $x_0 + U_0$.

Using (3), (4), (5) and (6), we can conclude that for every $a \in A''$, $a \succ a_0$, the following formula holds:

$$\begin{aligned}
f_a'(x_a)\, h_0 - f_a'(x_a)\, h_0 = \\
&= t_0^{-1}[f_a(x_a + t_0 h_0) - f_a(x_a) - r_{f_a}(x_a, t_0 h_0)] - \\
&- t_0^{-1}[f_a(x_0 + t_0 h_0) - f_a(x_0) - r_{f_a}(x_0, t_0 h_0)] = \\
&= t_0^{-1}[f_a(x_a + t_0 h_0) - f_a(x_0 + t_0 h_0)] - \\
&- t_0^{1}[f_a(x_a) - f_a(x_0)] - t_0^{-1} r_{f_a}(x_a, t_0 h_0) + \\
&+ t_0^{-1} r_{f_a}(x_0, t_0 h_0) \in V_0 + V_0 + V_0 + V_0 \subset W \cap Y .
\end{aligned} \tag{7}$$

According to the construction of the net $(f_a'(x_a) - f_a'(x_0), A'')$, it is

$$[f_a'(x_a) - f_a'(x_0)]\, h_0 \to z_0(h_0) \quad (a \in A'')$$

in the induced topology of $\hat{Y}$ and hence (7) implies $z_0(h_0) \in \overline{W}$, which contradicts (2).

The theorem is proved.

The following assertion is a direct consequence of the last theorem and two preceding propositions.

Corollary 3. Let $\mathscr{F}$ be a weak-pseudouniformly Gâteaux equidifferentiable at a point $x_0 \in M$, let both $\mathscr{F}$ and $\mathscr{F}'$ be collectively locally precompact in a weak neighbourhood of x_0 in M. Then $\mathscr{F}'$ is strongly equicontinuous at x_0.

Corollary 4. Let both $\mathscr{F}$ and $\mathscr{F}'$ be collectively locally precompact in M. Then $\mathscr{F}'$ is strongly equicontinuous on each convex subset N of M such that $\overline{N} \subset M$ and that $\mathscr{F}$ is weak-pseudouniformly Gâteaux equidifferentiable on it.

In view of the definition of local compactness, this corollary can be reformulated in the form of a necessary and sufficient condition for strong equicontinuity of derivatives as follows:

Corollary 5. Let $\mathscr{F}$ be collectively locally precompact and weak-pseudouniformly Gâteaux equidifferentiable in X (i.e. $M = X$ is assumed). Then $\mathscr{F}'$ is strongly equicontinuous in X if and only if it is collectively locally precompact in X.

Our Theorem 3 asserts that weak-pseudouniform Fréchet equidifferentiability of $\mathscr{F}$ at x_0 – for families $\mathscr{F}$ strongly equicontinuous uniformly on bounded sets, at least – is a sufficient condition for $\mathscr{F}'$ to be strongly equicontinuous at x_0. We shall now show that this property is also necessary.

Theorem 5. Let $\mathscr{F}$ be a family of mappings Gâteaux differentiable in a convex set M and suppose the corresponding family $\mathscr{F}'$ is strongly equicontinuous at $x_0 \in M$. Then $\mathscr{F}$ is weak-pseudouniformly Fréchet equidifferentiable at x_0.

Proof. Suppose to the contrary that there is a family $\mathscr{F}$ which satisfies the assumptions of our Theorem but is not weak-pseudouniformly Fréchet equidifferentiable at x_0; it means a neighbourhood $V \in \mathscr{V}$ of 0 in Y, a bounded set $B \subset X$ and a bounded net (x_c, C) in M weakly convergent to x_0 can be found such that for every $\delta > 0$ and $c \in C$, there exist $x_{c,\delta} \in \{x_b : b \in C,\ b \succ c\}$, $h_{c,\delta} \in B$, $t_{c,\delta} \in (0, \delta)$ and $f_{c,\delta} \in \mathscr{F}$ such that $x_{c,\delta} + t_{c,\delta} h_{c,\delta} \in M$ and

$$r_{c,\delta}(x_{c,\delta}, t_{c,\delta} h_{c,\delta}) \notin t_{c,\delta} V$$

where

$$r_{c,\delta}(x, th) = f_{c,\delta}(x + th) - f_{c,\delta}(x) - t f'_{c,\delta}(x)\, h\,.$$

In other words, considering the natural partial order in the set $A = C \times \mathbf{R}^+$ defined by the relation

$$(c_1, \delta_1) \succ (c_2, \delta_2) \Leftrightarrow c_1 \succ c_2\ \&\ \delta_1 \leqq \delta_2\,,$$

there are bounded nets (x_a, A) in M, (h_a, A) in B, (t_a, A) in $\mathbf{R}^+$ and a net (f_a, A) in $\mathscr{F}$ such that $x_a \rightharpoonup x_0$ $(a \in A)$, $t_a \to 0$ $(a \in A)$ and $x_a + t_a h_a \in M$ and

$$r_a(x_a, t_a h_a) \notin t_a V \tag{8}$$

for each $a \in A$.

On the other hand, according to the well-known mean value theorem (see e.g. [6]), the following inclusion holds for every $a \in A$:

$$\begin{aligned} r_a(x_a, t_a h_a) &= f_a(x_a + t_a h_a) - f_a(x_a) - f'_a(x_a)\, t_a h_a \in \\ &\in \overline{\operatorname{co}}\, \{f'_a(x_a + \tau t_a h_a)\, t_a h_a - f'_a(x_a)\, t_a h_a : \tau \in [0, 1]\} = \\ &= \overline{\operatorname{co}}\, \{[f'_a(x_a + \tau t_a h_a) - f'_a(x_0)]\, t_a h_a : \tau \in [0, 1]\} - \\ &- [f'_a(x_a) - f'_a(x_0)]\, t_a h_a = \\ &= \overline{\operatorname{co}}\, \{g(a, \tau)\, t_a h_a : \tau \in [0, 1]\} - g(a, 0) \end{aligned} \tag{9}$$

where

$$g(a, \tau) = f'_a(x_a + \tau t_a h_a) - f'_a(x_0)$$

Let us consider now the set $D = A \times [0, 1]$ with a partial order on it given by the relation

$$(a_1, \tau_1) \succ (a_2, \tau_2) \Leftrightarrow a_1 \succ a_2\,;$$

the set of all points $u_{(a,\tau)} = x_a + \tau t_a h_a$, $(a, \tau) \in D$ then forms a net, which is bounded and converges weakly to x_0, and hence the corresponding net $(g(a, \tau), (a, \tau) \in D)$

converges to 0 in $\mathscr{L}(X, Y)$ due to the strong equicontinuity of $\mathscr{F}'$ at x_0. Choosing a closed convex balanced neighbourhood W of 0 in X so that $W + W \subset V$, we can therefore find an $a_0 \in A$ such that $g(a, \tau) \in (B, W)$ whenever $(a, \tau) \in D$ and $a \succ a_0$. However, it follows then from (9) that

$$r_a(x_a, t_a h_a) \in \overline{\text{co}}\,(t_a W) - t_a W \subset t_a V,$$

which contradicts (8) and so proves the theorem.

The results of Theorem 5, Theorem 3 and Corollary 3 can be now summarized as follows:

Corollary 6. Let $\mathscr{F}$ be a family of mappings Gâteaux differentiable in a convex set M, let either $\mathscr{F}$ be strongly equicontinuous uniformly on bounded subsets of M or both $\mathscr{F}$ and $\mathscr{F}'$ be collectively locally precompact on a weak neighbourhood of a point $x_0 \in M$. Then the family $\mathscr{F}'$ of derivatives is strongly equicontinuous at $x_0 \in M$ if and only if $\mathscr{F}$ is weak-pseudouniformly Fréchet equidifferentiable at x_0.

Some particular cases

In case the spaces X or Y in assertions of the preceding section are of a special type, we can formulate some of our results in a simplier way. For instance if X is semireflexive, then each strongly continuous mapping from X into Y is automatically uniformly strongly continuous on every bounded subset of X due to the relative weak compactness of bounded sets in such space X. Therefore, the following theorem immediately follows from Corollary 6:

Theorem 6. Let X be semireflexive, let $\mathscr{F}$ be strongly equicontinuous and Gâteaux differentiable in a convex set $M \subset X$. Then $\mathscr{F}'$ is strongly equicontinuous at a point $x_0 \in M$ if and only if $\mathscr{F}$ is weak-pseudouniformly Fréchet equidifferentiable at x_0.

Let us remark that a similar assertion, concerning the case of a single real functional in a reflexive Banach space, appeared in a sligthly different formulation in [9, Lemma 2].

If we restricted ourselves to the case of real functionals on X, i.e. if $Y = \mathbf{R}$, we can formulate several other useful consequences of our theorems. So, by means of Theorem 4, Proposition 1 and the fact that weak and norm convergences in $\mathbf{R}$ coincide, the theorems below can be easily proved:

Theorem 7. Let $X, M, \mathscr{F}$ be as in the preceding section but $Y = \mathbf{R}$, i.e. $\mathscr{F}$ be a family of Gâteaux differentiable real functionals; suppose $\mathscr{F}$ is weak-pseudouniformly Gâteaux equidifferentiable at a point $x_0 \in M$. If $\mathscr{F}'$ is collectively locally precompact in a weak neighbourhood of x_0 in M, then it is strongly equicontinuous at x_0.

Corollary 7. Let $\mathscr{F}$ be a family of Gâteaux differentiable real functionals on $M \subset X$, let the family $\mathscr{F}'$ be collectively locally precompact in M. Then $\mathscr{F}'$ is strongly equicontinuous on every convex set $N, \bar{N} \subset M$, on which $\mathscr{F}$ is weak-pseudouniformly Gâteaux equidifferentiable.

A similar assertion, but with more restrictive assumptions and concerning real functionals in Banach spaces only, was proved — using different arguments — in [9, Theorem 1(i)]; that assertion (as well as the well-known classical results of [8]) follows now immediately from our Corollary 7.

We shall finish with an assertion that combines the results of our Corollary 5 and Corollary 6 in the case of $Y = \mathbf{R}$.

Corollary 8. Let $\mathscr{F}$ be a family of Gâteaux differentiable real functionals on a locally convex space X. Then any two of the following three properties imply the rest one (of course, the property (iii) trivially implies (ii)):

(i) $\mathscr{F}$ is weak-pseudouniformly Gâteaux equidifferentiable in X;

(ii) $\mathscr{F}'$ is collectively locally precompact in X;

(iii) $\mathscr{F}'$ is strongly equicontinuous in X.

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