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On Bumby's Equation $(x^2 - 2)^2 - 2 = 2y^2$: a Solution via Pythagorean triples

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An elementary solution of the Bumby's equation is given.

Je udáno elementární řešení Bumbyovy rovnice.

Дано элементарное решение уравнения Бумбы.

R. J. Stroeker devotes a recent paper [3] to a survey of methods which can be used to solve the equation mentioned in the title. He considers also the possibility of finding an entirely elementary solution but after a while he concludes his analysis with the words "... we soon realize that our line of reasoning most likely leads nowhere." No doubt most independently minded readers will find it tempting to say, "But perhaps there is an elementary approach which leads somewhere."

It is shown below that the equation can be solved by an elementary technique which is the oldest one known in Diophantine analysis (cf. [1]).

Recall first the following two basic facts: (i) If $a \in \mathbb{Z}$, then $a^2 \equiv 1 \pmod 4$ or $a^2 \equiv 0 \pmod 4$. (ii) If $c^2 = a^2 + b^2$ and $a, b \in \mathbb{Z}$ are relatively prime, then there are relatively prime integers p, q such that $\pm c = p^2 + q^2$ and $\{a, b\} = \{2pq, p^2 - q^2\}$. If, moreover, $a, b, c \in \mathbb{N}$, then $p, q \in \mathbb{N}$ (cf. [2]).

Now

$$(x^2 - 2)^2 = 2(y^2 + 1)\,, \tag{1}$$

and, by (i), $y = 2k + 1$, $k \in \mathbb{Z}$. Hence $(x^2 - 2)^2 = 4(2k^2 + 2k + 1)$ and, therefore, $x = 2m$, $m \in \mathbb{Z}$. Accordingly, $(2m^2 - 1)^2 = k^2 + (k + 1)^2$. By (ii), $\pm(2m^2 - 1) = p^2 + q^2$ and $\{k, k + 1\} = \{2pq, p^2 - q^2\}$. If $1 - 2m^2 = p^2 + q^2 \geqq 0$, then $m = 0$ and this yields two solutions (x, y) with $x = 0$ and $y = \pm 1$. Let us further consider the case where $2m^2 - 1 = p^2 + q^2$, $k = 2pq$ and $k + 1 = p^2 - q^2$. Then $2m^2 = p^2 + q^2 + 1 = 2p^2 - 2pq$ and so $m^2 = p(p - q)$. Since p and $p - q$

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are relatively prime, $p = r^2$ and $p - q = s^2$. It follows that $1 = p^2 - q^2 - 2pq = -2(r^2 - s^2)^2 + s^4$. Putting $t = r^2 - s^2$, we get

$$s^4 = 1 + 2t^2 . \tag{2}$$

Now, the equation (2) can be solved by Pythagorean triangles. In fact, it suffices to rewrite (2) as $s^4 + t^4 = (1 + t^2)^2$ and to suppose that $s, t \in \mathbb{N}$. Referring to (2) and (i) we first see that s is odd and that t is even. In view of (ii) we also have $t^2 = 2PQ$, $s^2 = P^2 - Q^2$ and $1 + t^2 = P^2 + Q^2$ where $P \geqq Q$ are relatively prime natural numbers. From $1 = P^2 + Q^2 - 2PQ$ we conclude that $P = Q + 1$. Thus $s^2 = (Q + 1)^2 - Q^2 = 2Q + 1$. From $P^2 - Q^2 = s^2$ and (i) we infer that Q is even, i.e., P and $2Q$ are relatively prime. Suppose $t \neq 0$. Then $t^2 = 2PQ$ implies $2Q = a^2$ where $a \in \mathbb{N}$. However, it is easy to check that two consecutive integers $2Q$ and $2Q + 1$ cannot be squares of natural numbers. Hence, $0 = t = r^2 - s^2$ and $1 = s^2$. This gives the solutions $(-2, -1)$, $(-2, 1)$, $(2, -1)$, $(2, 1)$ of (1).

The remaining case $2m^2 - 1 = p^2 + q^2$, $k = p^2 - q^2$ and $k + 1 = 2pq$ can be treated analogously; details are left to the interested reader.

Consequently, all the solutions of (1) are characterized by $y = \pm 1$.

Note. Another elementary solution of (2) can be found in [2; p. 98].

References

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