Acta Universitatis Carolinae. Mathematica et Physica

Helena Lazosová

A note on reinsurance contracts with multiple reinstatements

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 43 (2002), No. 1, 49--56

Persistent URL: http://dml.cz/dmlcz/142716

A Note on Reinsurance Contracts with Multiple Reinstatements

HELENA LAZOSOVÁ

Praha

Received 12. November 2000

Zabýváme se zajistnými smlouvami s vícenásobnou saturací a saturačním pojistným, které se stanovuje podle podílu plnění a zbylé doby krytí. Jsou prezentovány vzorce netto a brutto zajistného. Preference pojišťovny jsou analyzovány pomocí principu směrodatné odchylky.

Reinsurance contracts with multiple reinstatements and the reinstatement premium proportional to the payments obtained and to the residual time of coverage are considered. Formulae for the netto and risk loaded reinsurance premiums are presented. The preferences of the ceding insurer are analysed using the standard deviation principle.

1. Introduction

Reinsurance contracts with multiple reinstatements for which the reinstatement premium is calculated pro rata capti et temporis, i.e. in proportion to the payments obtained and to the residual time of coverage, are employed in catastrophic XL-reinsurance. They are less frequent in working – layer cover ([1], [5]).

Nevertheless the reinsurance pricing according to this principle is evidently more adequate and suitable to serve as a standard for similar reinsurance arrangements. It is the purpose of this note to show that its mathematical analysis can be performed for the compound Poisson model by means of explicit formulae leading to tables having some value for the risk theory.

Ceska pojistovna, joint-stock company, Spálená 16, 130 10 Praha 1, Czech Republic

Keywords: Reinsurance; Cover reinstatement; Compound Poisson process; Ordering of risks

2. Netto reinsurance premium

We consider the loss excedents in a reinsurance layer of width L in time interval $[0, 1]$.

The loss excedents are assumed to form a compound Poisson process of intensity λ. We denote the occurrence times of the claims σ_k; $k = 1, 2, \ldots$.

The standardised amounts of the corresponding excedents Y_k; $k = 1, 2, \ldots$, are the actual amounts divided by L.

Given that there are r claims in the layer during $[0, 1]$ then their occurrence times represent an ordered sample from the uniform distribution. The probability density of the time of the k-th claim equals

$$r\binom{r-1}{k-1} t^{k-1}(1-t)^{r-k}. \tag{1}$$

We assume that the reinsurance contract admits at most n reinstatements and the insurer is obliged to perform the reinstatements up to this number. Moreover,

reinstatement premium $= \pi \cdot$ cover burnt $\cdot$ residual time,

where π is the netto premium for the layer.

The expected total amount paid by the cedent to the reinsurer equals

$$\pi\left(1 + E\sum_{k=1}^{N\wedge n} Y_k(1-\sigma_k)\right), \tag{2}$$

where N denotes the total number of claims in the layer. According to the assumption N has Poisson distribution. Let $EN = \lambda$.

(2) can be written as

$$\pi\left(1 + \sum_{r=1}^{\infty}\frac{\lambda^r}{r!}e^{-\lambda}E\left\{\sum_{k=1}^{N\wedge n} Y_k(1-\sigma_k)\,|\,N = r\right\}\right),$$

and further using (1) as

$$\pi\left(1 + \sum_{r=1}^{\infty}\frac{\lambda^r}{r!}e^{-\lambda}\sum_{k=1}^{r\wedge n} EY\left(1 - \frac{k}{r+1}\right)\right).$$

Finally, this can be adjusted as follows

$$\pi\left(1 + EY\left(\frac{\lambda}{2}P(n-2) + n(1 - P(n-1)) - \frac{n(n+1)}{2\lambda}(1 - P(n))\right)\right), \tag{3}$$

where

$$P(n) = \sum_{j=0}^{n}\frac{\lambda^j}{j!}e^{-\lambda}. \tag{4}$$

For the expected amount of the reinsurance company settlements one obtains

$$EY\left(\sum_{j=1}^{n} j\frac{\lambda^j}{j!}e^{-\lambda} + (n+1)\sum_{j=n+1}^{\infty}\frac{\lambda^j}{j!}e^{-\lambda}\right) = EY(\lambda P(n-1) + (n+1)(1-P(n))).$$

The netto reinsurance premium will be determined equating the expected amount of cedent's payments to the expected amount of reinsurance company settlements

$$\pi E\left(1 + \sum_{k=1}^{N\wedge n} Y_k(1-\sigma_k)\right) = E\sum_{k=1}^{N\wedge(n+1)} Y_k. \tag{5}$$

It results

$$\pi = \frac{\lambda P(n-1) + (n+1)(1-P(n))}{\frac{1}{EY} + \frac{\lambda}{2}P(n-2) + n(1-P(n-1)) - \frac{n(n+1)}{2\lambda}(1-P(n))}. \tag{6}$$

3. An estimation method

Often the distribution in a layer is modelled by the Truncated Pareto distribution, i.e.

$$P(Y > x) = \left(\frac{d}{d+x}\right)^q, \qquad 0 \le x < 1. \tag{7}$$

The following relation involving the median $x_{0,5}$ and the third quartile $x_{0,75}$ may be of interest when estimating the parameters q, d.

Namely from (7) we obtain

$$\frac{x_{0,5}}{x_{0,75}} = \frac{(0,5)^{-1/q} - 1}{(0,25)^{-1/q} - 1},$$

which yields

$$0,5^{-1/q} = \frac{x_{0,75}}{x_{0,5}} - 1.$$

From

$$d = (0,5)^{1/q}(d + x_{0,5})$$

it follows $d = x_{0,5}^2/(x_{0,75} - 2x_{0,5})$.

Remark

In case of variable intensity λ_s of the claim occurrence times one has to replace in the formulae $1 - \sigma_k$ by $\int_{\sigma_k}^{1} \lambda_s \, ds$.

4. Risk loaded reinsurance premium

To determine the reinsurance premium we use the standard deviation principle throughout this part, i.e. for a risk X we define

$$\Pi^{risk} = EX + \beta\sqrt{\operatorname{var} X}, \tag{8}$$

where β is a weight constant.

The principle will be somewhat modified since the amount paid by the cedent is random and depends in the loss experience.

For the sake of simplicity we also assume that both the insurer and the reinsurer have the same expense loading C.

The brutto reinsurance premium therefore equals

$$\Pi^{brutto} = \frac{\Pi^{risk}}{1 - C}.$$

We also set $L = 1$.

With regard to (5) we denote

$$\xi_i = 1 + \sum_{k=1}^{N \wedge n} Y_k(1 - \sigma_k), \qquad \eta_r = \sum_{k=1}^{N \wedge (n+1)} Y_k.$$

Hence, $\pi E\xi_i = E\eta_r$.

The risk loaded reinsurance premium will depend on the standard deviation of the balance of the reinsurer. With regard to (2) it is determined by the equation

$$\Pi^{risk} = \pi + \beta\sqrt{\operatorname{var}(\pi\xi_i - \eta_r)}/E\xi_i. \tag{9}$$

We have to calculate

$$\operatorname{var}(\pi\xi_i - \eta_r).$$

The following formula is obtained

$$\pi^2 P(n) + \lambda\left(2EY\pi\left[\frac{\pi}{2} - 1\right] + E(Y^2)\frac{\pi^2}{12} + \operatorname{var} Y\left(\frac{\pi}{2} - 1\right)^2 + (EY)^2\left[\frac{\pi}{2} - 1\right]^2\right)$$

$$P(n-1) + \lambda^2(EY)^2\left[\frac{\pi}{2} - 1\right]^2 P(n-2) + \left(\operatorname{var} Y + (n \operatorname{var} Y + n^2(EY)^2)(1-\pi)^2 + \right.$$
$$+ 2(\pi - EY)nEY(\pi - 1) + (\pi - EY)^2\big)(1 - P(n)) + ((n \operatorname{var} Y + n^2(EY)^2)\pi(1-\pi) -$$
$$- (\pi - EY)\, nEY\pi)(n+1)\frac{1}{\lambda}(1 - P(n+1)) + \left(n\pi^2 E(Y^2)\frac{1}{12} + (n \operatorname{var} Y + \right.$$
$$\left. + n^2(EY)^2)\frac{\pi^2}{4}\right)(n+1)(n+2)\frac{1}{\lambda^2}(1 - P(n+2)).$$

5. Insurer's strategy

Given the reinsurance pricing as described in Section 2 the insurer selects the contract according to the preferences given as follows. He prefers risk Z_1 to risk Z_2, if it holds

$$EZ_1 + \gamma\sqrt{\operatorname{var} Z_1} \le EZ_2 + \gamma\sqrt{\operatorname{var} Z_2}$$

where γ is a weight constant.

The total payment Z of the insurer has the form

$$Z - \Pi^{risk}\left(1 + \sum_{k=1}^{N\wedge n} Y_k(1-\sigma_k)\right) + \sum_{k=n+2}^{N} Y_k. \tag{10}$$

Denoting $\eta_i = \sum_{k=n+2}^{N} Y_k$,

we can write $Z = \Pi^{risk}\xi_i + \eta_i$.

The following formulae are obtained

$$EZ = \Pi^{risk}\left(1 + EY\left(\frac{\lambda}{2}P(n-2) + n(1-P(n-1)) - \frac{n(n+1)}{2\lambda}(1-P(n))\right)\right) +$$
$$+ EY[\lambda(1-P(n)) - (n+1)(1-P(n+1))],$$
$$\operatorname{var} Z = (\Pi^{risk})^2 \operatorname{var}\xi_i + 2\Pi^{risk}\operatorname{cov}(\xi_i,\eta_i) + \operatorname{var}\eta_i,$$

where

$$\operatorname{var}\xi_i = P(n) + \lambda\left(EY + \frac{1}{3}\operatorname{var} Y + \frac{1}{3}(EY)^2\right)P(n-1) + \lambda^2(EY)^2\frac{1}{4}P(n-2) +$$
$$+ (n\operatorname{var} Y + n^2(EY)^2 + 2nEY + 1)(1-P(n)) - (n+1)\frac{1}{\lambda}(1-P(n+1))(n^2(EY)^2 + nEY +$$
$$+ n\operatorname{var} Y) + \frac{1}{\lambda^2}(n+1)(n+2)\left(1-P(n+2)\right)\left(nE(Y^2)\frac{1}{12} + (n\operatorname{var} Y + n^2(EY)^2)\frac{1}{4}\right) -$$
$$-\left(1 + EY\frac{\lambda}{2}P(n-2) + n(1-P(n-1)) - \frac{n(n+1)}{2\lambda}(1-P(n))\right)^2$$

$$\operatorname{cov}(\xi_i,\eta_i) = -(1-P(n+1))[(1+nEY)(n+1)EY + (EY)^2 n(n+1)/2] +$$
$$+ \lambda(1-P(n))(1+nEY)EY + \frac{1}{\lambda}(1-P(n+2))(n+2)(EY)^2\frac{n(n+1)}{2} -$$
$$-\left(1 + EY\left[\frac{\lambda}{2}P(n-2) + n(1-P(n-1)) - \frac{n(n+1)}{2\lambda}(1-P(n))\right]\right)(EY[\lambda(1-P(n)) -$$
$$-(n+1)(1-P(n+1))])$$

$$\operatorname{var}\eta_i =$$
$$= [\lambda(1-P(n)) - (n+1)(1-P(n+1))]\operatorname{var} Y + (EY)^2[\lambda^2(1-P(n-1)) - (2n+1)$$
$$\lambda(1-P(n)) + (n+1)^2(1-P(n+1)) - (\lambda(1-P(n)) - (n+1)(1-P(n+1)))^2].$$

6. Numerical illustration

For illustration we consider the choice of the insurer between the contract without reinstatement and with one reinstatement. In Tables 1, 2 we present the

Table 1. **Netto reinsurance premium**

$n = 1$

λ/EY	0,1	0,2	0,3	0,4	0,5
0,1	0,0099	0,0198	0,0295	0,0392	0,0487
0,5	0,0474	0,0928	0,1364	0,1783	0,2186
1	0,0865	0,1670	0,2422	0,3126	0,3786
1,5	0,1163	0,2224	0,3195	0,4088	0,4911
2	0,1380	0,2620	0,3739	0,4755	0,5681

Table 2. **Risk loaded reinsurance premium**

var $Y = 0,35$ $\beta = 0,05$
$n = 1$

λ/EY	0,1	0,2	0,3	0,4	0,5
0,1	0,0193	0,0295	0,0397	0,0500	0,0604
0,5	0,0673	0,1127	0,1566	0,1991	0,2402
1	0,1128	0,1923	0,2670	0,3373	0,4034
1,5	0,1463	0,2504	0,3463	0,4348	0,5167
2	0,1702	0,2916	0,4016	0,5018	0,5934

Table 3. **Insurer's strategy**

var $Y = 0,35$ $\beta = 0,05$
$n = 0$ $\gamma = 0,4$

λ/EY	0,1	0,2	0,3	0,4	0,5
0,1	0,0359	0,0469	0,0585	0,0705	0,0829
0,5	0,1472	0,2009	0,2564	0,3129	0,3698
1	0,2696	0,3764	0,4860	0,5969	0,7079
1,5	0,3809	0,5405	0,7040	0,8692	1,0343
2	0,4842	0,6963	0,9133	1,1326	1,3521

Table 4.

var $Y = 0,35$ $\beta = 0,05$
$n = 1$ $\gamma = 0,4$

λ/EY	0,1	0,2	0,3	0,4	0,5
0,1	0,0226	0,0332	0,0442	0,0555	0,0671
0,5	0,1018	0,1549	0,2100	0,2668	0,3251
1	0,2063	0,3135	0,4249	0,5398	0,6576
1,5	0,3115	0,4727	0,6407	0,8137	0,9907
2	0,4144	0,6295	0,8533	1,0838	1,3191

netto reinsurance premium and the risk loaded premium for a contract allowing one reinstatement for a set parameter values. Tables 3, 4 contain the insurer's evaluation of (10) by means of

$$EZ + \gamma\sqrt{\operatorname{var} Z}.$$

It results that the insurer will prefer the contract with one reinstatement.

7. An unlimited number of saturations

Pro rata capti et temporis

The netto reinsurance premium, which is determined by equating the expected amount of the cedent's payments to the expected amount of reinsurance company settlements or by letting $n \to \infty$ in (6), equals

$$\pi = \frac{\lambda EY}{1 + EY\dfrac{\lambda}{2}}.$$

The modified variance is calculated using the properties of the compound Poisson distribution as

$$\operatorname{var}(\pi\xi_p - \eta_z) = \operatorname{var}\left(\pi + \sum_{k=1}^{N} Y_k(\pi(1 - \sigma_k) - 1)\right) =$$

$$= \operatorname{var}\left(\sum_{k=1}^{N} Y_k(\pi(1 - \sigma_k) - 1)\right) = \lambda E((\pi\sigma' - 1)^2 Y^2) = \lambda E(\pi\sigma' - 1)^2 EY^2,$$

where σ' has uniform distribution in the interval $[0, 1]$.
Hence,

$$\operatorname{var}(\pi\xi_p - \eta_z) = \lambda\left(\frac{\pi^2}{12} + \left(\frac{\pi}{2} - 1\right)^2\right)(\operatorname{var} Y + (EY)^2).$$

To determine the reinsurance premium we use the standard deviation principle (see (9)). We define

$$\Pi^{risk} = \pi + \frac{\beta}{1 + EY\dfrac{\lambda}{2}}\sqrt{\lambda\left(\frac{\pi^2}{12} + \left(\frac{\pi}{2} - 1\right)^2\right)(\operatorname{var} Y + (EY)^2)},$$

where β is a weight constant.

The insurer's risk Z is

$$Z = \xi_p\Pi^{risk}.$$

For the variance of risk Z it holds

$$(\Pi^{risk})^2 \operatorname{var} \xi_p = \operatorname{var}\left(\Pi^{risk} + \sum_{k=1}^{N} Y_k \Pi^{risk}(1 - \sigma_k)\right) =$$

$$= \lambda\left(\frac{(\Pi^{risk})^2}{12} + \left(\frac{\Pi^{risk}}{2}\right)^2\right)(\operatorname{var} Y + (EY)^2) = \lambda \frac{(\Pi^{risk})^2}{3}(\operatorname{var} Y + (EY)^2).$$

The formula for evaluating risk Z by the standard deviation principle is

$$\Pi^{risk}\left(\left(1 + EY\frac{\lambda}{2}\right) + \gamma\sqrt{\lambda\frac{(\operatorname{var} Y + (EY)^2)}{3}}\right),$$

where γ is a weight constant.

References

[1] *General Insurance*, Actuarial Education Service, Oxford (1995).

[2] *Foundations of Casualty Actuarial Science*, Casualty Actuarial Society. New York (1990).

[3] LAZOSOVÁ H., *Pricing of non proportional reinsurance* (in Czech). M.s.c. thesis, Charles University. Prague (1999).

[4] MACK T., *Schadenversicherungsmathematik*. Verlag Versicherungswirtschaft, Karlsruhe (1997).

[5] PFEIFFER C., *Einführung in die Rückversicherung*. Gabler, Wiesbaden (1994).