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A Note on Some Separable Location Problems – A Multicriterial Approach

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A multicriterial approach to solving so called separable location problem with n service centres and m customers is investigated. The problem consists in locating each of the n service centres T_j, on exactly one road connecting two places A_j, B_j $(j = 1, \ldots, n)$. The centres are supposed to serve m customers C_i, $i = 1, \ldots, m$. The distances $\varrho(A_j, C_i)$, $\varrho(B_j, C_i)$, $\varrho(A_j, B_j)$ are given. The problems are called separable, because they split up in n one-dimensional optimization problems. The multicriterial approach takes into account three objective functions. Suggestions for further research are briefly discussed.

1. Introduction

In this article, we consider a location problem with n service centres T_j, $j = 1, \ldots, n$ and m customers C_i, $i = 1, \ldots, m$, which are served from centres T_j. Centre T_j must be placed on a road connecting two places A_j, B_j with known distances from C_i. The distance between A_j and B_j is also known. The position of T_j on the road A_jB_j is uniquely given by the distance x_j of T_j from A_j. If C_i is served from T_j, then customers C_i can be reached from T_j either via A_j or via B_j and we assume that the shortest route out of $T_jA_jC_i$ and $T_jB_jC_i$ can always be chosen (see Fig. 1). The aim is to determine the locations of T_j on roads A_jB_j (i.e. the distances x_j of T_j from A_j) in such a way that a reasonable balance (or compromise) among three criteria will be found. The framework of the location problem considered in

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this article is similar to that used in one-criterion problems considered in [1], [2], [3], [5], [6]. We chose also a different approach than that one described for a bi-criteria problem in [4].

2. Problem Formulation

Let $\varrho(X, Y)$ denote the distance between two points in a plane and let us introduce the following notations for all $i \in S := \{1, \ldots, m\}$, $j \in N := \{1, \ldots, n\}$: $d_j := \varrho(A_j, B_j)$, $a_{ij} := \varrho(C_i, A_j)$, $b_{ij} := d_j + \varrho(C_i, B_j)$. If $x_j = \varrho(A_j, T_j)$, then the length of the route $T_jA_jC_i$ is equal to $x_j + a_{ij}$ and the length of $T_jB_jC_i$ is equal to $d_j - x_j + \varrho(C_i, B_j) = b_{ij} - x_j$ (see Fig. 1). Therefore if a location $x_j \in [0, d_j]$ of T_j on A_jB_j is chosen, then the distance to be covered in order that C_i may be reached from T_j is given by function $r_{ij}(x_j) := \min(a_{ij} + x_j, b_{ij} - x_j)$ (i.e. we assume that the shorter of the two possible routes is chosen).

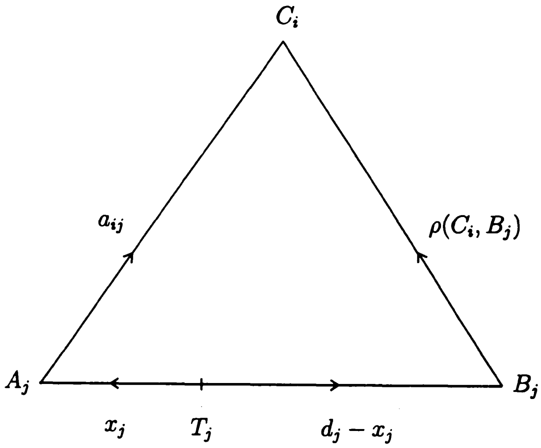

Figure 1

We shall consider three objective functions evaluating the performance quality of the system of service centres T_j with the location given by $\varrho(A_j, T_j) = x_j$ $x_j \in [0, d_j]$ for $j \in N$. The objective functions will be defined as follows:

$$f(\mathbf{x}) = \max_{j \in N} \max_{i \in S} r_{ij}(x_j) = \max_{j \in N} u_j(x_j), \tag{2.1}$$

$$g(\mathbf{x}) = \min_{j \in N} \min_{i \in S} r_{ij}(x_j) = \min_{j \in N} l_j(x_j), \tag{2.2}$$

$$h(\mathbf{x}) = \max_{j \in N} (u_j(x_j) - l_j(x_j)). \tag{2.3}$$

The first function (2.1) can be interpreted as a "pessimistic" performance evaluation, its value gives for each $j \in N$ the greatest distance between T_j and C_i over all customers C_i, $i \in S$. The second function (2.2) can be interpreted as a "optimistic" evaluation and its value gives for each j the smallest distance between T_j and C_i over all C_i, $i \in S$. In the ideal case centre T_j will serve the closest customer at a distance $l_j(x_j)$ but it may happen that T_j must serve the farthest customer at a distance $u_j(x_j)$, because e.g. all other centres, which may be closer to this customer are occupied. In such situation, it may be reasonable to require that $u_j(x_j) - l_j(x_j) \leq \lambda$, for a given λ. A similar situation arises if service centre T_j is "obnoxious" to some extent and it may be desirable that $l_j(x_j) \geq \beta$ for a given positiv β. On the other hand, we may require for a given positive α that $u_j(x_j) \leq \alpha$, which together with $l_j(x_j) \geq \beta$ gives again a restriction $p_j(x_j) = u_j(x_j) - l_j(x_j) \leq \lambda = \alpha - \beta$. That's why we included the objective function $h(\mathrm{x})$. We shall first investigate the behaviour of functions $u_j(x_j)$, $l_j(x_j)$ and $p_j(x_j)$ on $[0, d_j]$. Using these results several optimization problems will be solved. The optimal solutions will represent various types of compromises among the three criteria represented by objective functions f, g, p. Hints for further research will be briefly discussed.

In the next paragraph we shall investigate the properties of $p_j(x_j)$ for a fixed $j \in N$. These properties will make possible to solve easily some optimization problems, the optimal solutions of which represent a compromise among the three objective functions f, g, h.

3. The Properties of $p_j(x_j)$

We shall investigate the properties of function

$$p_j(x_j) := u_j(x_j) - l_j(x_j) \tag{3.1}$$

on the interval $[0, d_j]$. We shall assume in the sequel that j is an arbitrary fixed index from N.

It holds:

$$l_j(x_j) = \begin{cases} v_j + x_j & \text{for} \quad x_j \in \left[0, \min\left(d_j, \dfrac{w_j - v_j}{2}\right)\right] \\ w_j - x_j & \text{for} \quad x_j \in \left[\max\left(0, \dfrac{w_j - v_j}{2}\right), d_j\right] \end{cases}$$

where

$$v_j := \min_{i \in S} a_{ij}, \qquad w_j := \min_{i \in S} b_{ij}$$

Further, it may happen that for some $i \in S$ the inequality $r_{ij}(x_j) < u_j(x_j)$ holds for all $x_j \in [0, d_j]$. Such functions are "redundant" for the definition of $u_j(x_j)$ and will be excluded from further consideration by the following

Algorithm "reduction"

[1] $S_1 := \emptyset;\ S := \{1, \dots, m\}$

[2] $T := \{k \mid a_{kj} = \max_{i \in S} a_{ij}\};\ y_p := \max_{k \in T} \frac{1}{2}(a_{kj} + b_{kj})$

[3] $S_2 := \{i \in S \mid a_{ij} \leq a_{pj}\ \&\ b_{ij} \leq b_{pj}\}$

[4] $S_1 := S_1 \cup \{p\}, S := S \setminus S_2$

[5] If $S \neq \emptyset$, go to [2]

[6] S_1 is the reduced set of indices i

The set S_1 in step [6] contains only such i that $u_j(x_j) = r_{ij}(x_j)$ for some subset of $[0, d_j]$ (so called non-redundant r_{ij}'s).

In Example 5.1 below functions r_{5j}, r_{6j} are redundant (Fig. 2). The determination of the piecewise linear function $u_j(x_j)$ needs to determine all local minima and maxima of this function on $[0, d_j]$. This will be done by making use of the following.

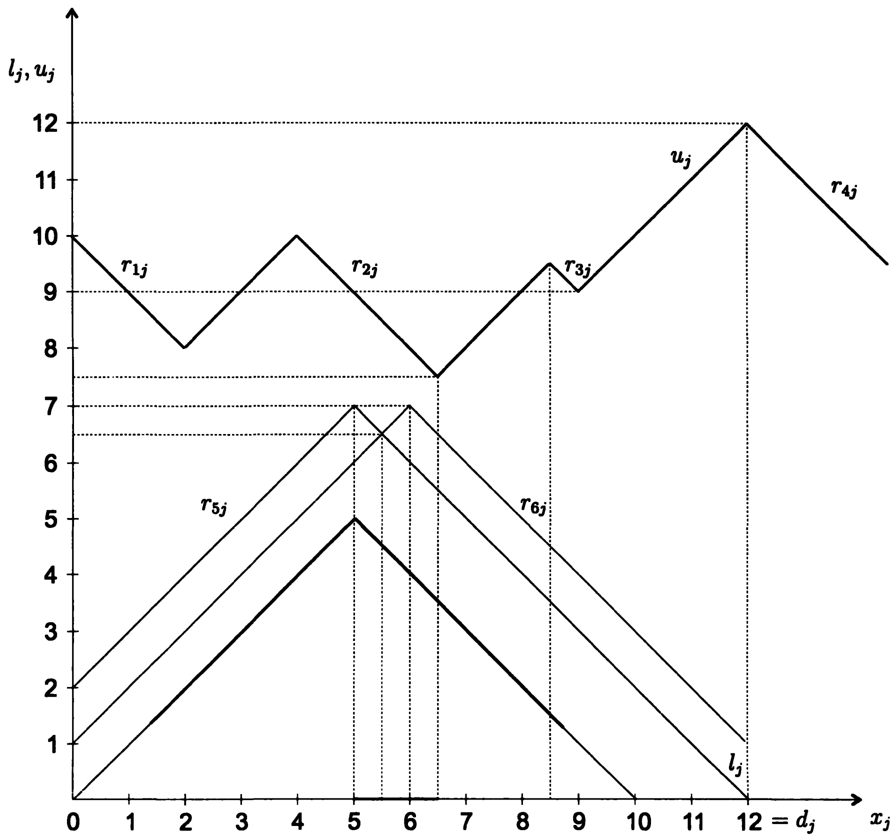

Figure 2

Algorithm – "local extrema"

1. Compute local maxima $\bar{x}_{ij} := \frac{b_{ij}-a_{ij}}{2}$ for all $i \in S_1$ and consider only $\bar{x}_{ij} \in [0, d_j]$ and $\bar{x}_{hj} := \max\{\bar{x}_{ij} \mid \bar{x}_{ij} < 0\}$ (if $\{\bar{x}_{ij} \mid \bar{x}_{ij} < 0\} \neq \emptyset$) $\bar{x}_{ij} := \min\{\bar{x}_{ij} \mid \bar{x}_{ij} > d_j\}$ (if $\{\bar{x}_{ij} \mid \bar{x}_{ij} > d_j\} \neq \emptyset$). Let $\{i_1, \dots, i_r\}$ be such sequence of indices. We shall assume w.l.o.g. that $i_j = j$ for j, $1 \leq j \leq r$ so that it holds $\bar{x}_{1j} < \bar{x}_{2j} < \dots < \bar{x}_{rj}$ with $r \leq m$.
2. If $\bar{x}_{1j} < 0$ and $b_{ij} > 0$, set $\bar{x}_{1j} := 0$, otherwise set $\underline{x}_{1j} = 0$;
   If $\bar{x}_{rj} > 0$ and $a_{rj} > -d$, set $\bar{x}_{rj} := d_j$, otherwise set $\underline{x}_{r+1j} = d_j$.
3. Compute local minima $\underline{x}_{t-1j} := \frac{b_{i-1j}-a_{ij}}{2}$ for $t = 2, \dots, r+1$.

Exactly one of the following cases occurs:

(3.2) $$\bar{x}_{1j} = 0 < \underline{x}_{1j} < \bar{x}_{2j} < \dots < \underline{x}_{r-1j} < \bar{x}_{rj} = d_j$$

(3.3) $$\underline{x}_{1j} = 0 < \bar{x}_{1j} < \underline{x}_{2j} < \dots < \underline{x}_{r-1j} < \bar{x}_{rj} = d_j$$

(3.4) $$\bar{x}_{1j} = 0 < \underline{x}_{1j} < \bar{x}_{2j} < \dots < \underline{x}_{r-1j} < \bar{x}_{rj} = d_j$$

(3.5) $$\underline{x}_{1j} = 0 < \bar{x}_{1j} < \underline{x}_{2j} < \dots < \bar{x}_{rj} < \underline{x}_{r-1j} = d_j$$

(compare Example 5.1, Fig. 2, where $r = 4$ and case (3.2) occurs).

It follows immediately from the properties of $u_j(x_j)$ and $l_j(x_j)$ that $p_j(x_j) := u_j(x_j) - l_j(x_j)$ is a piecewise linear quasiconvex function, the first linear piece of which is decreasing in case (3.2) a (3.4) and constant in cases (3.3) and (3.5). The minimum of $p_j(x_j)$ is attained either on $[\hat{x}, \underline{x}_{tj}]$ where $\hat{x}_j = \frac{1}{2}(w_j - v_j)$ and $\underline{x}_{tj} = \min\{\underline{x}_{tj} \mid \underline{x}_{tj} \geq \hat{x}_j\}$ and $\hat{x}_j \geq \bar{x}_{l-1j}$, or on the whole interval $[\underline{x}_{tj}, \hat{x}_j]$, where $\underline{x}_{tj} = \max\{\underline{x}_{tj} \mid \underline{x}_{ij} \leq \hat{x}_j\}$ and $\hat{x}_j \leq \bar{x}_{lj}$. In Example 5.1 $l = 2$ and $p_j(x_j) = 4$ on $[\hat{x}_j, \underline{x}_{2j}] = [5, 6.5]$. If $\hat{x}_j \leq 0$, then $p_j(x_j)$ is nondecreasing on $[0, d_j]$ and if $\hat{x}_j \geq d_j$, function $p_j(x_j)$ is nonincreasing on $[0, d_j]$.

Let us denote $\lambda_j := \min\{p_j(x_j) \mid x_j \in [0, d_j]\}$ and $F_j(\lambda) := \{x_j \in [0, d_j] \mid p_j(x_j) \leq \lambda\}$. Then $F_j(\lambda)$ is closed subinterval of $[0, d_j]$ and $F_j(\lambda) = \emptyset$, if $\lambda < \lambda_j$.

Let $\alpha_j := \min\{u_j(x_j) \mid x_j \in F_j(\lambda)\}$ and $\beta_j := \max\{l_j(x_j) \mid x_j \in F_j(\lambda)\}$. Then $u_j(x_j) - l_j(x_j) \leq \alpha - \beta$ can be satisfied on $[0, d_j]$ if $\alpha - \beta \geq \lambda_j$, $\alpha \geq \alpha_j$ and $\beta \geq \beta_j$. The determination of α_j, β_j is a purely technical problem and is not described in detail here. The same holds for the determining of subintervals $F_j(\lambda)$ (compare Tab. 5 in Example 5.1).

4. Some Optimization Problems

Let us define for each $j \in N$ the value $x_j^{\text{opt}} \in F_j(\lambda_j)$ as follows: $x_j^{\text{opt}} = \underline{x}_{lj}$, where the index l, $\hat{x}_j$ and λ_j are defined as in the preceding paragraph. It holds then

(4.1) $$u_j(x_j^{\text{opt}}) = \min\{u_j(x_j) \mid x_j \in F_j(\lambda_j)\}.$$

Similarly if we set $\tilde{x}_j^{\text{opt}} = \hat{x}_j \; \forall j \in N$ then it holds:

$$(4.2) \qquad l_j(\tilde{x}_j^{\text{opt}}) = \max \{ l_j(x_j) \mid x_j \in F_j(\lambda_j) \}$$

(compare Example 5.1, Fig. 2, where $\lambda_j = 4$, $l = 2$, $x_j^{\text{opt}} = 5$, $\tilde{x}_j^{\text{opt}} = 6.5$).

Let us consider the following optimization problems:

$$\left.\begin{array}{ll} \text{minimize} & f(\mathrm{x}) \\ \text{s.t.} & x_j \in F_j(\lambda_j) \; \forall j \in N \end{array}\right\} \text{(P1)}$$

$$\left.\begin{array}{ll} \text{minimize} & f(\mathrm{x}) \\ \text{s.t.} & x_j \in F_j(\lambda) \; \forall j \in N \end{array}\right\} \text{(P2)}$$

$$\left.\begin{array}{ll} \text{minimize} & g(\mathrm{x}) \\ \text{s.t.} & x_j \in F_j(\lambda_j) \; \forall j \in N \end{array}\right\} \text{(P3)}$$

$$\left.\begin{array}{ll} \text{minimize} & h(\mathrm{x}) \\ \text{s.t.} & f(\mathrm{x}) \le \alpha \\ & 0 \le x_j \le d_j \; \forall j \in N \end{array}\right\} \text{(P4)}$$

It follows immediately that $\mathrm{x}^{\text{opt}} = (x_1^{\text{opt}}, \ldots, x_n^{\text{opt}})$ is the optimal solution of (P1), $\tilde{\mathrm{x}}^{\text{opt}} = (\tilde{x}_1^{\text{opt}}, \ldots, \tilde{x}_n^{\text{opt}})$ is the optimal solution of (P3). Optimal solution of (P2) can be found via the minimization of $u_j(x_j)$ on intervals $F_j(\lambda) \; \forall j \in N$ (note that the set of feasible solutions of (P2) is nonempty only for $\lambda \ge \max_{j \in N} \lambda_j$). The set of feasible solutions of (P4) is the Cartesian product of L_j, $j \in N$, where $L_j \subseteq [0, d_j]$, and each subset L_j is a union of at most m closed intervals. It remains therefore for each j to minimize function $p_j(x_j)$ on the closed intervals, the union of which is equal to L_j and then choose the minimal value of these minima.

Remark 4.1 *It is obvious that in this way we can formulate further easily solvable optimization problems, the optimal solutions of which are other types of compromise solutions among the three objective functions* $f(\mathrm{x})$, $g(\mathrm{x})$, $h(\mathrm{x})$ *(e.g. maximization of* $g(\mathrm{x})$ *s.t.* $f(\mathrm{x}) \le \alpha$ *and* $h(\mathrm{x}) \le \lambda$ *and so on).*

Remark 4.2 *The case, in which the weighting coefficients expressing the importance of customers are included can be a subject of further research. In such case, instead of functions* r_{ij}, *functions* $\tilde{r}_{ij}(x_j) = \min(a_{ij} + w_{ij}x_j, b_{ij} - w_{ij}x_j)$ *would be used.*

Remark 4.3 *Another subject of further research could be the usage of the results described above for finding approximate solutions for nonseparable location problems with the objective function* $s(\mathrm{x}) := \max_{i \in S} \min_{j \in N} r_{ij}(x_j)$, *which are in general NP-hard ([3]).*

Remark 4.4 *The inclusion of stochastics can be another direction for further investigation. We could study e.g. the case in which for each pair* (C_i, T_j)

there a probability $p_{ij} \in (0, 1)$ is given, that customer C_i will be accepted by centre T_j.

Remark 4.5 *The procedures suggested above can be adjusted to discrete problems, in which only finite subsets of positions in the connecting roads A_jB_j are available for locating centres T_j.*

5. Numerical Example

Example 5.1

We shall assume that $j \in N$ is an arbitrary fixed index, $m = 6$, $d_j = 12$, $r_{ij}(x_j) = \min(a_{ij} + x_j, b_{ij} - x_j)$, where $b_{ij} = b'_{ij} + d_j$, $u_j(x_j) = \max_{1 \le i \le 6} r_{ij}(x_j)$, $l_j(x_j) = \min_{1 \le i \le 6} r_{ij}(x_j)$, a_{ij}, b_{ij} are given in Tab. 1

i	1	2	3	4	5	6
a_{ij}	14	6	1	0	2	1
b_{ij}	10	14	18	24	12	13

Table 1

The graphs of $u_j(x_j)$, $l_j(x_j)$ are presented in Fig. 2. The explicit expression of $u_j(x_j)$ and $l_j(x_j)$ is given in Tab. 2 (functions $r_{5j}(x_j)$, $r_{6j}(x_j)$ are redundant for the definition of $u_j(x_j)$, so that S_1 obtained from the algorithm "reduction" is equal to $\{1, 2, 3, 4\}$ with $r = 4$.

$x_j \in$	$[0, 2)$	$[2, 4)$	$[4, 6.5)$	$[6.5, 8.5)$	$[8.5, 9.5)$	$[9.5, 12]$
$u_j(x_j)$	$10 - x_j$	$6 + x_j$	$14 - x_j$	$1 + x_j$	$18 - x_j$	x_j
$x_j \in$	$[0, 5]$			$[5, 12]$		
$l_j(x_j)$	x_j			$10 - x_j$		

Table 2

$$\max_{0 \le x_j \le 12} l_j(x_j) = l_j(\hat{x}_j) = l_j(5) = 5$$

Local maxima $\underline{x}_{ij}$, $\bar{x}_{ij}$ of $u_j(x_j)$ are given in Tab. 3

i	1	2	3	4
$\bar{x}_{ij}$	0	4	8.5	12
$\underline{x}_{ij}$	2	6.5	9	–

Table 3

We see that in this case $\bar{x}_{ij} \leq \bar{x}_{i+1j}$ for $i = 1, \ldots, 3$ so that no reordering of indices $i \in S_1$ is necessary. The explicit expression of function $p_j(x_j) = u_j(x_j) - l_j(x_j)$ is given in Tab. 4. Its graph is presented in Fig. 3.

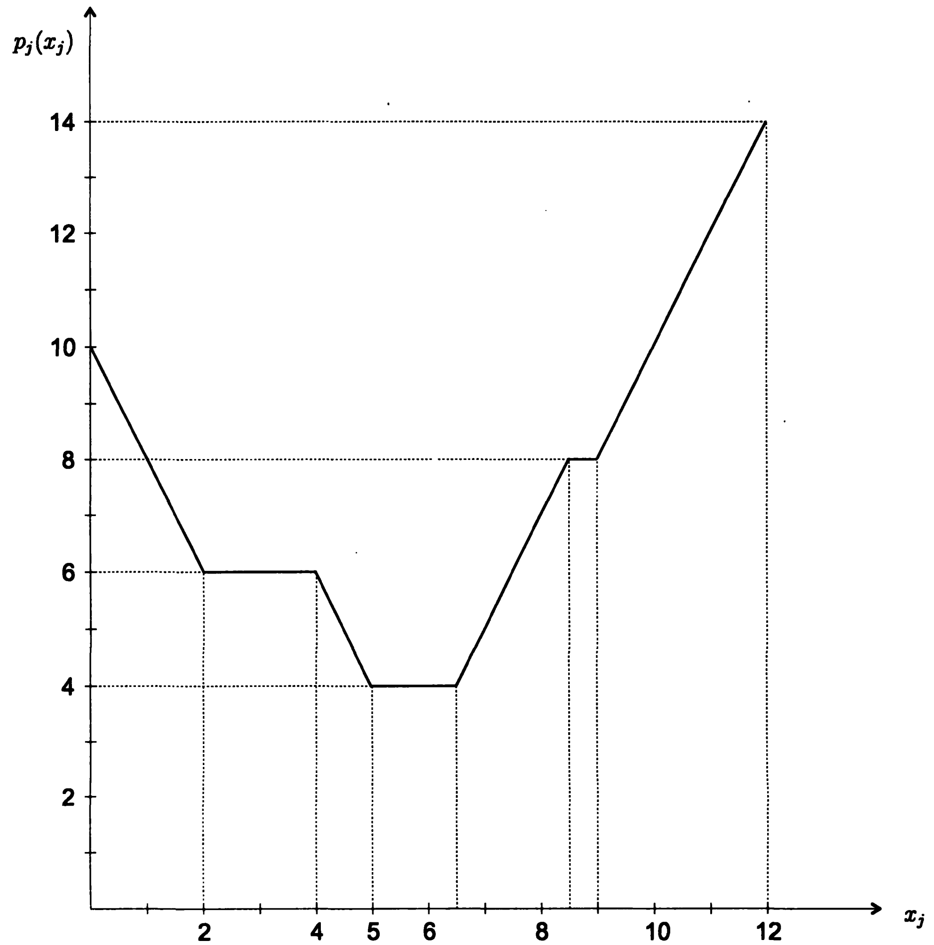

Figure 3

$x_j \in$	$[0, 2]$	$(2, 4]$	$(4, 5]$	$(5, 6.5]$	$(6.5, 8.5]$	$(8.5, 9]$	$[9, 12]$
$p_j(x_j)$	$10 - 2x_j$	6	$14 - 2x_j$	4	$-9 + 2x_j$	8	$-10 + 2x_j$

Table 4

We see that $\min\limits_{0 \le x_j \le 12} p_j(x_j) = \lambda_j = 4$ and $p_j(x_j) = \lambda_j = 4$ for all $x_j \in [5, 6.5]$. Let us define $F_j(\lambda_j) = \{x_j \mid p_j(x_j) \le \lambda_j \ \& \ x_j \in [0, 14]\}$; the form of point-to-set mapping $F_j(\lambda_j)$ follows from Tab. 5.

If $\lambda_j < 4$, $F_j(\lambda_j) = \emptyset$; if $\lambda_j \ge 14$, $F_j(\lambda_j) = [0, 14]$

λ_j	$F_j(\lambda_j)$
$[4, 6]$	$\left[\frac{14 - \lambda_j}{2}, \frac{\lambda_j + 9}{2}\right]$
$[6, 8]$	$\left[\frac{10 - \lambda_j}{2}, \frac{\lambda_j + 9}{2}\right]$
$[8, 10]$	$\left[\frac{10 - \lambda_j}{2}, \frac{\lambda_j + 9}{2}\right]$
$[10, 14]$	$\left[0, \frac{\lambda_j + 10}{2}\right]$

Table 5

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