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Petr Dostál

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Optimal Trading Strategies with Transaction Costs Paid Only for the First Stock

PETR DOSTÁL

Praha

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We consider an agent who trades with n stocks, where n is not so large, and pays proportional transaction costs. It is assumed that the vector of stock market prices is an n dimensional geometric Brownian motion and suppose that the agent is interested in a question what is the optimal investment policy for the i-th stock provided that he/she pays the transaction costs only for trades with this asset. We restrict ourselves to HARA utility functions and derive the first term in Taylor's expansion of function connecting the transaction tax and the width of no-trade region.

1. Introduction

We consider an investment problem with proportional transaction costs without consumption. We seek for a strategy that maximizes the asymptotics of expected utility of the portfolio market price similarly as in [2], [4], in contrast to [1], [3], [5], [8], [10], where the agent maximizes the expected value of the Laplace transform of the utility of consumption at a point chosen according to his/her time preferences. It is shown in [2] that the investment problem is a limiting case of the investment-consumption problem in case of logarithmic utility as the parameter of the Laplace transform goes to zero, i.e. when the consumption is postponed to the

Charles University in Prague, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail adress: dostal@karlin.mff.cuni.cz

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future as much as it is possible. If there were no transaction costs, both approaches lead to the same optimal policy: to keep a constant proportion of the portfolio market price invested in each stock. Such a vector is called Merton proportion here, see [9]. We focus on the case when we pay the transaction costs only for the trades with one of several assets.

We suppose that the vector of stock market prices $X(t)$ is a n-dimensional geometric Brownian motion driven by an n-dimensional $\mathscr{F}_t$-Wiener process $W(t)$ as follows

$$(1) \qquad dX(t) = \mathbb{X}(t)\mu\, dt + \mathbb{X}(t)\Sigma^{\frac{1}{2}}\, dW(t), \qquad X(0) = \mathrm{x}_0 \in (0,\infty)^n,$$

where $\mu \in \mathbb{R}^n$ and $\Sigma^{\frac{1}{2}} \in \mathbb{R}^{n\times n}$ is a positively definite matrix such that $\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}} =: \Sigma$ and $\mathbb{X}(t) := \mathit{diag}\, X(t)$ denotes the diagonal matrix of the vector $X(t)$ and where $(\mathscr{F}_t, t \geq 0)$ is an augmented filtration. Further, we denote by $Y(t)$ the portfolio market price at time $t \geq 0$, by $H(t)$ the vector of numbers of shares of each stock and by $G(t)$ the vector of positions in the market. Then $Y(t)G(t) = \mathbb{X}(t)H(t)$.

We will consider only utility functions with hyperbolic absolute risk aversion (HARA) $\mathscr{U}_\gamma(x) = \frac{1}{\gamma}x^\gamma$ if $\gamma < 0$ and $\mathscr{U}_0(x) = \ln x$. The case $\gamma \in (0,1]$ is omitted, since it leads to a too aggressive strategy. Further, denote $\boldsymbol{e}_\gamma(x) := \mathscr{U}_\gamma(e^x)$ and assume that $Y(0) = y_0 > 0$ and $G(0) = g_0 \in \mathbb{R}^n$ are deterministic random variables. We assume that the deposit part is not discounted and that we pay $(1+b)$-multiple of the stock market price in order to obtain the first stock. On the other hand, we obtain $(1-c)$-multiple of the stock market price, when we sell it, where $b \in (0,\infty)$ and $c \in (0,1)$. The aim of this paper is to find $f \in C^2(-1/b, 1/c)$, $v \in \mathbb{R}$ and a special strategy such that

$$(2) \qquad \boldsymbol{e}_\gamma(\ln Y(t) - f(G_1(t)) - vt) \text{ is a supermartingale,}$$

when considering a wide class of strategies, and such that (2) is a martingale if we consider the special one. We restrict ourselves to the strategies that keep the vector of positions $G(t)$ within a compact set in $(-1/b, 1/c) \times \mathbb{R}^{n-1}$. Then

$$v = \lim_{t\to\infty} \frac{1}{t}\boldsymbol{e}_\gamma^{-1} E\boldsymbol{e}_\gamma(\ln \hat{Y}(t)) \geq \limsup_{t\to\infty} \frac{1}{t}\boldsymbol{e}_\gamma^{-1} E\boldsymbol{e}_\gamma(\ln Y(t)),$$

where $\hat{Y}(t)$ denotes the portfolio market price corresponding to the special strategy here.

2. Dynamics

We restrict ourselves to the strategies such that $Y(t) > 0$ and $G_1(t) \in (-1/b, 1/c)$ hold for every $t \geq 0$ almost surely and we always assume that the transaction costs at time t are paid at the next moment after t. Let $H_1^+(t)$ and $H_1^-(t)$ denote the sum of shares of the first stock bought and sold on the time interval $[0,t)$, respectively. We assume that these processes are non-decreasing $\mathscr{F}_t$-adapted left-continuous with right-hand limits and that $H_i(t)$, $G_i(t)$ and $Y(t)$ are locally bounded $\mathscr{F}_t$-progressive measurable processes for $1 \leq i \leq n$. We are going to compute with stochastic differentials as every integrator was a continuous process, see remark 3.2 in [4] how to remove this assumption, and we refer the reader, who is not familiar with stochastic integration, to Chapter 3 in [7] for the corresponding theory. We need to distinguish between the derivative of f usually denoted by f' and the transposition x' of vector x and therefore we write $\dot{f}$ instead of f' for the derivative of f. Later on, we will need some statements derived in [4] that relate to the dynamics of certain processes. By lemma 3.1 in [4], $H_1^+(t)$, $H_1^-(t) < \infty$ hold almost surely and

$$dY(t) = H(t)'\, dX(t) - bX_1(t)\, dH_1^+(t) - cX_1(t)\, dH_1^-(t).$$

Let $\mathrm{e}_1 \in \mathbb{R}^n$ denote the column vector with 1 in the first row and 0 in the remaining ones and let $\mathbb{x} := diag\, \mathrm{x} \in \mathbb{R}^{n\times n}$ denote the diagonal matrix of the vector $\mathrm{x} \in R^n$.

By lemma 3.3 in [4],

$$dG_1(t) = \mathbb{B}_1(G(t))\, dt + \mathbb{S}_1(G(t))\, dW(t) + dG_1^+(t) - dG_1^-(t),$$

where $\mathbb{B}_1(\mathrm{x}) = \mathrm{e}_1'\mathbb{B}(\mathrm{x})$, $\mathbb{S}_1(\mathrm{x}) = \mathrm{e}_1'\,\mathbb{S}(\mathrm{x})$ and

$$\text{(3)} \qquad \mathbb{B}(\mathrm{x}) = [\mathbb{x} - \mathrm{x}\mathrm{x}'][\mu - \Sigma\mathrm{x}], \quad \mathbb{S}(\mathrm{x}) = [\mathbb{x} - \mathrm{x}\mathrm{x}']\Sigma^{\frac{1}{2}},$$

where the processes $G_1^+(t)$, $G_1^-(t)$ are non-decreasing $\mathscr{F}_t$-adapted such that

$$Y(t)\, dG_1^+(t) = (1 + bG_1(t))\, X_1(t)\, dH_1^+(t),$$
$$Y(t)\, dG_1^-(t) = (1 - cG_1(t))\, X_1(t)\, dH_1^-(t).$$

Further, we will abbreviate the notation

$$h_\pm(G_1(t)) * dG_1^\pm(t) := h_+(G_1(t))\, dG_1^+(t) + h_-(G_1(t))\, dG_1^-(t)$$

whenever h_+ and h_- are continuous functions on $(-1/b,\, 1/c)$. By lemma 3.4 in [4], $U(t) := \ln Y(t) - f(G_1(t)) - vt$ is an $\mathscr{F}_t$-semimartingale with

$$\frac{de_\gamma(U(t))}{e^{\gamma U(t)}} = d_f^v(G(t))\, dt + v_f(G(t))\, dW(t) + \delta_\pm^f(G_1(t)) * dG_1^\pm(t),$$

whenever $f \in C^2(-1/b, 1/c)$, where $x := \mathrm{e}_1'\mathrm{x}$,

$$\text{(4)} \qquad v_f(\mathrm{x}) = \mathrm{x}'\Sigma^{1/2} - \dot{f}(x)\,\mathbb{S}_1(\mathrm{x}), \quad \delta_\pm^f(x) = -\vartheta_\pm(x) \mp \dot{f}(x)$$

$$\text{(5)} \qquad d_f^v(\mathrm{x}) = \mathrm{d}(\mathrm{x}) - v - \dot{f}(x)\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x}) - \frac{1}{2}[\ddot{f}(\mathrm{x}) - \gamma\dot{f}(\mathrm{x})^2]\,\mathbb{S}_1(\mathrm{x})\,\mathbb{S}_1(\mathrm{x})'$$

$$\text{(6)} \qquad \tilde{\mathbb{B}}(\mathrm{x}) = [\mathbb{x} - \mathrm{x}\mathrm{x}'][\mu - (1-\gamma)\Sigma\mathrm{x}], \quad \mathrm{d}(\mathrm{x}) = \mu'\mathrm{x} - \frac{1-\gamma}{2}\mathrm{x}'\Sigma\mathrm{x}$$

and where $\vartheta_+(x) = \frac{b}{1+bx}$, $\vartheta_-(x) = \frac{c}{1-cx}$. Moreover, if we restrict ourselves to the strategies that keep the position $G(t)$ within a compact set in $(-1/b,\, 1/c) \times \mathbb{R}^{n-1}$, we obtain by lemma 3.5 in [4] that

(i) If $G_1(t) \in [\alpha, \beta] \subseteq (-1/b, 1/c)$, $d_f^\nu(G(t)) = 0$ and

$$\int_0^t \delta_+^f(G_1(s))\,dG_1^+(s) = \int_0^t \delta_-^f(G_1(s))\,dG_1^-(s) = 0 \tag{7}$$

hold for every $t \geq 0$ (almost surely, where $\nu \in \mathbb{R}$, then $EY(t)^\delta < \infty$ holds for every $\delta < 0$ and $t \geq 0$.

(ii) If $EY(t)^\delta < \infty$ holds for every $\delta < 0$ and $t \geq 0$, then

$$V := e_\gamma(U(0)) + \int \exp\{\gamma U(s)\} v_f(G(s))\,dW(s) \tag{8}$$

is an $\mathcal{F}_t$-martingale.

Let us consider the case when the transaction costs are zero and let us assume that the position $G(t)$ is kept within some bounded set in $\mathbb{R}^n$. Then $Y(t)^{-1}dY(t) = G(t)'[\mu\,dt + \Sigma^{\frac{1}{2}}dW(t)]$ and therefore $\mathcal{U}_\gamma(Y(t))$ is a.s. equal to the product of an exponential martingale

$$\mathcal{E}_t := \exp\left\{\gamma \int_0^t G(s)'\,\Sigma^{\frac{1}{2}}dW(s) - \frac{1}{2}\gamma^2 \int_0^t G(s)'\Sigma G(s)\,ds\right\}$$

and the process $e_\gamma\{\ln Y(0) + \int_0^t \mathrm{d}(G(s))\,ds\}$. The function d attains its maximum $\bar{\nu} := \frac{1-\gamma}{2}\Theta_\gamma'\Sigma\Theta_\gamma$ at the Merton proportion $\mathrm{x} = \Theta_\gamma := \frac{\Theta}{1-\gamma}$, where $\Theta := \Sigma^{-1}\mu$. Hence, the process $e_\gamma(\ln Y(t) - \bar{\nu}t) = \mathcal{E}_t \cdot e_\gamma(Z(t))$ is a supermartingale and it is a martingale only in case that $G_t = \Theta_\gamma$ holds for almost all t a.s., where $Z(t) := \ln Y(0) + \int_0^t \mathrm{d}(G(s))\,ds - \bar{\nu}t$ is a non-increasing process which is constant if and only if $G(t) = \Theta_\gamma$ holds for almost all $t \geq 0$.

3. Preliminary computations

The aim of this section is to show, how one can obtain some heuristic results. The obtained formulas also serve as the reference points for the next section. The key point of this paper is that the optimality condition (26) is of the same form as in the one-dimensional case.

If x denotes the variable representing the position $G(t)$, we write

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} := \mathrm{x} \quad \begin{pmatrix} \sigma_1^2 & L' \\ L & \hat{\Sigma} \end{pmatrix} := \Sigma, \quad \begin{pmatrix} \mu_1 \\ \hat{\mu} \end{pmatrix} := \mu.$$

One can imagine that we would trade with the remaining assets in order to maximize the function d(x). Then we would keep $\hat{x}$ on $\hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx$, where $\hat{\theta}_\gamma := \frac{1}{1-\gamma}\hat{\theta}$ and $\hat{\theta} := \hat{\Sigma}^{-1}\hat{\mu}$, and the function d is at every $\mathrm{x} = (x, \hat{x}')'$ such that $\hat{x} = \hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx$ holds of the form

$$\frac{1-\gamma}{2}\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + x\tilde{\mu} - \frac{1-\gamma}{2}x^2\tilde{\sigma}^2, \tag{9}$$

where $\tilde{\mu} := \mu_1 - L'\hat{\theta}$ and $\tilde{\sigma}^2 := \sigma_1^2 - L'\hat{\Sigma}^{-1}L$. The function (9) is obviously maximal at $x := \theta_\gamma := \frac{1}{1-\gamma}\tilde{\sigma}^{-2}\tilde{\mu}$. It can be useful to compare the Merton proportion Θ_γ and the values θ_γ and $\hat{\theta}_\gamma$. Obviously,

$$\Sigma^{-1} = \begin{pmatrix} \tilde{\sigma}^{-2} & -\tilde{\sigma}^{-2}L'\hat{\Sigma}^{-1} \\ -\hat{\Sigma}^{-1}L\tilde{\sigma}^{-2} & \hat{\Sigma}^{-1} + \hat{\Sigma}^{-1}L\tilde{\sigma}^{-2}L'\hat{\Sigma}^{-1} \end{pmatrix}$$

and therefore the Merton proportion Θ_γ is equal to

$$\Theta_\gamma = \frac{\Sigma^{-1}\mu}{1-\gamma} = \begin{pmatrix} \frac{\tilde{\sigma}^{-2}(\mu_1 - L'(\hat{\theta})}{1-\gamma} \\ \frac{\hat{\theta} - \tilde{\sigma}^{-2}\hat{\Sigma}^{-1}L(\mu_1 - L'\hat{\theta})}{1-\gamma} \end{pmatrix} = \begin{pmatrix} \theta_\gamma \\ \hat{\theta}_\gamma - \hat{\Sigma}^{-1}L\theta_\gamma \end{pmatrix}.$$

The above considered strategy would be close to the optimal one only in case of small transaction costs, so we have to correct it. We define $x \mapsto \varepsilon_x \in \mathbb{R}^{n-1}$ so that $\hat{x} = \hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx + \varepsilon_x$. Then

$$d(x) := \mathrm{d}(x) = \frac{1-\gamma}{2}\left[\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \tilde{\sigma}^2(2\theta_\gamma x - x^2) - \varepsilon_x'\hat{\Sigma}\varepsilon_x\right]. \tag{10}$$

Further, the diffusion coefficient of $G_1(t)$ is of the form $S_1(\mathrm{x})S_1(\mathrm{x})'$ and it can be rewritten into the form

$$S^2(x) := x^2\left[\tilde{\sigma}^2(1-x)^2 + (\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + \varepsilon_x)'\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + \varepsilon_x)\right].$$

The first element of $\tilde{\mathbb{B}}(\mathrm{x})$ is of the form

$$\tilde{B}_1(x) := (1-\gamma)x\left[\tilde{\sigma}^2(1-x)(\theta_\gamma - x) + (\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + \varepsilon_x)'\hat{\Sigma}\varepsilon_x\right].$$

The martingale condition is of the form

$$\frac{1}{2}[\ddot{f}(x) - \gamma\dot{f}(x)^2]S^2(x) + \dot{f}(x)\tilde{B}_1(x) - [d(x) - v] = 0. \tag{11}$$

The optimal boundary condition for the remaining assets is of the form

$$\frac{1}{2}[\ddot{f}(x) - \gamma\dot{f}(x)^2]\frac{\partial S^2}{\partial\varepsilon}(x) + \dot{f}(x)\frac{\partial\tilde{B}_1}{\partial\varepsilon}(x) - \frac{\partial}{\partial\varepsilon}[d(x) - v] = 0 \tag{12}$$

and the boundary conditions for the first asset with the smoothness of fit are of the form

$$\dot{f}(\alpha) = -\vartheta_+(\alpha), \qquad \ddot{f}(\alpha) = -\dot{\vartheta}_+(\alpha) = \vartheta_+(\alpha)^2, \tag{13}$$

$$\dot{f}(\beta) = +\vartheta_-(\beta), \qquad \ddot{f}(\beta) = +\dot{\vartheta}_-(\beta) = \vartheta_-(\beta)^2, \tag{14}$$

provided that the special strategy will just keep the first position within the interval $[\alpha, \beta] \subseteq (-1/b, 1/c)$ and the remaining ones on values that depend on the value of the first position. Obviously,

$$(15) \qquad \frac{1}{2}\frac{\partial S^2}{\partial \varepsilon}(x) = x^2\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + \varepsilon_x),$$

$$(16) \qquad \frac{\partial \tilde{B}_1}{\partial \varepsilon}(x) = (1-\gamma)x\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + 2\varepsilon_x), \quad \frac{\partial d}{\partial \varepsilon}(x) = -(1-\gamma)\hat{\Sigma}\varepsilon_x$$

and therefore the condition (12) is of the form

$$[\ddot{f}(x) - \gamma\dot{f}(x)^2]x^2\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + \varepsilon_x) +$$
$$+ \dot{f}(x)(1-\gamma)x\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L + 2\varepsilon_x) + (1-\gamma)\hat{\Sigma}\varepsilon_x = 0.$$

Since $\hat{\Sigma}$ is a regular matrix, the above condition is also of the form

$$(17) \qquad \varepsilon_x\left[x^2\frac{\ddot{f}(x) - \dot{f}(x)^2}{1-\gamma} + (1 + x\dot{f}(x))^2\right] =$$

$$(18) \qquad = (\hat{\Sigma}^{-1}L - \hat{\theta}_\gamma)\left[x^2\frac{\ddot{f}(x) - \dot{f}(x)^2}{1-\gamma} + x\dot{f}(x)(1 + x\dot{f}(x))\right]$$

If expression in the brackets in (17) is not equal to zero, we can introduce e_x so that

$$(19) \qquad \varepsilon_x = (\hat{\Sigma}^{-1}L - \hat{\theta}_\gamma)e_x.$$

If $\hat{\theta}_\gamma \neq \hat{\Sigma}^{-1}L$ and there exist e_x such that (19) holds, then the condition (12) is of the form

$$(20) \qquad 1 + x\dot{f}(x) - (1 - e_x)\left[x^2\frac{\ddot{f}(x) - \dot{f}(x)^2}{1-\gamma} + (1 + x\dot{f}(x))^2\right] = 0.$$

Then the diffusion coefficient of $G_1(t)$ is of the form

$$(21) \qquad S^2(x) = \tilde{\sigma}^2x^2[(1-x)^2 + (1-e_x)^2\kappa_\gamma^2],$$

where $\kappa_\gamma^2 := \tilde{\sigma}^{-2}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L)'\hat{\Sigma}(\hat{\theta}_\gamma - \hat{\Sigma}^{-1}L) \neq 0$ if and only $\hat{\theta}_\gamma \neq \hat{\Sigma}^{-1}L$. The modified drift coefficient is then of the form

$$(22) \qquad \tilde{B}_1(x) = \tilde{\sigma}^2(1-\gamma)x[(1-x)(\theta_\gamma - x) - e_x(1-e_x)\kappa_\gamma^2],$$

$$(23) \qquad d(x) = \frac{1-\gamma}{2}\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \frac{1-\gamma}{2}\tilde{\sigma}^2[2\theta_\gamma x - x^2 - \kappa_\gamma^2 e_x^2].$$

The maximal value of v that can be reached corresponds to the case of zero transaction costs and the strategy that keeps the position on the Merton proportion. Hence, we obtain the right-hand inequality in (24). Further, we introduce $\omega_\gamma^2 \geq 0$ such that we have the left-hand equality in (24)

$$(24) \qquad \frac{1-\gamma}{2}\{\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2)\} = v \leq \frac{1-\gamma}{2}(\tilde{\sigma}^2\theta_\gamma^2 + \hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma).$$

Then $d(x) - v = \frac{1-\gamma}{2}\tilde{\sigma}^2[\omega_\gamma^2 - (\theta_\gamma - x)^2 - \kappa_\gamma^2 e_x^2]$. Now, the martingale condition is of the form

$$x^2\left[(1-x)^2+(1-e_x)^2\kappa_\gamma^2\right]\frac{\ddot f(x)-\gamma\dot f(x)^2}{1-\gamma}+$$
$$+\,2x\left[(1-x)(\theta_\gamma-x)-e_x(1-e_x)\kappa_\gamma^2\right]\dot f(x)+(\theta_\gamma-x)^2+\kappa_\gamma^2e_x^2-\omega_\gamma^2=0.$$

The conditions (13) and (14) at the boundary points α, β together with the martingale condition give the following one

$$\omega_\gamma^2=(\theta_\gamma-\xi_\alpha)^2+\kappa_\gamma^2\left(\frac{e_\alpha+b\alpha}{1+b\alpha}\right)=(\theta_\gamma-\xi_\beta)^2+\kappa_\gamma^2\left(\frac{e_\beta-c\beta}{1-c\beta}\right)^2, \tag{25}$$

where $\xi_\alpha:=\xi_+(\alpha)$, $\xi_\beta:=\xi_-(\beta)$ and $\xi_+(x):=x\frac{1+b}{1+bx}$, $\xi_-(x):=x\frac{1-c}{1-cx}$. The condition (20) together with the conditions (13) and (14) give the following condition on e_x at the points α, β

$$\begin{pmatrix}e_\alpha\\ e_\beta\end{pmatrix}=1-\frac{1}{1\mp x\vartheta_\pm(x)}=\frac{x\vartheta_\pm(x)}{x\vartheta_\pm(x)\mp1}=\begin{pmatrix}-b\alpha\\ c\beta\end{pmatrix}.$$

Then the condition (25) is of the form

$$\omega_\gamma^2=(\theta_\gamma-\xi_\alpha)^2=(\theta_\gamma-\xi_\beta)^2. \tag{26}$$

4. Existence of function f

This section provides the technical results necessary for the next section. It also introduces a notation used later on. The aim of this section is to prove theorem 4.24. It is recommended to skip this section up to theorem 4.24 for the first reading. We always assume that $\kappa_\gamma^2>0$ in this section.

Lemma 4.1 *Let* $\varkappa_\gamma^2>0$, $\theta_\gamma\neq0$ *and* $b>0$, $c\in(0,1)$. *Denote* $\xi(x,h):=$ $:=x\left[1-(1-x)h\right]$ *and*

$$F(x,h,w,\eta):=\left[\eta+(1+xh)^2\right]\left[(1-x)^2\eta+(\theta_\gamma-\xi(x,h))^2-w^2\right]+\kappa_\gamma^2\eta.$$

Then $\eta\mapsto F(x,h,w,\eta)$ *is a quadratic function. Denote* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}:\mathbb{R}^3\to\mathbb{R}$ *the corresponding coefficient so that*

$$F(x,h,w,\eta)=\mathscr{A}(x,h,w)\eta^2+\mathscr{B}(x,h,w)\eta+C(x,h,w).$$

Further, denote

$$\Phi:=\{u\in\mathbb{R}^3:u_1\neq0,\,1+u_1u_2>0,\,u_3^2<\aleph_\gamma,\,\mathscr{B}^2(u)>4\mathscr{A}(u)\mathscr{C}(u)\},$$

where $\aleph_\gamma:=\theta_\gamma^2\wedge\left[(1-\theta_\gamma)^2+\kappa_\gamma^2\right]$, *and*

$$\phi:(x,h,w)\in\Phi\mapsto\max\{\eta\in\mathbb{R},\,F(x,h,w,\eta)=0\}.$$

Then $\phi\in C^\infty(\Phi)$. *Moreover, if* $u\in\mathbb{R}^3$ *is such that*

$$u_1\neq0,\,1+u_1u_2>0\quad and\quad(\theta_\gamma-\xi(u_1,u_2))^2\le u_3^2<\aleph_\gamma, \tag{27}$$

then $u \in \Phi$ and $\eta := \phi(u) \geq 0$ is the only non-negative solution of the equation $F(u,\eta) = 0$.

Proof: Let $\tilde{u} \in \Phi$. If $\mathscr{A}(\tilde{u}) > 0$, then

$$\phi(u) = \frac{-\mathscr{B}(u) + \sqrt{\mathscr{B}^2(u) - 4\mathscr{A}(u)\mathscr{C}(u)}}{2\mathscr{A}(u)} \tag{28}$$

is an infinitely differentiable function in a neighbourhood of $\tilde{u}$. Obviously, $\mathscr{A}(\tilde{u}) = (1 - \tilde{u}_1)^2$. If $u \in \mathbb{R}^3$ is such that $u_3^2 < \aleph_\gamma$ and $\mathscr{A}(u) \leq 0$, i.e., $u_1 = 1$, then $\mathscr{A}(u) = 0$ and $\mathscr{B}(u) = (\theta_\gamma - 1)^2 + \kappa_\gamma^2 - u_3^2 > 0$. Hence, we get that

$$\phi(u) = \frac{-2\mathscr{C}(u)}{\mathscr{B}(u) + \sqrt{\mathscr{B}^2(u) - 4\mathscr{A}(u)\mathscr{C}(u)}} \tag{29}$$

is an infinitely differentiable function in a neighbourhood of $\tilde{u}$ in case that $\mathscr{A}(\tilde{u}) \leq 0$.

Let $u \in \mathbb{R}^3$ be such that (27) is satisfied. We are going to show that $\mathscr{B}^2(u) > 4\mathscr{A}(u)\mathscr{C}(u)$. Since $\mathscr{C}(u) = F(u,0) \leq 0$ and $\mathscr{A}(u) \geq 0$, we get that $4\mathscr{A}(u)\mathscr{C}(u) \leq 0$. Hence, it is sufficient to show that $\mathscr{B}(u) > 0$ in case that $\mathscr{A}(u) = 0$ or $\mathscr{C}(u) = 0$. Since we have showed that $\mathscr{B}(u) > 0$ in case $\mathscr{A}(u) = 0$, we are only to consider the case $\mathscr{C}(u) = 0$. Then $(\theta_\gamma - \xi(u_1,u_2))^2 = u_3^2$ and therefore $\mathscr{B}(u) = (1 + u_1u_2)^2(1 - u_1)^2 + \kappa_\gamma^2 > 0$.

Now, we are going to show that $\eta := \phi(u) \geq 0$. If $\mathscr{A}(u) > 0$, then we obtain that $\phi(u) \geq 0$ from (28) since $\mathscr{C}(u) \leq 0$. If $\mathscr{A}(u) = 0$, then $\mathscr{B}(u) > 0$ as showed above and we get from (29) that $\phi(u) = -\mathscr{C}(u)/\mathscr{B}(u) \geq 0$, since $\mathscr{C}(u) \leq 0$. Since $\mathscr{C}(u) \leq 0 \leq \mathscr{A}(u)$, there cannot be another non-negative root $0 \leq \tilde{\eta} \neq \eta$ of the equation $F(u,\tilde{\eta}) = 0$. □

Lemma 4.2 *Let $\omega_\gamma^2 < \aleph_\gamma$, $\kappa_\gamma^2 > 0$ and $b > 0$, $c \in (0,1)$. Let h be an infinitely differentiable function satisfying $G(x,h(x),\eta_h(x)) = \omega_\gamma^2$ in a neighbourhood of $x_0 \in \mathbb{R}$ and $\eta_h(x_0) = 0$, where*

$$G(x,h,\eta) := (1 - x)^2\eta + [\theta_\gamma - \xi(x,h)]^2 + \kappa_\gamma^2\eta([1 + xh]^2 + \eta)^{-1}$$

and $\eta_h(x) := x^2\frac{\dot{h}(x) - h^2(x)}{1-\gamma}$. Let $x_0 \neq 0$ be such that

$$1 + x_0h(x_0) > 0 \quad \textit{and} \quad 1 - (1 - x_0)h(x_0) > 0. \tag{30}$$

Then $[\theta_\gamma - \xi(x_0,h(x_0))]^2 = \omega_\gamma^2$ and sign $\dot{\eta}_h(x_0) =$ sign $[\theta_\gamma - \xi(x_0,h(x_0))]$.

Moreover, if $\theta_\gamma = \xi(x_0,h(x_0))$, then there exists $\delta > 0$ such that $\eta_h(x) < 0$ holds for every $x \in \mathbb{R}\backslash\{x_0\}$ such that $|x - x_0| < \delta$.

Proof: Since $\eta_h(x_0) = 0$, we get that $\omega_\gamma^2 = G(x_0,h(x_0),0) = [\theta_\gamma - \xi(x_0,h(x_0))]^2$ by the definition of G. Since $\eta_h(x_0) = 0$ and $dG(x,h(x),\eta_h(x))/dx = d\omega_\gamma^2/\delta x = 0$, we get that

$$(31)\qquad 0 = \frac{d}{dx}[\theta_\gamma - \xi(x,h(x))]^2 + \dot\eta_h(x)[(1-x)^2 + \kappa_\gamma^2(1+xh(x))^{-2}]$$

holds at $x = x_0$. A straightforward computation using $\dot h(x_0) = h^2(x_0)$ shows that

$$(32)\qquad \frac{d}{dx}[\xi(x,h(x)) - \theta_\gamma] = [1 + x_0 h(x_0)][1 - (1-x_0)h(x_0)] > 0$$

holds at $x = x_0$ by (30). Then the equality of signs in the first part of the statement follows from (31) and (32). If $\theta_\gamma = \xi(x_0, h(x_0))$, then $\omega_\gamma = 0$ and we obtain from (32) that there exists $\delta > 0$ such that $\xi(x,h(x)) \neq \theta_\gamma$ holds on $(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$. If $x \neq x_0$ is such that $|x - x_0| < \delta$, we obtain that $\eta_h(x) < 0 = \eta_h(x_0)$ since the case $\eta_h(x) \geq 0$ leads to a contradiction $\omega_\gamma^2 = G(x,h(x),\eta_h(x)) \geq (\theta_\gamma - \xi(x,h(x)))^2 > 0 = \omega_\gamma^2$. □

Lemma 4.3 *Let* $\omega_\gamma^2 < \aleph_\gamma$, $\kappa_\gamma^2 > 0$ *and* $b > 0$, $c \in (0,1)$. *(i) Let* $x_0 > 0$, $h(x_0) = -\vartheta_+(x_0)$ *or* $x_0 < 0$, $h(x_0) = \vartheta_-(x_0)$, *then* (30) *holds.*

(ii) Let $0 < x_1 < x_2 < \infty$ *and* $h \in C^1[x_1,x_2]$ *be such that* $\eta_h \geq 0$ *holds on* (x_1,x_2) *and such that* $h(x_1) = -\vartheta_+(x_1)$. *Then*

$$1 + xh(x) \geq (1 \wedge (bx_2)^{-1})/2 > 0$$

holds for every $x \in [x_1,x_2]$.

(iii) Let $h \in C^1[x_1,x_2]$ *be such that* $x_1 > 0$ *and* $h(x_1) = -\vartheta_+(x_1)$ *in case* $\theta_\gamma > 0$ *and such that* $x_2 < 0$ *and* $h(x_2) = \vartheta_-(x_2)$ *in case* $\theta_\gamma < 0$. *Further, assume that*

$$G(x,h(x),\eta_h(x)) = \omega_\gamma^2 \quad \text{and} \quad \eta_h(x) \geq 0$$

hold on $[x_1,x_2]$. *Then* (30) *holds for every* $x_0 \in [x_1,x_2]$.

Proof: (i) Obviously,

$$1 \mp x_0\vartheta_\pm(x_0) = \begin{pmatrix} \frac{1}{1+bx_0} \\ \frac{1}{1-cx_0} \end{pmatrix},\ 1 \pm (1-x_0)\vartheta_\pm(x_0) = \begin{pmatrix} \frac{1+b}{1+bx_0} \\ \frac{1-c}{1-cx_0} \end{pmatrix}.$$

(ii) It follows from the assumption $\eta_h \geq 0$ on (x_1,x_2) that $\dot h(x) \geq h^2(x) \geq 0$ and therefore h is a non-decreasing function on $[x_1,x_2]$. Further, if $x \in (x_1,x_2)$ is such that $h(x) \neq 0$, then

$$\dot r(x) = \frac{\dot h(x) - h^2(x)}{h^2(x)} \geq 0, \quad \text{where} \quad r(x) := \frac{1 + xh(x)}{-h(x)}.$$

If $x \in [x_1,x_2]$ is such that $h(x) \geq -1/(2x_2)$, then $h(x) \geq -1/(2x)$ and therefore $1 + xh(x) \geq 1/2$.

Further, we can assume that $h(x_1) < -1/(2x_2)$. If $x_0 \in (x_1, x_2]$ is such that $h(x) < -1/(2x_2)$ holds for every $x \in [x_1, x_0)$, then

$$1 + xh(x) = -h(x)r(x) \geq -h(x)r(x_1) \geq 1/(2bx_2)$$

holds for every $x \in [x_1, x_0)$, since r is a non-decreasing function on $[x_1, x_0)$ with $r(x_1) = 1/b$.

(iii) Since $1 - (1 - x)h(x)$ is a continuous function on $[x_1, x_2]$ attaining by (i) a positive value at x_1 or at x_2, respectively, it is sufficient to show that it does not attain the value zero in order to verify that it is positive on $[x_1, x_2]$, respectively, it is sufficient to show that it does not attain the value zero in order to verify that it is positive on $[x_1, x_2]$. Let us assume that $x \in [x_1, x_2]$ is such that $1 = (1 - x)h(x)$. Then $\xi(x, h(x)) = 0$ and we obtain a contradiction $\omega_\gamma^2 = G(x, h(x), \eta_h(x)) \geq \theta_\gamma^2 > \omega_\gamma^2$, since $\eta_h(x) \geq 0$.

If $\theta_\gamma > 0$, we assume that $x_1 > 0$ and therefore the first inequality in (30) at every $x_0 \in [x_1, x_2]$ follows from (ii). If $\theta_\gamma < 0$, we assume that $x_2 < 0$. If $x \in [x_1, x_2]$, we obtain from the previous part of the proof that $h(x) < 1/(1 - x) < 1/x$ and therefore $1 + xh(x) > 0$. □

Lemma 4.4 *Let* $\omega_\gamma^2 < \aleph_\gamma$, $\kappa_\gamma^2 > 0$ *and* $b > 0$, $c \in (0, 1)$. *(i) Let* $(x_0, h_0) \in \mathbb{R}^2$ *be such that*

$$(33) \qquad x_0 \neq 0,\ 1 + x_0 h_0 > 0, \quad \textit{and} \quad (\theta_\gamma - \xi(x_0, h_0))^2 \leq \omega_\gamma^2.$$

Then there exist $\delta > 0$ *and* $h \in C^\infty(x_0 - \delta, x_0 + \delta)$ *such that*

$$(34) \qquad \dot{h}(x) = \varphi(x, h(x), \omega_\gamma), \quad (x, h(x), \omega_\gamma) \in \Phi, \quad h(x_0) = h_0$$

hold whenever $|x - x_0| < \delta$, *where* $\varphi(x, h, w) := h^2 + (1 - \gamma)x^{-2}\phi(x, h, w)$.

(ii) Let $h_1(z), h_2(z)$ *solve* (34) *whenever* $z \in [x, y]$ *and* $i \in \{1, 2\}$. *If there exists* $z \in [x, y]$ *such that* $h_1(z) = h_2(z)$, *then* $h_1 = h_2$ *holds on* $[x, y]$.

(iii) Let $\mathscr{I}$ *be a bounded open interval with an extreme point* $\tilde{x}_0$ *and* $h \in C^\infty(\mathscr{I})$ *satisfying* (34) *on* $\mathscr{I}$. *Let* $h_0 := \lim_{\mathscr{I} \ni x \to \tilde{x}_0} h(x) \in \mathbb{R}$ *be such that* (33) *is satisfied if* x_0 *is replaced by* $\tilde{x}_0$. *Then there exist an open interval* $\tilde{\mathscr{I}} \supseteq \mathscr{I}$ *containing* $\tilde{x}_0$ *and* $\tilde{h} \in C^\infty(\tilde{\mathscr{I}})$ *satisfying* (34) *on* $\tilde{\mathscr{I}}$ *such that* $\tilde{h} = h$ *on* $\mathscr{I}$.

Proof: If (33) is satisfied, then $(x_0, h_0, \omega_\gamma) \in \Phi$ by lemma 4.1. Then (i)-(iii) follow from the theorem on existence and uniqueness of ODE, since $\varphi \in C^\infty(\Phi)$ by lemma 4.1. □

Lemma 4.5 *Let* $\omega_\gamma^2 < \aleph_\gamma$, $\kappa_\gamma^2 > 0$ *and* $b > 0$, $c \in (0, 1)$. *Let* $h_0 = \mp \vartheta_\pm(x_0)$ *and* $\xi(x_0, h_0) = \theta_\gamma \mp \omega_\gamma$, *where* x_0, θ_γ *are positive numbers in the upper case and negative in the lower one.*

Then there exists a unique infinitely differentiable function h *with convex open domain* $\mathscr{D}_h$ *such that* h *is a maximal solution of* (34) *and* $\eta_h(x_0) = 0$. *Further, assume that* $\omega_\gamma > 0$ *and denote*

$$v_1(x_0) := x_1 := \inf\{x \in \mathscr{D}_h \cap (x_0, \infty), \dot{h}(x) = h^2(x)\} \quad \textit{if } \theta_\gamma > 0$$

$$v_2(x_0) := x_2 := \sup\{x \in \mathscr{D}_h \cap (-\infty, x_0), \dot{h}(x) = h^2(x)\} \quad \textit{if } \theta_\gamma < 0.$$

Then (i) $x_1 > x_0 > 0$ *and* $\dot{h}(x) > h^2(x)$ *holds on* (x_0, x_1) *in case that* $\theta_\gamma > 0$ *and* $x_2 < x_0 < 0$ *and* $\dot{h}(x) > h^2(x)$ *holds on* (x_2, x_0) *if* $\theta_\gamma < 0$.

(ii) If $\theta_\gamma < 0$, *then* $x_2 = -\infty$ *and* $h > 0$ *on* $(-\infty, x_0]$ *or* $\dot{h}(x_2) = h^2(x_2)$. *If* $\theta_\gamma > 0$, *then* $x_1 = \infty$ *and* $h < 0$ *on* $[x_0, \infty)$ *or* $x_1 = 1$ *and* $h(1_-) = \infty$ *or* $\dot{h}(x_1) = h^2(x_1)$.

Proof: By lemma 4.3 (i), (30) holds with $h(x_0) := h_0$. By assumption $G(x_0, h_0, 0) = (\theta_\gamma - \xi(x_0, h_0))^2 = \omega_\gamma^2$. By lemma 4.4 (i), there exist $\delta > 0$ and $\tilde{h} \in C^\infty(x_0 - \delta, x_0 + \delta)$ satisfying (34) on $(x_0 - \delta, x_0 + \delta)$. All infinitely differentiable solutions h of (34) with convex open domain form a set arranged by inclusion so that every chain has an upper bound. By Zorn's lemma, there exists h a maximal solution of (34) with convex open domain $\mathscr{D}_h$ such that $h = \tilde{h}$ holds on $(x_0 - \delta, x_0 + \delta)$. If $\hat{h}$ is another function with such properties and domain $\mathscr{D}_{\hat{h}}$, we obtain that $h = \hat{h}$ on $\mathscr{D}_h \cap \mathscr{D}_{\hat{h}} \ni x_0$ by lemma 4.4 (ii). Considering $h \cup \hat{h} \in C^\infty(\mathscr{D}_h \cup \mathscr{D}_{\hat{h}})$ with convex open domain $\mathscr{D}_h \cup \mathscr{D}_{\hat{h}}$, we obtain that $\mathscr{D}_h = \mathscr{D}_{\hat{h}}$, since h and $\hat{h}$ are both maximal solutions.

Since $G(x_0, h_0, 0) = \omega_\gamma^2$, we obtain that $F(u, 0) = 0$, where $u := (x_0, h_0, \omega_\gamma)$. By lemma 4.1, $\phi(u) = 0$ and therefore $\eta_h(x_0) = \phi(u) = 0$ as $h(x_0) = h_0$.

(i) By lemma 4.2 and the assumption $\theta_\gamma - \xi(x_0, h(x_0)) = \pm\omega_\gamma$, we get that $\dot{\eta}_h(x_0) > 0$ and therefore $x_1 > x_0$ if $\theta_\gamma > 0$ and $\dot{\eta}_h(x_0) < 0$ and therefore $x_2 < x_0$ if $\theta_\gamma < 0$. Since η_h is a continuous function, we obtain from the definition of x_1 or x_2 that $\eta_h > 0$ holds on (x_0, x_1) if $\theta_\gamma > 0$ and on (x_2, x_0) if $\theta_\gamma < 0$. In particular $\dot{h} > h^2 \geq 0$ holds on the above mentioned interval and therefore there exists $h(x_{1-})$ or $h(x_{2+})$, respectively.

(ii) If $x_1 = \infty$ and $\theta_\gamma > 0$ or if $x_2 = -\infty$ and $\theta_\gamma < 0$, we obtain that

$$(35) \quad \omega_\gamma^2 = \lim_{x \to \pm\infty} G(x, h(x), \eta_h(x)) \geq \limsup_{x \to \pm\infty} [\theta_\gamma - x(1 - (1 - x)h(x))]^2$$

and therefore $h(\pm\infty_\mp) = 0$. Since h increases on (x_0, x_1) or (x_2, x_0), we then obtain that $h < 0$ on (x_0, x_1) or $h > 0$ on (x_2, x_0), respectively. The same argument as in (35) shows that $x_1 = 1$ if $h(x_{1-}) = +\infty$ and $\theta_\gamma > 0$ and that $h(x_{2+}) = -\infty$ is impossible if $\theta_\gamma < 0$.

Now, let us assume that none of the previous cases happens. If $\theta_\gamma > 0$, we obtain from lemma 4.3 (ii) that

$$1 + x_1 h(x_{1-}) = \lim_{x \uparrow x_1} 1 + x h(x) \geq \lim_{x \uparrow x_1} \frac{1 \wedge (bx)^{-1}}{2} = \frac{1 \wedge (bx_1)^{-1}}{2} > 0.$$

If $\theta_\gamma < 0$, we have by lemma 4.3 (iii) that $1 \geq (1 - x)h(x)$ holds on $[x_3, x_0]$ for every $x_3 \in (x_2, x_0)$, where $x_0 < 0$. Then

$$1 + x_2 h(x_{2+}) \geq 1 + x_2/(1 - x_2) = 1/(1 - x_2) > 0.$$

The right-hand inequality in (33) with x_0 replaced by x_1 or x_2 and $h_0 := h(x_{1-})$ or $h_0 := h(x_{2+})$ can be obtained by modifying (35), where $\pm\infty$ is replaced by x_{1-} or x_{2+}, respectively. By lemma 4.4, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subseteq \subseteq \mathscr{D}_h$, where $i = 1$ if $\theta_\gamma > 0$ and $i = 2$ if $\theta_\gamma > 0$. By definition of x_1 and x_2, we get that $\dot{h}(x_1) = h^2(x_1)$ holds in the upper case and $\dot{h}(x_2) = h^2(x_2)$ in the lower one. □

Remark 4.6 Denote

$$\omega_\gamma(x) := \pm(\theta_\gamma - \xi(x, \mp\vartheta_\pm(x))),$$

where the upper case is considered when $\theta_\gamma > 0$ and the lower one when $\theta_\gamma < 0$. Further, denote

$$I := \{x > 0, 0 < \omega_\gamma(x) < \sqrt{\aleph_\gamma}\} \quad \text{if} \quad \theta_\gamma > 0,$$
$$J := \{x < 0, 0 < \omega_\gamma(x) < \sqrt{\aleph_\gamma}\} \quad \text{if} \quad \theta_\gamma < 0.$$

Let $\kappa_\gamma^2 > 0$ and x_0, θ_γ be both positive or negative numbers such that $\omega_\gamma^2(x_0) < \aleph_\gamma$. By h_{x_0} we denote the maximal solution of (34) with ω_γ^2 equal to $\omega_\gamma^2(x_0)$ and $h_0 := -\vartheta_+(x_0)$ if $\theta_\gamma > 0$ and $h_0 := \vartheta_-(x_0)$ if $\theta_\gamma < 0$. Since $(x_0, x, h) \mapsto \mapsto \varphi(x, h, \omega_\gamma(x_0))$ is an infinitely differentiable function on $\{(x_0, x, h) \in \mathbb{R}^3; (x, h, \omega_\gamma(x_0)) \in \Phi\}$, we obtain from the theorem on stability of ODE that $(x, y) \mapsto \mapsto h_x(y)$ is an infinitely differentiable function at all points (x, y) such that h_x is defined at y. Put

$$\mathscr{D} := \{(x,y) \in \mathbb{R}^2 : h_x(y) \quad \text{is defined \&} \quad (x, h_x(y), \omega_\gamma(x)) \in \Phi\}.$$

Further, denote by I_1 the set of all $x_0 \in I$ such that $v_1(x_0) < \infty$ and $h_{x_0}(v_1(x_0)_-) < < \infty$, $I_2 := \{x_0 \in I, \ v_1(x_0) = \infty\}$, $I_3 := I \setminus (I_1 \cup I_2)$. Similarly, we put $J_1 := := \{x_0 \in J, v_2(x_0) > -\infty\}$ and $J_2 := J \setminus J_1$.

Lemma 4.7 *(i) The function* $(x,t) \in \mathscr{D} \mapsto \eta_{h_x}(t) := x^{2\frac{\dot{h}_x(t) - h_x^2(t)}{1-\gamma}}$ *is infinitely differentiable on* $\mathscr{D}$*. Moreover, if* $x_1 \in I_1$ *or* $x_2 \in J_1$*, then* $\eta_{h_{x_1}}(v_1(x_1)) = 0$ *or* $\eta_{h_{x_2}}(v_2(x_2)) = 0$*, respectively.*

(ii) Further, the following function is infinitely differentiable on $\mathscr{D}$

$$Z : (x,t) \in \mathscr{D} \mapsto \dot{\eta}_{h_x}(t). \tag{36}$$

If $x_1 \in I$*, then* $Z(x_1, x_1) > 0$*. Further* $Z(x_1, v_1(x_1)) < 0$ *holds in case that* $x_1 \in I_1$*. Similarly, we have that* $Z(x_2, x_2) < 0$ *if* $x_2 \in J$ *and* $Z(x_2, v_2(x_2)) > 0$ *if* $x_2 \in J_1$*.*

Proof: (i) The first part follows from remark 4.6 as $(x, y) \mapsto h_x(y)$ is infinitely differentiable function. The moreover part follows from lemma 4.5 (ii).

(ii) The first part follows from remark 4.6 again. If $x \in I$ or $x \in J$, then $\eta_{h_x}(x) = 0$ by lemma 4.5. Since $h_x(x) = -\vartheta_+(x)$ if $x \in I$ and $h_x(x) = \vartheta_-(x)$ if $x \in J$, we get by lemma 4.3 (i), that the inequalities in (30) are satisfied with $x_0 := x, h := h_x$. If $x \in I$ or $x \in J$, then $\theta_\gamma - \xi(x, h_x(x))$ is equal to $\omega_\gamma(x)$ or

$-\omega_y(x)$, respectively, where $\omega_y(x) > 0$. By lemma 4.2, $\dot{\eta}_{h_x}(x) > 0$ if $x \in I$ and $\dot{\eta}_{h_x}(x) < 0$ if $x \in J$.

If $x \in I_1$ or $x \in J_1$, then lemma 4.3 (iii) gives that the inequalities in (30) are satisfied with $x_0 := v_i(x), h := h_x$, where $i = 1$ or $i = 2$ respectively. By lemma 4.2, $\dot{\eta}_{h_x}(v_i(x)) \neq 0$. By lemma 4.5, η_{h_x} cannot increase at $v_1(x)$ and therefore $\dot{\eta}_{h_x}(v_1(x)) < 0$ in case that $x \in I_1$ and η_{h_x} cannot decrease at $v_2(x)$ and therefore $\dot{\eta}_{h_x}(v_2(x)) > 0$ in case that $x \in J_1$. □

Lemma 4.8 *If $x_0 \in I$, then $v_1(x)$ is lower semi-continuous at x_0. If $x_0 \in J$, then v_2 is upper semi-continuous at x_0.*

Moreover, if $x \in I$ is a cluster point of I_3 such that $v_1(x) \geq 1$, then $x \in I_3$.

Proof: We are going to show that v_1 is lower semi-continuous at every $x_0 \in I$. The upper semi-continuity of v_2 on J can be proved similarly. We will show that

$$w_i := \liminf_{I_i \ni x \to x_0} v_1(x) \geq v_1(x_0)$$

holds for $i = 1, 2, 3$.

(i) We are going to show that $w_1 < v_1(x_0)$ leads to a contradiction. Since $w_1 < \infty$, x_0 is a cluster point of I_1. Let $\{x_m\} \subseteq I_1$ be a sequence tending to x_0 such that $v_1(x_m) \to w_1$. By lemma 4.7(ii),

$$0 > Z(x_m, v_1(x_m)) \to Z(x_0, w_1) \quad \& \quad Z(x_0, x_0) > 0.$$

Hence, $w_1 \neq x_0$. Since $w_1 \leftarrow v_1(x_m) > x_m \to x_0$ as $m \to \infty$ we get that $w_1 \geq x_0$ and therefore $w_1 \in (x_0, v_1(x_0))$. Then $(x_0, w_1) \in \mathscr{D}$ and we obtain from lemma 4.7 (i) that $0 = \eta_{h_{x_m}}(v_1(x_m)) \to \eta_{h_{x_0}}(w_1)$ and therefore $\dot{h}_{x_0}(w_1) = h^2_{x_0}(w_1)$. This leads to a contradiction with the definition of $v_1(x_0) > w_1$.

(ii) Since $v_1(x) = \infty$ holds for every $x \in I_2$, we get that $w_2 = \infty \geq v_1(x_0)$ immediately.

(iii) Let $x_m \in I_3$ be such that $x_m \to x_0$ and $v_1(x_m) \to w_3$ as $m \to \infty$. Then $v_1(x_m) = 1$ holds for every $m \in \mathbb{N}$. We are going to show that $x_0 \in I_3$ provided that $v_1(x_0) \geq 1 = w_1$. Let us assume that $v_1(x_0) \geq 1$. Since $x_m < v_1(x_m) = 1$ and $h_{x_m}(1_-) = \infty$ hold for every $m \in \mathbb{N}$, there exists a sequence $z_m \in (x_m, 1)$ tending to 1 such that $h_{x_m}(z_m) \to \infty$ as $m \to \infty$. If $x_0 \notin I_3$ and $v_1(x_0) \geq 1$ held, then $v_1(x_0) > 1 > x_0$ or $v_1(x_0) = 1$ and $\dot{h}_{x_0}(1) = h^2_{x_0}(1)$ by lemma 4.5. In particular, we get that $(x_0, 1) \in \mathscr{D}$ holds in both cases and we are able to obtain a contradiction $\infty > h_{x_0}(1) = \lim_{m \to \infty} h_{x_m}(z_m) = \infty$, since $(x, y) \mapsto h_x(y)$ is continuous at $(x_0, 1)$ by remark 4.6. Hence, $x_0 \in I_3$ and therefore $v_1(x_0) = 1$ in case that $v_1(x_0) \geq 1$ and that x_0 is a cluster point of I_3. If x_0 is not a cluster point of I_3 or if $v_1(x_0) < 1$, then $w_3 \geq v_1(x_0)$ obviously holds. It follows from (i)-(iii) that

$$\liminf_{I \ni x \to x_0} v_1(x) \geq w_1 \wedge w_2 \wedge w_3 \geq v_1(x_0).$$

If $x_0 \in J$, it is sufficient to modify (i) and (ii) in order to show that

$$\limsup_{J \ni x \to x_0} v_2(x) \leq \max_{i=1,2} \limsup_{J_i \ni x \to x_0} v_2(x) \leq v_2(x_0).$$

□

Lemma 4.9 *If $x \in I$, then v_1 is an upper semi-continuous function at x. If $x \in J$, then v_2 is a lower semi-continuous function at x.*

Moreover, if x_∞ is such that $\omega_\gamma(x_\infty) = 0$, then $v_i(x) \to x_\infty$ as $I \ni x \to x_\infty$ in case that $i = 1$ and as $J \ni x \to x_\infty$ in case that $i = 2$.

Proof: Similarly as in the proof of lemma 4.8, we focus on the case $x \in I$ and $x = x_\infty > 0$. Let $I \ni x_m \to x$. We are going to show that

$$w_\infty := \limsup_{m \to \infty} v_1(x_m) \leq v_1(x) \tag{37}$$

in case that $x \in I$ and that $w_\infty \leq x$ in case $x = x_\infty$.

(i) If $x \in I_2$, then $v_1(x) = \infty$ and therefore we obtain $w_\infty \leq v_1(x)$ immediately.

In the remaining cases, we will show that $w_\infty > v_1(x)$ leads to a contradiction. Let $\{y_m\}$ be a subsequence of $\{x_m\}$ such that $v_1(y_m) \to w_\infty > v_1(x)$. Since $y_m \to x < v_1(x)$, there exist $m_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that $y_m < v_1(x) - \delta_0$ and $v_1(x) + \delta_0 < v_1(y_m)$ hold for every $m \geq m_0$.

(ii) Now, consider the case $x \in I_3$. Then $v_1(x) = 1$. Since $\dot{h}_{y_m} > h_{y_m}^2$ holds on $(y_m, v_1(y_m)) \ni 1$ for every $m \geq m_0$, we get for every $\delta \in (0, \delta_0)$ that

$$\liminf_{m \to \infty} h_{y_m}(1) \geq \lim_{m \to \infty} h_{y_m}(1 - \delta) = h_x(1 - \delta).$$

Since $h_x(1 - \delta) \to \infty$ as $\delta \to 0^+$, we obtain that $h_{y_m}(1) \to \infty$ as $m \to \infty$. Let $m_1 \geq m_0$ be such that $h_{y_m}(1) > 0$ holds whenever $m \geq m_1$. Since $y_m < 1 < v_1(y_m)$ holds for every $m \geq m_1$, we get that $\dot{h}_{y_m} > h_{y_m}^2$ holds on $(1, v_1(y_m))$ for every $m \geq m_1$ and therefore

$$v_1(y_m) - 1 \leq \int_1^{v_1^m} \frac{\dot{h}_{y_m}(y)}{h_{y_m}^2(y)} dy = \frac{1}{h_{y_m}(1)} - \frac{1}{h_{y_m}(v_{1-}^m)} \leq \frac{1}{h_{y_m}(1)} \to 0$$

as $m_1 \leq n \to \infty$, since $h_{y_m}(v_{1-}^m) \geq h_{y_m}(1) > 0$ holds for every $m \geq m_1$, where $v_1^m := v_1(y_m)$. Hence, we get that $w_\infty = \lim_{m \to \infty} v_1(y_m) \leq 1 = v_1(x)$.

(iii) Consider the case $x \in I_1$ or $x = x_\infty$. By lemma 4.5, $\eta_{h_x}(x) = 0$ and if $x \in I_1$, then $\eta_{h_x}(v_1(x)) = 0$. By lemma 4.7, $\dot{\eta}_{h_x}(v_1(x)) < 0$ in case $x \in I_1$ and therefore there exists $\delta \in (0, \delta_0)$ such that $(x, z_\delta) \in \mathscr{D}$ and $\eta_{h_x}(z_\delta) < 0$, where $z_\delta := v_1(x) + \delta$. Such $\delta \in (0, \delta_0)$ exists also in case $x = x_\infty$ by lemma 4.2. If $m \geq m_0$, then $y_m < z_\delta < v_1(y_m)$ and therefore $(y_m, z_\delta) \in \mathscr{D}$. By lemma 4.7, $\eta_{h_{y_m}}(z_\delta) \to \eta_{h_x}(z_\delta) < 0$ as $m \to \infty$ and therefore there exists $m_1 \geq m_0$ such that $\eta_{h_{y_m}}(z_\delta) < 0$ holds for every $\geq m_1$. Hence, we obtain $\dot{h}_{y_m}(z_\delta) < h_{y_m}^2(z_\delta)$ for every $m \geq m_1$ which is a contradiction with lemma 4.5 (i) as $v_1(y_m) > z_\delta > y_m$.

(iv) We are to show the moreover part. By (37) with $v_1(x)$ replaced by x_∞, we get that $x_\infty = \lim x \leq \limsup v_1(x) \leq x_\infty$ as $I \ni x \to x_\infty$.

The proof of the remaining part of the statement can be obtained by modifying (i), (iii) and (iv). □

Corollary 4.10 *The function v_1 is a continuous on I and v_2 is continuous on J. In particular, I_2 is a closed set in I and J_2 in J. Further, I_3 is a closed set in I by the moreover part of lemma 4.8.*

Lemma 4.11 *The set $I_3 \cup I_4$ is relatively closed in I and J_4 in J, where*

$$I_4 := \{x \in I_1, h_x(v_1(x)) \geq \vartheta_-(v_1(x)) \,\&\, v_1(x) < 1/c\},$$
$$J_4 := \{x \in J_1, h_x(v_2(x)) \leq -\vartheta_+(v_2(x)) \,\&\, v_2(x) > -1/b\}.$$

Proof: By corollary 4.10, I_3 is a closed set in I. Let $I \ni x \leftarrow x_m \in I_4$ as $m \to \infty$, then $v_1(x) = \lim_{m\to\infty} v_1(x_m) \leq 1/c < \infty$ and therefore $x \notin I_2$. Since I is a disjoint union of sets I_1, I_2, I_3 and $I_3 \subseteq I_3 \cup I_4$, we are to show that $x \in I_4$ provided that $x \in I_1$. Let $x \in I_1$. If $v_1(x) \neq 1/c$, then $v_1(x) < 1/c$ and we obtain from continuity of

$$\mathscr{I} : y \in \{\tilde{y} \in I_1; v_1(\tilde{y}) < 1/c\} \mapsto h_y(v_1(y)) - \vartheta_-(v_1(y)) \tag{38}$$

at $y = x$ that $\mathscr{I}(x) = \lim_{m\to\infty} \mathscr{I}(x_m) \geq 0$ and therefore $x \in I_4$. Now, we are to show that $v_1(x) = 1/c$ is not possible. Since $x \in I_1$, we get a contradiction

$$\infty > h_x(v_1(x)) = \lim_{m\to\infty} h_{x_m}(v_1(x_m)) \geq \lim_{m\to\infty} \vartheta_-(v_1(x_m)) = \infty$$

in case $v_1(x) = 1/c$. The proof that J_4 is relatively closed in J would be quite similar. □

Lemma 4.12 *The set $I_2 \cup I_5$ is relatively closed in I and $J_2 \cup J_5$ in J, where*

$$I_5 := \{x \in I_1, h_x(v_1(x)) \leq \vartheta_-(v_1(x)) \quad or \quad v_1(x) \geq 1/c\}$$
$$J_5 := \{x \in J_1, h_x(v_2(x)) \geq -\vartheta_+(v_2(x)) \quad or \quad v_2(x) \leq -1/b\}.$$

Proof: By corollary 4.10, I_2 is a closed set in I. Let $I \ni x \leftarrow x_m \in I_5$. We are to show that $x \in I_5$ provided that $x \notin I_2$. If $v_1(x) \geq 1/c$, we have that $v_1(x) \neq 1$ and therefore $x \notin I_3$. Hence, $x \in I \setminus (I_2 \cup I_3) \subseteq I_1$ is such that $v_1(x) \geq 1/c$ and therefore $x \in I_5$. If $v_1(x) < 1/c$, we obtain from the continuity of v_1 that the same inequality holds for x_m if m is large enough, and we obtain from continuity of $\mathscr{I}$ given by (38) that $0 \geq \lim_{m\to\infty} \mathscr{I}(x_m) = \mathscr{I}(x)$ and therefore $x \in I_5$. The proof that $J_2 \cup J_5$ is closed in J would be similar. □

Lemma 4.13 *(i) Let $\theta_\gamma > 0$, then $I = (\underline{x}, \bar{x}) \neq \emptyset$, where*

$$\bar{x} = \infty \quad if \quad \theta_\gamma \geq 1 + 1/b, \qquad \bar{x} = \xi_+^{-1}(\theta_\gamma) \quad if \quad \theta_\gamma < 1 + 1/b,$$
$$\underline{x} = 0 \quad if \quad \theta_\gamma = \sqrt{\aleph_\gamma}, \quad \underline{x} = \xi_+^{-1}(\theta_\gamma - \sqrt{\aleph_\gamma}) \quad if \quad \theta_\gamma > \sqrt{\aleph_\gamma}.$$

Let $\theta_\gamma < 0$, *then* $J = (\underline{x}, 0) \neq \emptyset$, *where*

$$\underline{x} = -\infty \quad \text{if} \quad \theta_\gamma \leq 1 - 1/c, \quad \underline{x} = \xi_-^{-1}(\theta_\gamma) \quad \text{if} \quad 1 - 1/c < \theta_\gamma < 0.$$

(ii) If $\theta_\gamma > 0$, *then* $I_2 \cup I_5 \neq \emptyset$. *If* $\theta_\gamma < 0$, *then* $J_2 \cup J_5 \neq \emptyset$.

Proof: Let $\theta_\gamma > 0$. By the definition, I is an intersection of two open intervals. The first one corresponds to the conditions $x > 0$, $0 < \omega_\gamma(x)$ and it is of the form $(0, \bar{x})$, where $\bar{x} = \infty$ in case that $\theta_\gamma \geq 1 + 1/b$, i.e. in case that $\theta_\gamma > x\frac{1+b}{1+bx}$ holds for every $x \in (0, \infty)$, and where $\bar{x} \in (0, \infty)$ is the unique solution of $\theta_\gamma = x\frac{1+b}{1+bx}$ on $(0, \infty)$ in case that $\theta_\gamma < 1 + 1/b$. The second interval corresponds to the condition $\omega_\gamma(x) < \sqrt{\aleph_\gamma}$ and it is of the form $(\underline{x}, \infty)$, where $\underline{x} = 0$ in case $\theta_\gamma \leq [(1 - \theta_\gamma)^2 + \kappa_\gamma^2]^{1/2}$, i.e. in case that $\theta_\gamma = \sqrt{\aleph_\gamma}$, and where $\underline{x} \in (0, \infty)$ is the unique solution of the equation

$$\omega_\gamma(x) = \theta_\gamma - x\frac{1 + b}{1 + bx} = \sqrt{(1 - \theta_\gamma)^2 + \kappa_\gamma^2} = \sqrt{\aleph_\gamma}$$

on $(0, \infty)$ in case that $\theta_\gamma > [(1 - \theta_\gamma)^2 + \kappa_\gamma^2]^{1/2} = \sqrt{\aleph_\gamma}$. Since

$$\theta_\gamma - (1 + 1/b) < \theta_\gamma - 1 \leq \sqrt{(1 - \theta_\gamma)^2 + \kappa_\gamma^2} = \sqrt{\aleph_\gamma},$$

we obtain that $\theta_\gamma - \sqrt{\aleph_\gamma} < 1 + 1/b$ in case that $\theta_\gamma > \sqrt{\aleph_\gamma}$ and therefore

$$\underline{x}\frac{1 + b}{1 + b\underline{x}} = \theta_\gamma - \sqrt{\aleph_\gamma} < \theta_\gamma \wedge [1 + 1/b] = \bar{x}\frac{1 + b}{1 + b\bar{x}}.$$

Since $x \in [0, \infty] \mapsto x\frac{1+b}{1+bx}$ is an increasing function, we obtain that $\underline{x} < \bar{x}$ and therefore $I = (\underline{x}, \bar{x}) \neq \emptyset$.

(ii) If $\theta_\gamma \geq 1 + 1/b$, we put $\check{x} := 1 + (1/c \vee \underline{x})$. Then $v_1(\check{x}) > \check{x} \geq 1/c$ and so $\check{x} \in I \setminus (I_3 \cup I_4) \subseteq I_2 \cup I_5$.

Now, consider the case when $\theta_\gamma < 1 + 1/b$. Then $\bar{x} < \infty$ and $\omega_\gamma(x) \downarrow 0 = \omega_\gamma(\bar{x})$ as $x \uparrow \bar{x}$. By the moreover part of lemma 4.9, $v_1(x) \to \bar{x}$ as $I \ni x \to \bar{x}$. Let $I \ni x_m \to \bar{x}$ and $z_m \in (x_m, v_1(x_m))$ be sequences. Then $z_m \to \bar{x}$ and therefore $0 > -\vartheta_+(\bar{x}) = h_{\bar{x}}(\bar{x}) \leftarrow h_{x_m}(z_m)$ as $m \to \infty$. Since $z_m \in (x_m, v_1(x_m))$ was arbitrary sequence, we get that $h_{x_m}(v_1(x_m)_-) < 0$ holds for m large enough and therefore $x_m \in I_2 \cup I_5$.

The proof of the part of the statement corresponding to the case $\theta_\gamma < 0$ would be similar and a little bit more simple, since $(1 - \theta_\gamma)^2 + \kappa_\gamma^2 > \theta_\gamma^2$ in this case. □

Let $x \in \mathbb{R}$ and f be C^2 in a neighbourhood of x. We denote

$$(\mathscr{D}f)(\mathrm{x}) := \frac{1}{2}\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1[\ddot{f}(x) - \gamma\dot{f}(x)^2] + \mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x})\dot{f}(x) - \mathrm{d}(\mathrm{x}),$$

where $\mathrm{x} = (x, \tilde{x}')'$ and where $\mathbb{S}(\mathrm{x})$, $\tilde{\mathbb{B}}(\mathrm{x})$, $\mathrm{d}(\mathrm{x})$ are defined by (3) and (6). If $\eta_{\dot{f}}(x) + [1 + x\dot{f}(x)]^2 \neq 0$, we define

(39) $$e_x := \frac{\eta_f(x) + x\dot{f}(x)[1 + x\dot{f}(x)]}{\eta_f(x) + [1 + x\dot{f}(x)]^2} \quad \text{and put}$$

(40) $$(\mathscr{D}_e f)(x) = \frac{1}{2}S^2(x)[\ddot{f}(x) - \gamma\dot{f}(x)^2] + \tilde{B}(x)\dot{f}(x) - d(x),$$

where $S^2(x)$, $\tilde{B}(x)$, $d(x)$ are defined by (21), (22), (23).

Lemma 4.14 *Let $c \in (0,1)$; b, $\kappa_\gamma^2 > 0$ and $\theta_\gamma \notin \{0, 1 + \kappa_\gamma^2\}$. Then there exist*

(41) $$f \in C^2(-1/b, 1/c),\ \varepsilon : x \in [\alpha,\beta] \subseteq (-1/b, 1/c) \mapsto \varepsilon_x \in \mathbb{R}^{n-1},$$

(42) $$\hat{v} > \frac{1-\gamma}{2}\{\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \tilde{\sigma}^2[\theta_\gamma^2 - ((\theta_\gamma - 1)^2 + \kappa_\gamma^2)] \vee 0\} =: v_0$$

such that

(i) $\alpha > \inf I$ if $\theta_\gamma > 0$ and $\beta < 0 = \sup J$ in case that $\theta_\gamma < 0$,

(ii) $\dot{f}(x) = -\vartheta_+(x)$ holds on $(-1/b, \alpha]$, $\dot{f}(x) = \vartheta_-(x)$ on $[\beta, 1/c)$,

(iii) $(\mathscr{D}f)(\mathrm{x}) + \hat{v} = 0$ holds for every $\mathrm{x} = (x, \hat{x}')'$ such that $x \in [\alpha,\beta]$ and $\hat{x} = \hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx + \varepsilon_x$.

Proof: We choose ε_x in order to satisfy the following equation $\hat{x}_x :=$ $:= \hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx + \varepsilon_x = (1 - x)\hat{\theta}_\gamma$. Denote $\bar{\sigma}^2 := \tilde{\sigma}^2(1 + \kappa_\gamma^2)$ and $\bar{\theta}_\gamma := \theta_\gamma/(1 + \kappa_\gamma^2)$. Then $\bar{\theta}_\gamma \notin \{0,1\}$ and

(43) $$\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1 = \bar{\sigma}^2x^2(1-x)^2 =: \mathscr{S}^2(x),$$

(44) $$\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x}) = (1-\gamma)x(1-x)\bar{\sigma}^2[\bar{\theta}_\gamma - x] =: \tilde{\mathscr{B}}(x),$$

(45) $$\mathrm{d}(\mathrm{x}) = \frac{1-\gamma}{2}\{\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \bar{\sigma}^2[2\bar{\theta}_\gamma x - x^2]\} =: d_0(x)$$

if $\mathrm{x} = (x, \hat{x}_x)$. Since $\mathscr{S}^2$, $\tilde{\mathscr{B}}$ and $d_0 - \frac{1-\gamma}{2}\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma$ are of a special form, we obtain by [4, corollary 6.5] that there exist $\bar{\omega}_\gamma \in (0, |\bar{\theta}_\gamma| \wedge |1 - \bar{\theta}_\gamma|)$ such that $-1/b < \alpha :=$ $:= \xi_+^{-1}(\bar{\theta}_\gamma - \bar{\omega}_\gamma) < \xi_-^{-1}(\bar{\theta}_\gamma + \bar{\omega}_\gamma) =: \beta < 1/c$, that $0,1 \notin [\alpha,\beta]$ and $f \in C^2(-1/b, 1/c)$ such that $\delta_+^f(y) = \delta_-^f(z) = d_f^v(x) = 0$ hold whenever $y \in (-1/b, \alpha]$, $z \in [\beta, 1/c)$, $x \in [\alpha,\beta]$ and that $\delta_+^f(y) < 0$, $\delta_-^f(z) < 0$, $d_f^v(x) < 0$ whenever $x \in (-1/b, 1/c) \setminus [\alpha,\beta]$, $y \in (\alpha, 1/c)$, $z \in (-1/b, \beta)$ with $v = \frac{1-\gamma}{2}\bar{\sigma}^2(\bar{\theta}_\gamma^2 - \bar{\omega}_\gamma^2)$. The conditions (ii) and (iii) are obviously satisfied with $\hat{v} := \frac{1-\gamma}{2}\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + v > v_0$. If $\theta_\gamma < 0$, then $\bar{\theta}_\gamma < 0$ and $\bar{\omega}_\gamma \in (\bar{\theta}_\gamma, 0)$ and we get that $\beta = \xi_-^{-1}(\bar{\theta}_\gamma + \bar{\omega}_\gamma) < \xi_-^{-1}(0) = 0 = \sup J$, since ξ_- is an increasing function on $(-\infty, 1/c)$. Let $\theta_\gamma > 0$. We will show that $\bar{\theta}_\gamma - \bar{\omega}_\gamma > \theta_\gamma - \sqrt{\aleph_\gamma}$ and then will we obtain that $\alpha = \xi_+^{-1}(\bar{\theta}_\gamma - \bar{\omega}_\gamma) >$ $> \xi_+^{-1}(\theta_\gamma - \sqrt{\aleph_\gamma}) = \inf I$, since ξ_+ is an increasing function on $(-1/b, \infty) \ni \alpha$.

If $\bar{\theta}_\gamma \leq \frac{1}{2}$, then $\aleph_\gamma = \theta_\gamma^2 \leq (1 - \theta_\gamma)^2 + \kappa_\gamma^2$ and therefore the desired inequality obviously holds, since $\theta_\gamma = \sqrt{\aleph_\gamma}$ in this case. If $\bar{\theta}_\gamma > \frac{1}{2}$, then $\aleph_\gamma = (1 - \theta_\gamma)^2 + \kappa_\gamma^2$ and $\aleph_\gamma - (\theta_\gamma + (1 - 2\bar{\theta}_\gamma))^2 = \kappa_\gamma^2(1 - 2\bar{\theta}_\gamma)^2 \geq 0$. If $\bar{\theta}_\gamma \in (\frac{1}{2}, 1)$, then we get from the previous inequality that $\theta_\gamma - \sqrt{\aleph_\gamma} \leq 2\bar{\theta}_\gamma - 1 = \bar{\theta}_\gamma - |1 - \bar{\theta}_\gamma| < \bar{\theta}_\gamma - \bar{\omega}_\gamma$. If $\bar{\theta}_\gamma \geq 1$, then $\bar{\theta}_\gamma - \bar{\omega}_\gamma > \bar{\theta}_\gamma - |\bar{\theta}_\gamma - 1| = 1 \geq \theta_\gamma - \sqrt{\aleph_\gamma}$, since $\aleph_\gamma \geq (1 - \theta_\gamma)^2$. □

Lemma 4.15 *Let $b > 0$, $c \in (0,1)$ and $-1/b < \alpha < \beta < 1/c$ be such that $0 \notin [\alpha, \beta]$. Further, assume that $\alpha = x_0$, $\beta = v_1(\alpha)$, $h_{x_0}(\beta) = \vartheta_-(\beta)$ if $\theta_\gamma > 0$ and that $x_0 = \beta$, $\alpha = v_2(\beta)$, $h_{x_0}(\alpha) = -\vartheta_+(\alpha)$ if $\theta_\gamma < 0$. Then there exists $f \in C^2(-1/b, 1/c)$ such that $\dot{f}(x) = h_{x_0}(x)$ on (α, β) and*

(i) $\dot{f}(x) = -\vartheta_+(x)$ on $(-1/b, \alpha]$, $\dot{f}(x) = \vartheta_-(x)$ on $[\beta, 1/c)$

(ii) $1 + x\dot{f}(x) > 0$ holds on $(-1/b, 1/c)$ and

$$G(x, \dot{f}(x), \eta_{\dot{f}}(x)) \geq \omega_\gamma^2(x_0), \quad \ddot{f}(x) \geq \dot{f}(x)^2 \tag{46}$$

hold on $(-1/b, 1/c)$ so that the left-hand inequality is strict if and only if $x \notin [\alpha, \beta]$ and the right-hand one if and only if $x \in (\alpha, \beta)$. Further,

$$(\mathscr{D}_e f)(x) + \tilde{v} = \tilde{\sigma}^2 \frac{1-\gamma}{2}\left[G(x, \dot{f}(x), \eta_{\dot{f}}(x)) - \omega_\gamma^2(x_0)\right] \geq 0 \tag{47}$$

holds for every $x \in (-1/b, 1/c)$, where

$$\tilde{v} := \frac{1-\gamma}{2}\{\hat{\theta}_\gamma' \hat{\Sigma} \hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2(x_0))\}. \tag{48}$$

Remark 4.16 (i) The reader can see from the proof of lemma 4.15 that $\xi_-(\beta) = \theta_\gamma + \omega_\gamma(x_0)$ in case that $c > 0$ and $\theta_\gamma > 0$. Similarly, we would have that $\xi_+(\alpha) = \theta_\gamma - \omega_\gamma(x_0)$ in case that $b > 0$ and $\theta_\gamma < 0$.

(ii) If $\theta_\gamma > 0$ and $c \leq 0$ or if $\theta_\gamma < 0$ and $b \leq 0$, then the statement of lemma 4.15 remains valid and the proof correct if we place $1/c$ by ∞ in case $\theta_\gamma > 0$ and $-1/b$ by $-\infty$ in case $\theta_\gamma < 0$ provided that $b + c > 0$.

(iii) The statement of lemma 4.15 remains valid also in case that $\theta_\gamma > 0$ and $\alpha \in I_2$ or if $\theta_\gamma < 0$ and $\beta \in J_2$ provided that we remove the assumptions $h_{x_0}(\beta) = \vartheta_-(\beta)$, $\beta < 1/c$ and we replace $1/c$ by ∞ in case that $\theta_\gamma > 0$ and we remove assumptions $h_{x_0}(\alpha) = -\vartheta_+(\alpha)$, $\alpha > -1/b$ and that we replace $-1/b$ by $-\infty$ in case that $\theta_\gamma < 0$. It is very easy to modify the proof of lemma 4.15 in order to obtain the corresponding proof and therefore it is left to the reader.

Proof of lemma 4.15: We are going to define f as a primitive function to some $h \in C^1(-1/b, 1/c)$. The condition (i) tells us, how to define h on $(-1/b, 1/c) \setminus (\alpha, \beta)$. If $x \in [\alpha, \beta]$, we put $h(x) := h_{x_0}(x)$. The assumptions of lemma 4.15 ensure that the definition is correct and that $h \in C(-1/b, 1/c)$. To show that $h \in C^1(-1/b, 1/c)$, we need to verify that $\dot{h}(\alpha_+) = \dot{h}(\alpha_-)$ and $\dot{h}(\beta_+) = \dot{h}(\beta_-)$. It follows from assumptions that $x_0 \in I_1$ or $x_0 \in J_1$, respectively. By lemma 4.5 and lemma 4.7, $\eta_{h_{x_0}}(y) = 0$ holds at each $y \in \{\alpha, \beta\}$ and we get that

$$\dot{h}(\alpha_+) = \dot{h}_{x_0}(\alpha_+) = h_{x_0}(\alpha)^2 = \vartheta_+^2(\alpha) = -\dot{\vartheta}_+(\alpha) = \dot{h}(\alpha_-)$$
$$\dot{h}(\beta_-) = \dot{h}_{x_0}(\beta_-)^2 = h_{x_0}(\beta)^2 = \vartheta_-^2(\beta) = +\dot{\vartheta}_-(\beta) = \dot{h}(\beta_+).$$

It follows from lemma 4.5 that $\ddot{f}(x) > \dot{f}(x)^2$ and that $1 + x\dot{f}(x) > 0$ hold on (α, β), see the definition of Φ containing all points of the form $(x, h_{x_0}(x), \omega_\gamma(x_0))$ and the definition of $\dot{f}(x)$ for $x \in (\alpha, \beta)$. The equality $\ddot{f}(x) = \dot{f}(x)^2$ and the inequality $1 + x\dot{f}(x) > 0$ on $(-1/b, 1/c) \setminus (\alpha, \beta)$ immediately follows from the definition of $\dot{f}$ on this set. If $\tilde{x} \in [\alpha, \beta]$, then h solves the equation $\dot{h}(x) = \varphi(x, h(x), \omega_\gamma(x_0))$ at $\tilde{x}$ and therefore $\eta_h(x) = \phi(x, h(x), \omega_\gamma(x_0))$ holds at $x = \tilde{x}$. Hence, the equality sign holds in the left-hand inequality in (46) for every $\tilde{x} \in [\alpha, \beta]$. Now, we are going to show that

$$G(x, \dot{f}(x), \eta_f(x)) = (\theta_\gamma - \xi(x, \dot{f}(x)))^2 > \omega_\gamma^2(x_0), \tag{49}$$

hold whenever $x \in (-1/b, 1/c) \setminus [\alpha, \beta]$. The left-hand equality in (49) holds on $(-1/b, 1/c) \setminus (\alpha, \beta)$, since $\eta_f = 0$ there. Moreover, $\xi(x, \dot{f}(x)) = \xi_+(x)$ if $x \in (-1/b, \alpha]$ and $\xi(x, \dot{f}(x)) = \xi_-(x)$ if $x \in [\beta, 1/c)$ and therefore it is sufficient to show that

$$\xi_+(x) > \xi_+(\alpha) = \theta_\gamma - \omega_\gamma(x_0), \quad \theta_\gamma + \omega_\gamma(x_0) = \xi_-(\beta) < \xi_-(y) \tag{50}$$

hold, whenever $-1/b < x < \alpha$ and $\beta < y < 1/c$ in order to verify the right-hand inequality in (49). By lemma 4.5 (i), $\eta_f > 0$ holds on (α, β). By lemma 4.2, $(\theta_\gamma - \xi(z, \dot{f}(z)))^2 = \omega_\gamma^2(x_0) \neq 0$ and sign $\dot{\eta}_f(z) = \operatorname{sign}[\theta_\gamma - \xi(z, \dot{f}(z))] \neq 0$ holds for every $z \in \{\alpha, \beta\}$. Since $\dot{\eta}_f$ cannot be negative or zero at α and positive or zero at β, we get the equalities in (50). The inequalities follows immediately, since ξ_+, ξ_- are increasing functions on $(-1/b, 1/c)$. Now, it remains to show that the left-hand equality in (47) holds, but it follows from remark 4.17 (see below) and (39), since $e_x - x(1 - e_x)\dot{f}(x) = \eta_f(x)/(\eta_f(x) + [1 + x\dot{f}(x)]^2)$. □

Remark 4.17 If $v = \frac{1-\gamma}{2}\{\hat{\theta}_\gamma' \Sigma \hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2)\}$ and f is C^2 in the neighbourhood of x and $\eta_f(x) + [1 + x\dot{f}(x)]^2 \neq 0$, then

$$(\mathscr{D}_e f)(x) + v = \frac{1}{2} S^2(x)[\ddot{f}(x) - \dot{f}(x)^2]$$
$$+ \frac{1-\gamma}{2} \tilde{\sigma}^2 \{[\theta_\gamma - \xi(x, \dot{f}(x))]^2 + \kappa_\gamma^2 \zeta(x, e_x, \dot{f}(x))^2 - \omega_\gamma^2\},$$

where $\zeta(x, e, h) = e - x(1 - e)h$ and where e_x is given by (39) and $S^2(x)$ by (21).

Lemma 4.18 *Let* $b > 0$, $c \in (0, 1)$ *and* $x_0 \in (-1/b, 1/c)$ *be such that* $x \in I_2 \cup I_5$ *if* $\theta_\gamma > 0$ *and* $x_0 \in J_2 \cup J_5$ *if* $\theta_\gamma < 0$. Let $\alpha = x_0$, $\beta = v_1(\alpha)$ *if* $\theta_\gamma > 0$ *and* $\beta = x_0$, $\alpha = v_2(\beta)$ *if* $\theta_\gamma < 0$. *Then* $h_{x_0} \in (-\vartheta_+(x), \vartheta_-(x))$ *holds for every* $x \in (\alpha, \beta) \cap \cap (-1/b, 1/c)$.

Further, $z(\beta) \in (-b, c)$ *if* $x_0 \in I_5 \setminus I_4$ *and* $z(\alpha) \in (-b, c)$ *if* $x_0 \in J_5 \setminus J_4$, *where* z *is given by formula* (51).

Proof: We focus on the case $\theta_\gamma > 0$. It follows from the definition of h_{x_0}, lemma 4.5 and the definition of Φ that $1 + xh_{x_0}(x) > 0$ holds for every $x \in (\alpha, \beta)$. Further, it follows from the properties of h_{x_0} that $h_{x_0}(\alpha) = -\vartheta_+(\alpha)$ and therefore we obtain for every $x \in (\alpha, \beta)$ that $z(x) = z(\alpha) + \int_\alpha^x \dot{z}(y)\,dy > z(\alpha) = -b$, where

$$(51) \qquad z(y) := \frac{h_{x_0}(y)}{1 + yh_{x_0}(y)} \quad \text{and} \quad \dot{z}(y) = \frac{\dot{h}_{x_0}(y) - h_{x_0}^2(y)}{(1 + yh_{x_0}(y))^2} > 0$$

holds for every $y \in (\alpha, \beta)$ by lemma 4.5 (i). If $x_0 \in I_1$, then $(x_0, \beta) \in \mathscr{D}$ and $(x_0, \beta, \omega_\gamma(x_0)) \in \Phi$. In particular, h_{x_0} is defined at β and $1 + \beta h_{x_0}(\beta) > 0$ and the same argument as for $x \in (\alpha, \beta)$ can be used for $x = \beta$ in order to obtain that $z(\beta) > -b$. A straightforward computation using $1 + xh_{x_0}(x) > 0$ gives that $h_{x_0}(x) > -\vartheta_+(x)$ holds for every $x \in (\alpha, \beta) \cap (-1/b, 1/c)$ and a similar computation shows that $z(x) < c$ holds if and only if $h_{x_0}(x) < \vartheta_-(x)$ whenever $x \in (\alpha, \beta) \cap$ $\cap (-1/b, 1/c)$. If $x_0 \in I_1$ and $\beta < 1/c$, we obtain by the same way that the same and the following equivalence $z(x) \leq c \equiv h_{x_0}(x) \leq \vartheta_-(x)$ holds also for $x = \beta$.

If $\beta < 1/c$, we obtain from the assumption $x_0 \in I_2 \cup I_5$ that $x_0 \in I_5 \subseteq I_1$ and that $h_{x_0}(\beta) \leq \vartheta_-(\beta)$. Then $z(\beta_-) = z(\beta) \leq 1/c$. Since z is an increasing function on (α, β), we get that $z(x) < z(\beta_-) \leq 1/c$ and therefore $h_{x_0}(x) < \vartheta_-(x)$ holds for every $x \in (\alpha, \beta) \subseteq (-1/b, 1/c)$.

Let $\beta \geq 1/c$. We recall that h_{x_0} is an increasing function on (α, β), since it satisfies $\dot{h}_{x_0} > h_{x_0}^2$ there. If $h_{x_0}(\beta_-) \leq 0$, then we get $h_{x_0}(x) \leq 0 < \vartheta_-(x)$ on $(\alpha, 1/c) = (\alpha, \beta) \cap (-1/b, 1/c)$ immediately and it covers the case when $x_0 \in I_2$. Hence, we can assume that $x_0 \in I_5 \subseteq I_1$ and that $h_{x_0}(\beta) = h_{x_0}(\beta_-) > 0$. Then we obtain that

$$z(x) < z(\beta) = \frac{h_{x_0}(\beta)}{1 + \beta h_{x_0}(\beta)} < \frac{h_{x_0}(\beta)}{\beta h_{x_0}(\beta)} = \frac{1}{\beta} \leq c$$

holds for every $x \in (\alpha, \beta) \cap (-1/b, 1/c)$. □

Lemma 4.19 *Let* $b > 0$, $c \in (0, 1)$. *Let* $x_0 \in (-1/b, 1/c)$ *be such that* $x_0 \in I \setminus (I_3 \cup I_4)$ *if* $\theta_\gamma > 0$ *and* $x_0 \in J \setminus J_4$ *if* $\theta_\gamma < 0$. *Then there exists* $f \in C^2(-1/b, 1/c)$ *such that*

(i) $\mathscr{D}_c f + \tilde{v} \geq 0$ *holds on* $(-1/b, 1/c)$, *where* $\tilde{v}$ *is given by* (48),

(ii) $\dot{f} = -\vartheta_+$ *on* $(-1/b, x_0]$ *if* $\theta_\gamma > 0$, $\dot{f} = \vartheta_-$ *on* $[x_0, 1/c)$ *if* $\theta_\gamma < 0$,

(iii) $\dot{f} \in (-\vartheta_+, \vartheta_-)$ *on* $(x_0, 1/c)$ *if* $\theta_\gamma > 0$ *and on* $(-1/b, x_0)$ *if* $\theta_\gamma < 0$.

Proof: We focus on the case $\theta_\gamma > 0$. Then $x_0 \in (-1/b, 1/c) \cap [I \setminus (I_3 \cup I_4)]$. If $x_0 \in I_2$, we apply remark remark 4.16 (iii) in order to obtain $f \in C^2(-1/b, \infty)$ such that $\dot{f} = -\vartheta_+$ holds on $(-1/b, \alpha]$, that (47) hold on $(-1/b, \infty)$ and $\dot{f}(x) = h_{x_0}(x)$ on (α, ∞). Then (i), (ii) are satisfied and (iii) follows from lemma 4.18.

Let $x_0 \in I_1$. Then $x_0 \in I_1 \setminus (I_3 \cup I_4) \subseteq I_5 \setminus I_4$. By lemma 4.18, $h_{x_0}(x) \in (-\vartheta_+(x), \vartheta_-(x))$ holds for every $x \in (\alpha, \beta) \cap (-1/b, 1/c)$ and $\tilde{c} := z(\beta) = h_{x_0}(\beta)/(1 + \beta h_{x_0}(\beta)) \in$ $\in (-b, c)$. Then $h_{x_0}(\beta) = \tilde{c}/(1 - \tilde{c} h_{x_0}(\beta))$ and we obtain from lemma 4.15 or remark 4.16 (ii) with c replaced by $\tilde{c}$ that $(h_{x_0}(x), x \in \alpha, \beta))$ can be extended to a function with a primitive function $f \in C^2(-1/b, 1/c)$ satisfying (i) and (ii) in the statement of lemma 4.15, where $1/\tilde{c}$ is replaced by ∞ in case that $\tilde{c} \leq 0$. Then (i), (ii) are satisfied and we get that $\dot{f}(x) = h_{x_0}(x) \in (-\vartheta_+(x), \vartheta_-(x))$ holds for every

$x \in (\alpha, \beta)$. If $x \in [\beta, 1/c)$, then $\dot{f}(x) = \tilde{c}/(1 - \tilde{c}x) \in (-\vartheta_+(x), \vartheta_-(x))$, since $\tilde{c} \in (-b, c)$ and therefore (iii) is verified. □

Lemma 4.20 *Let* $x \in (-1/b, 1/c)$, *f be C^2 in a neighbourhood of x. Let* $\mathrm{x} = (x, \mathcal{X}')' \in \mathbb{R}^n$ *be such that* $1 + x\dot{f}(x) > 0$ *and* $\ddot{f}(x) \geq \dot{f}(x)^2$.

Then $(\mathcal{D}f)(\mathrm{x}) \geq (\mathcal{D}_e f)(x)$ *and both sides are equal to each other if and only if* $\mathcal{X} = \hat{\theta}_y - \hat{\Sigma}^{-1}Lx + (\hat{\Sigma}^{-1}L - \hat{\theta}_y)e_x$, *where e_x is given by* (39).

Proof: By assumptions, $\eta_f(x) + [1 + x\dot{f}(x)]^2 > 0$ and therefore e_x is defined correctly. We introduce $\tilde{\varepsilon}_x$ uniquely defined by the following equation $\mathcal{X} = \hat{\theta}_y - \hat{\Sigma}^{-1}Lx + (\hat{\Sigma}^{-1}L - \hat{\theta}_y)e_x + \tilde{\varepsilon}_x$. Then

$$\frac{1}{2}\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1 = \frac{1}{2}S^2(x) + \tilde{\varepsilon}_x'\frac{\partial}{\partial\varepsilon}\frac{1}{2}\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1 + \frac{1}{2}x^2\tilde{\varepsilon}_x'\hat{\Sigma}\tilde{\varepsilon}_x,$$

$$\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x}) = \tilde{B}(x) + \tilde{\varepsilon}_x'\frac{\partial}{\partial\varepsilon}\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x}) + (1 - \gamma)x\tilde{\varepsilon}_x'\hat{\Sigma}\tilde{\varepsilon}_x,$$

$$\mathrm{d}(\mathrm{x}) = d(x) + \tilde{\varepsilon}_x'\frac{\partial}{\partial\varepsilon}\mathrm{d}(\mathrm{x}) - \frac{1 - \gamma}{2}\tilde{\varepsilon}_x'\hat{\Sigma}\tilde{\varepsilon}_x,$$

where $\frac{\partial}{\partial\varepsilon}\frac{1}{2}\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1$, $\frac{\partial}{\partial\varepsilon}\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x})$, $\frac{\partial}{\partial\varepsilon}\mathrm{d}(\mathrm{x})$ stand for the following expressions $\frac{1}{2}\frac{\partial S^2}{\partial\varepsilon}(x)$, $\frac{\partial \tilde{B}_1}{\partial\varepsilon}(x)$, $\frac{\partial d}{\partial\varepsilon}(x)$ given by (15) and (16), respectively. Since e_x satisfies (20), we have that

$$\tilde{\varepsilon}_x'\left[(\ddot{f}(x) - \gamma\dot{f}(x)^2)\frac{\partial}{\partial\varepsilon}\frac{1}{2}\mathrm{e}_1'\mathbb{S}(\mathrm{x})\mathbb{S}(\mathrm{x})'\mathrm{e}_1 + \dot{f}(x)\frac{\partial}{\partial\varepsilon}\mathrm{e}_1'\tilde{\mathbb{B}}(\mathrm{x}) + \frac{\partial}{\partial\varepsilon}\mathrm{d}(\mathrm{x})\right]$$

is equal to zero and therefore $(\mathcal{D}f)(\mathrm{x}) - (\mathcal{D}_e f)(x)$ is equal to

$$\frac{1 - \gamma}{2}\tilde{\varepsilon}_x'\hat{\Sigma}\tilde{\varepsilon}_x\{\eta_f(x) + [\dot{f}(x)x + 1]^2\} \geq 0. \tag{52}$$

Since the expression in braces is positive, we obtain that (52) is equal to zero if and only if $\tilde{\varepsilon}_x'\hat{\Sigma}\tilde{\varepsilon}_x = 0$ which happens if and only if $\tilde{\varepsilon}_x = 0$, since $\hat{\Sigma}$ is a positively definite matrix. □

Lemma 4.21 *Let* $\kappa_\gamma^2 > 0$ *and* $\theta_\gamma \notin \{0, 1 + \kappa_\gamma^2\}$. *If* $\theta_\gamma > 0$, *then* $I_3 \cup I_4 \neq \emptyset$. *Further,* $J_4 \neq \emptyset$ *in case that* $\theta_\gamma < 0$.

Proof: Let $I_3 \cup I_4 = \emptyset$ if $\theta_\gamma > 0$ or $J_4 = \emptyset$ if $\theta_\gamma < 0$. We will show hat this assumption leads to a contradiction. By lemma 4.14, there exist ϑ satisfying (42) and f, ε satisfying (41) and

$$0 \leq \xi_+^{-1}(\theta_\gamma - \sqrt{\aleph_\gamma}) = \underline{x} < \alpha < \beta < 1/c \text{ if } \theta_\gamma > 0$$

and $-1/b < \alpha < \beta < 0 = \sup J$ if $\theta_\gamma < 0$ such that (ii), (iii) in lemma 4.14 hold. Further, $0 < \omega_\gamma(x) \uparrow \sqrt{\aleph_\gamma}$ as $x \downarrow \underline{x}$ if $\theta_\gamma > 0$ and as $x \uparrow 0$ if $\theta_\gamma < 0$ and therefore we get that there exists $x_0 \in (\underline{x}, \alpha)$ in case $\theta_\gamma > 0$ and $x_0 \in (\beta, 0)$ in case $\theta_\gamma < 0$ such that

$$\tilde{v} := \frac{1-\gamma}{2}\{\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2(x_0))\} \in (v_0, \hat{v}).$$

By lemma 4.19, there exists $\tilde{f} \in C^2(-1/b, 1/c)$ such that

$$\dot{\tilde{f}} = -\vartheta_+ \text{ on } (-1/b, x_0], \quad \dot{\tilde{f}} \in (-\vartheta_+, \vartheta_-) \text{ on } (x_0, 1/c) \text{ if } \theta_\gamma > 0$$

$$\dot{\tilde{f}} = +\vartheta_- \text{ on } [x_0, 1/c), \quad \dot{\tilde{f}} \in (-\vartheta_+, \vartheta_-) \text{ on } (-1/b, x_0) \text{ if } \theta_\gamma < 0$$

and $\mathscr{D}_\varepsilon \tilde{f} + \tilde{v} \geq 0$ on $(-1/b, 1/c)$. Then

$$\dot{\tilde{f}}(\alpha) > -\vartheta_+(\alpha) = \dot{f}(\alpha) \;\&\; \dot{\tilde{f}}(\beta) < \vartheta_-(\beta) = \dot{f}(\beta).$$

Hence, there exists $\tilde{x} \in (\alpha, \beta)$ such that $\dot{f}(\tilde{x}) = \dot{\tilde{f}}(\tilde{x})$ and $\ddot{f}(\tilde{x}) \geq \ddot{\tilde{f}}(\tilde{x})$. Then we obtain a contradiction

$$0 = (\mathscr{D}f)(\tilde{\mathrm{x}}) + \hat{v} \geq (\mathscr{D}\tilde{f})(\tilde{\mathrm{x}}) + \hat{v} > (\mathscr{D}_\varepsilon \tilde{f})(\tilde{x}) + \tilde{v} \geq 0$$

by lemma 4.20, where $\tilde{\mathrm{x}} := (\tilde{x}, \hat{x}_{\tilde{x}}')'$ and $\hat{x}_{\tilde{x}} = \hat{\theta}_\gamma - \hat{\Sigma}^{-1}L\tilde{x} + \varepsilon_{\tilde{x}}$. □

Lemma 4.22 *Let* $\theta_\gamma = 1 + \kappa_\gamma^2 > 1$, *then* $I_3 \cup I_4 \neq \emptyset$.

Proof: Obviously, $\theta_\gamma^2 > \aleph_\gamma = (1-\theta_\gamma)^2 + \kappa_\gamma^2 > (1-\theta_\gamma)^2$. Hence, $0 < \theta_\gamma - \sqrt{\aleph_\gamma} < 1$ and therefore $0 < \underline{x} = \xi_+^{-1}(\theta_\gamma - \sqrt{\aleph_\gamma}) < 1$. Since $\theta_\gamma > 1$, we obtain from lemma 4.13 that $\bar{x} > 1$ and therefore $(\underline{x}, 1) \subseteq I$. Let $y \in (\underline{x}, 1)$. We are going to show that $v_1 := v_1(y) \geq 1$. If $y \in I_2 \cup I_3$, then we obtain the desired inequality immediately. Let $y \in I_1$, then we obtain from lemma 4.7 that $\eta_{h_y}(v_1(y)) = 0$. By lemma 4.5 (i), $\eta_{h_y} > 0$ holds on $(y, v_1(y))$ and therefore η_{h_y} does not increase at $v_1(y)$. By lemma 4.2,

$$\text{(53)} \qquad \operatorname{sign} \eta_{h_y}(v_1) = \operatorname{sign}[\theta_\gamma - \xi(v_1, h_y(v_1))] = \operatorname{sign}\{\kappa_\gamma^2 + \mathscr{V}\},$$

where $\mathscr{V} := (1 - v_1)[1 + v_1 h_y(v_1)]$. Since the left-hand side of (53) cannot be positive and $1 + v_1 h_y(v_1) > 0$ holds by lemma 4.3 (ii), we get that $v_1 \geq 1$ also in case that $y \in I_1 = I \backslash (I_2 \cup I_3)$. By lemma 4.3 (ii), lemma 4.15 and lemma 4.18,

$$0 < \frac{1 \wedge \frac{1}{b}}{2} \leq 1 + xh_y(x) < 1 + x\vartheta_-(x) = \frac{1}{1-cx} \leq \frac{1}{1-c} < \infty$$

holds for every $x \in (y, 1)$. Further, $\omega_\gamma^2(y) \uparrow \aleph_\gamma = \kappa_\gamma^2 + \kappa_\gamma^4$ as $y \downarrow \underline{x}$. Obviously, $\mathscr{A}(x, h, \omega_\gamma) = (1-x)^2 \geq 0$ and

$$\mathscr{B}(x, h, \omega_\gamma) = \kappa_\gamma^4 + \kappa_\gamma^2 - \omega_\gamma^2 + 2(1-x)(1+xh)[(1-x)(1+xh) + \kappa_\gamma^2],$$

$$\mathscr{C}(x, h, \omega_\gamma) = (1+xh)^2[(\kappa_\gamma^2 + (1+xh)(1-x))^2 - \omega_\gamma^2].$$

Since $\omega_\gamma^2(y) < \aleph_\gamma = \kappa_\gamma^4 + \kappa_\gamma^2$ holds whenever $y \in I$, we obtain that $\mathscr{B}(x, h_y(x), \omega_\gamma(y)) > 0$ holds whenever $\underline{x} < y < x < 1$. Since $\eta_{h_y}(x) > 0$ holds for every $x \in (y, 1)$, we get from the definition of G and the equation $G(x, h_y(x), \eta_{h_y}(x)) = \omega_\gamma^2(y)$ on $(y, 1)$ that $\mathscr{C}(x, h_y(x), \omega_\gamma(y)) < 0$. Obviously, $\limsup_{x \to 1-} \limsup_{y \to \underline{x}+} \mathscr{C}(x, h_y(x), \omega_\gamma(y)) < 0$,

$$\limsup_{y \to \underline{x}_+} \mathscr{B}(x, h_y(x), \omega_\gamma(y)) = O(1-x) \quad \text{as} \quad x \to 1_-$$

and the same formula holds provided that $\mathscr{B}$ is replaced by $\sqrt{\mathscr{B}^2 - 4\mathscr{A}\mathscr{C}}$ there. Since $\eta_{h_y}(x) = \phi(x, h_y(x), \omega_\gamma(y))$ holds for every $\underline{x} < y < x < 1$, where ϕ is defined in lemma 4.1, we obtain from formula (29) that

$$\liminf_{x \to 1_-} \liminf_{y \to \underline{x}_+} \eta_{h_y}(x)(1-x) > 0. \tag{54}$$

It follows from the definition of $\eta_{h_y}(x)$ that (54) holds provided that η_{h_y} is replaced by $\dot{h}_y$. Then $\dot{h}_y$ is not integrable from left at $x = 1$ if $y > \underline{x}_+$ is close to $\underline{x}_+$ enough and therefore $y \in I_3$ in this case. □

Corollary 4.23 *Let* $\kappa_\gamma^2 > 0$. *If* $\theta_\gamma > 0$, *then* $I_4 \cap I_5 \neq \emptyset$. *If* $\theta_\gamma < 0$, *then* $J_4 \cap J_5 \neq \emptyset$.

Proof: Let $\theta_\gamma > 0$. By lemma 4.13, $I_2 \cup I_5 \neq \emptyset$. By lemma 4.12, it is closed in I. By lemma 4.11, $I_3 \cup I_4$ is closed in I and lemma 4.21 or lemma 4.22 gives that $I_3 \cup I_4 \neq \emptyset$. Obviously $I = (I_2 \cup I_5) \cup (I_3 \cup I_4)$. By lemma 4.13, I is an open interval, i.e. a connected set, and therefore $(I_2 \cup I_5) \cap (I_3 \cup I_4) \neq \emptyset$. Further, if $x \in I_2$, then $x \notin I_3 \cup I_4$ and if $x \in I_3$, then $x \notin I_2 \cup I_5$. Hence, $I_4 \cap I_5 \neq \emptyset$ if $\theta_\gamma > 0$ and similar steps would lead to the conclusion that $J_4 \cap J_5 \neq \emptyset$ if $\theta_\gamma < 0$. □

Theorem 4.24 *Let* $\theta_\gamma \neq 0$, $\kappa_\gamma^2 > 0$ *and* $b > 0$, $c \in (0,1)$. *Then there exist* $f \in C^2(-1/b, 1/c)$, $-1/b < \alpha < \beta < 1/c$, $\omega_\gamma \in (0, |\theta_\gamma|)$ *and* $v \in \mathbb{R}$ *such that* $\alpha > 0$ *if* $\theta_\gamma > 0$ *and* $\beta < 0$ *if* $\theta_\gamma < 0$, *that* $1 + x\dot{f}(x) > 0$ *holds on* $(-1/b, 1/c)$ *and that*

(i) $\ddot{f} > \dot{f}^2$ *holds on* (α, β) *and* $\ddot{f} = \dot{f}^2$ *on* $(-1/b, 1/c) \setminus (\alpha, \beta)$

(ii) $(\mathscr{D}_e f) + v = 0$ *on* $[\alpha, \beta]$ *and* $(\mathscr{D}_e f) + v > 0$ *on* $(-1/b, 1/c) \setminus [\alpha, \beta]$

(iii) $\xi_+(\alpha) = \theta_\gamma - \omega_\gamma$, $\xi_-(\beta) = \theta_\gamma + \omega_\gamma$, $v = \frac{1-\gamma}{2}\{\hat{\theta}_\gamma' \Sigma \hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2)\}$

(iv) $\dot{f} = -\vartheta_+$ *on* $(-1/b, \alpha]$, $\dot{f} \in (-\vartheta_+, \vartheta_-)$ *on* (α, β), $\dot{f} = \vartheta_-$ *on* $[\beta, 1/c)$

Proof: We focus on the case $\theta_\gamma > 0$. By corollary 4.23, there exists $\alpha \in I_4 \cap I_5$. Hence, $\alpha \in I_1$ and $\alpha < \beta := v_1(\alpha) < 1/c$ is such that $h_\alpha(\beta) = \vartheta_-(\beta)$. By lemma 4.15 and lemma 4.18, there exists $f \in C^2(-1/b, 1/c)$ such that $1 + x\dot{f}(x) > 0$ holds on $(-1/b, 1/c)$ and that (i), (ii) and (iv) hold with $v := \tilde{v}$ given by (48). To verify (iii), it is sufficient to look at the definition of $\omega_\gamma(\alpha) =: \omega_\gamma$ and $v := \tilde{v}$ and at remark 4.16 (i). The proof corresponding to the case $\theta_\gamma < 0$ would be similar. □

5. Optimal strategies

In this section, we prove that the strategy given by theorem 4.24 is optimal, see theorem 5.2, and we derive the first term in Taylor's expansion of function which connects transaction costs and the width of no-trade region. The proof of existence of optimal strategies is left to the next section. We omit the singular case $\kappa_\gamma^2 = 0$, i.e. $\hat{\theta}_\gamma = \hat{\Sigma}^{-1}L$ and left it to the reader. In this singular case, it is optimal to keep the remaining positions on $\hat{x} = \hat{\theta}_\gamma(1 - x)$. Then every interval strategy $[(\alpha, \beta)]$ such that $0, 1 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$ can be explicitly evaluated and there exists the corresponding function f in explicit form similarly as in the one-dimensional case.

Remark 5.1 Let $0 \notin [\alpha, \beta] \subseteq (-1/b, 1/c)$. Denote by $[(\alpha, \beta)]_e$ the strategy that keeps the first position within the interval $[\alpha, \beta]$, that does not trade with the first stock when the first position is in (α, β) and that keeps the remaining positions on

$$\hat{x} = \hat{\theta}_\gamma - \hat{\Sigma}^{-1}Lx + (\hat{\Sigma}^{-1}L - \hat{\theta}_\gamma)e_x, \tag{55}$$

where e_x is given by (39). Further, if α, β, f and v are such as in the theorem 4.24, then $\delta_+^f(\alpha) = \delta_-^f(\beta) = 0$ and $-d_f^v(\mathrm{x}) = (\mathscr{D}f)(\mathrm{x}) + v = (\mathscr{D}_e f)(x) + v = 0$ holds at every $x \in [\alpha, \beta]$ provided that $\hat{x}$ satisfies (55).

Theorem 5.2 *Let $\theta_\gamma \neq 0$ and $\kappa_\gamma^2 > 0$. Let $b, c > 0$ and α, β, f and v be such as in theorem 4.24. Let us consider a strategy that keeps the position $G(t)$ within a compact set in $(-1/b, 1/c) \times \mathbb{R}^{n-1}$ and $E\, Y_t^\delta < \infty$ for all $\delta < 0$ and $t \geq 0$.*

Then $\boldsymbol{e}_\gamma(U(t))$ is a supermartingale. Moreover, if $[(\alpha, \beta)]_e$ is applied, then $\boldsymbol{e}_\gamma(U(t))$ is a martingale.

Proof: It follows from the properties of f and v that $(\mathscr{D}_e f) + v \geq 0$ and $\delta_+^f, \delta_-^f \leq 0$ hold on $(-1/b, 1/c)$. By lemma 4.20, $(\mathscr{D}f)(\mathrm{x}) \geq (\mathscr{D}_e f)(x)$ holds for every $x \in (-1/b, 1/c)$ and $\hat{x} \in \mathbb{R}^{n-1}$ and therefore $-d_f^v(x) = (\mathscr{D}f)(\mathrm{x}) + v \geq 0$ hold for every $\mathrm{x} \in \mathbb{R}^n$. By lemma 3.5 in [4], $V(t)$ is an $\mathscr{F}_t$-martingale. Further, we obtain from the inequalities $d_f^v, \delta_+^f, \delta_-^f \leq 0$ and lemma 3.4 in [4] that $\boldsymbol{e}_\gamma(U(t))$ is an $\mathscr{F}_t$-supermartingale provided that we show that it is an integrable process. In case $\gamma = 0$, we also obtain from the above-mentioned lemma and inequalities that $V(t) \geq \boldsymbol{e}_\gamma(U(t))$ holds a.s. and therefore $\boldsymbol{e}_\gamma(U(t))$ is a r.v. integrable from above for every $t \geq 0$. The same conclusion holds also for $\gamma < 0$, since $\boldsymbol{e}_\gamma(U(t)) \leq 0$ holds in this case. We are going to show that $\boldsymbol{e}_\gamma(U(t))$ is also integrable from below for every $t \geq 0 \geq \gamma$. By assumption, $Y(t)^\delta$ is an integrable r.v. for every $\delta < 0 \leq t$ and obviously $f(G_1(t)) + vt$ is a bounded r.v. Hence, $E\boldsymbol{e}_\delta(U(t)) =$ $= EY(t)^\delta \boldsymbol{e}_\delta(-[f(G_1(t)) + vt]) > -\infty$ holds for every $\delta < 0$. If $\gamma = 0$, we obtain from the inequality $x \geq \boldsymbol{e}_\delta(x)$ holding for every $\delta < 0$ and $x \in \mathbb{R}$ that $E\boldsymbol{e}_\gamma(U(t)) = EU(t) \geq E\boldsymbol{e}_\delta(U(t)) > -\infty$ and therefore $\boldsymbol{e}_\gamma(U(t))$ is integrable also from below.

Now, we are going to prove the moreover part. We have from theorem 4.24 (ii) that $(\mathscr{D}_e f) + v = 0$ holds on $[\alpha, \beta]$ and lemma 4.20 gives that $(\mathscr{D}f)(\mathrm{x}) = (\mathscr{D}_e f)(x)$ holds for every $x \in [\alpha, \beta]$, provided that $\hat{x}$ is such that (55) and (39) hold. It follows from the properties of the strategy $[(\alpha, \beta)]_e$ that $-d_f^v(G(t)) = (\mathscr{D}f)(G(t)) + v =$ $= (\mathscr{D}_e f)(G_1(t)) + v = 0$ holds for every $t \geq 0$ almost surely. Further properties of the strategy $[(\alpha, \beta)]_e$ ensure that conditions (7) are also satisfies for every $t \geq 0$

almost surely. By lemma 3.5 in [4], we get that $e_\gamma(U(t))$ is equal to an $\mathcal{F}_t$-martingale almost surely. $\square$

Remark 5.3 Similarly, as in [4, theorem 5.6], we could show that the optimality of the derived strategy is invariant under reasonable time change in model in case that $\gamma = 0$, since the martingales remain martingales and supermartingales remain supermartingales.

Lemma 5.4 *Let $\theta_\gamma \neq 0$, $\kappa_\gamma^2 > 0$ and $0 < b_1 \leq b_2 < \infty$, $0 < c_1 \leq c_2 < 1$. Let $(f_i, \alpha_i, \beta_i, \omega_{\gamma,i}, v_i)$ be as in theorem 4.24 with $b := b_i, c := c_i$. Let $b := b_1 = b_2$ if $\theta_\gamma > 0$ and $c := c_1 = c_2$ if $\theta_\gamma < 0$, then $v_1 \geq v_2$.*

In particular, the values $\alpha, \beta, \omega_\gamma$ and v in theorem 4.24 are unique.

Proof: We focus on the case $\theta_\gamma > 0$. Let $v_1 < v_2$. We are going to show that this assumption leads to a contradiction. Then $\omega_\gamma^2(\alpha_1) > \omega_\gamma^2(\alpha_2)$ and therefore

$$\theta_\gamma - \xi_+(\alpha_1) = \omega_\gamma(\alpha_1) > \omega_\gamma(\alpha_2) = \theta_\gamma - \xi_+(\alpha_2).$$

Since ξ_+ is an increasing function on $(-1/b, \infty) \ni \alpha_1, \alpha_2$, we get that $\alpha_1 < \alpha_2$. Since $\alpha_1 < \alpha_2 < \beta_2 < 1/c_2 \leq 1/c_1$ and

$$\dot{f}_1(\alpha_2) > -\vartheta_+(\alpha_2) = \dot{f}_2(\alpha_2), \qquad \dot{f}_1(\beta_2) \leq \frac{c_1}{1 - c_1\beta_2} \leq \frac{c_2}{1 - c_2\beta_2} = \dot{f}_2(\beta_2).$$

we obtain that $\dot{f}_1 - \dot{f}_2$ is a continuous function on the interval $[\alpha_2, \beta_2]$ with $(\dot{f}_1 - \dot{f}_2)(\alpha_2) > 0 \geq (\dot{f}_1 - \dot{f}_2)(\beta_2)$. Hence, there exists $x \in [\alpha_2, \beta_2]$ such that $\ddot{f}_2(x) \geq$ $\geq \ddot{f}_1(x)$ and $\dot{f}_2(x) = \dot{f}_1(x)$. Then $-v_2 = (\mathcal{D}_e f_2)(x) \geq (\mathcal{D}_e f_1)(x) \geq -v_1$, i.e. $v_2 \leq v_1$, a contradiction. Hence, we get that $v_1 \geq v_2$. If $b_1 = b_2$ and $c_1 = c_2$, we obtain that also $v_1 \leq v_2$ and therefore $v_1 = v_2$. Now, it is sufficient to realize that α_i, β_i and $\omega_{\gamma,i}$ are uniquely determined by the value v_i. $\square$

Lemma 5.5 *Let θ_γ, κ_γ^2, ω_γ, b, c, α, β, f and v be as in theorem 4.24. Let $\delta \in (-c, b)$. Put $\tilde{b} := \frac{b-\delta}{1+\delta} > 0$ and $\tilde{c} := \frac{c+\delta}{1+\delta} \in (0,1)$. Then $\lambda := \ln\frac{1+b}{1-c} = \ln\frac{1+\tilde{b}}{1-\tilde{c}}$. Further, put*

$$y := \tilde{y}(x) := \frac{(1+\delta)x}{1+\delta x}, \qquad \tilde{f}(y) := f(x) + \ln(1+\delta x) \tag{56}$$

and $\tilde{\alpha} := \tilde{y}(\alpha)$, $\tilde{\beta} := \tilde{y}(\beta)$. Then the statement of theorem 4.24 remains valid if we replace (b, c, α, β, f) by $(\tilde{b}, \tilde{c}, \tilde{\alpha}, \tilde{\beta}, \tilde{f})$.

In particular, v depends only on θ_γ, κ_γ^2 and λ. Further, v is by lemma 5.4 a non-increasing function in λ.

Proof: Obviously, $\tilde{y} : (-1/b, 1/c) \to (-1/\tilde{b}, 1/\tilde{c})$ is an increasing homeomorphism with $\dot{\tilde{y}}(x) = \frac{1+\delta}{(1+\delta x)^2}$ and $\ddot{\tilde{y}}(x) = -\frac{2\delta(1+\delta)}{(1+\delta x)^3}$. By (56), we get that

$$\dot{\tilde{f}}(y) = \frac{\dot{f}(x) + \frac{\delta}{1+\delta x}}{\dot{\tilde{y}}(x)}, \quad \ddot{\tilde{f}}(y) = \frac{\ddot{f}(x) + \frac{2\delta}{1+\delta x}\dot{f}(x) + \frac{\delta 2}{(1+\delta x)^2}}{\dot{\tilde{y}}(x)^2}.$$

Then

$$(57) \qquad \ddot{\tilde{f}}(y) - \dot{\tilde{f}}(y)^2 = \frac{(1 + \delta x)^4}{(1 + \delta)^2}[\ddot{f}(x) - \dot{f}(x)^2]$$

and therefore $\eta_{\tilde{f}}(y) = \eta_f(x)(1 + \delta x)^2$. Further, we get that $1 + y\dot{\tilde{f}}(y) = (1 + \delta x)[1 + x\dot{f}(x)]$ and therefore

$$1 - \tilde{e}_y = \frac{1 + y\dot{\tilde{f}}(y)}{\eta_{\tilde{f}}(y) + [1 + y\dot{\tilde{f}}(y)]^2} = \frac{[1 + x\dot{f}(x)](1 + \delta x)^{-1}}{\eta_f(x) + (1 + x\dot{f}(x))^2} = \frac{1 - e_x}{1 + \delta x}.$$

Since $1 - y = \frac{1-x}{1+\delta x}$, we get that $\tilde{S}^2(y) = \frac{(1+\delta)^2}{(1+\delta x)^4} S^2(x)$ and therefore

$$\tilde{S}^2(y)\frac{\ddot{\tilde{f}}(y) - \dot{\tilde{f}}(y)^2}{1 - \gamma} = S^2(x)\frac{\ddot{f}(x) - \dot{f}(x)^2}{1 - \gamma}.$$

Similarly, we obtain that $\xi(y,\dot{\tilde{f}}(y)) = \xi(x,\dot{f}(x))$ and $\zeta(y, \tilde{e}_y,\dot{\tilde{f}}(y)) = \zeta(x, e_x,\dot{f}(x))$. Hence, we get from the formulas above and remark 4.17 that $(\mathscr{D}_e\tilde{f})(\tilde{y}(x)) + v = (\mathscr{D}_e f)(x) + v$ holds for every $x \in (-1/c, 1/b)$. Since $\tilde{y}$ is an increasing homeomorphism mapping (α, β) onto $(\tilde{\alpha}, \tilde{\beta})$, we get that $(\tilde{\text{ii}})$ in theorem 4.24 holds. Further, we obtain $(\tilde{\text{i}})$ in theorem 4.24 from (57). Obviously, $\tilde{f} \in C^2(-1/\tilde{b}, 1/\tilde{c})$. If $y \in (-1/\tilde{b}, \tilde{\alpha}]$, then there exists $x \in (-1/b\, \alpha]$ such that $y = \tilde{y}(x)$. Then

$$\dot{\tilde{f}}(y) = \left[-\frac{b}{1 + bx} + \frac{\delta}{1 + \delta x}\right]\frac{(1 + \delta x)^2}{1 + \delta} = -\frac{\tilde{b}}{1 + \tilde{b}y}$$

and similarly we would obtain that $\dot{\tilde{f}}(y) = \frac{\tilde{c}}{1-\tilde{c}y}$ if $y \in [\tilde{\beta}, 1/\tilde{c})$. Hence, $(\tilde{\text{iv}})$ is verified. Further,

$$\theta_\gamma - \omega_\gamma = \xi_+(\alpha) = \xi(\alpha,\dot{f}(\alpha)) = \xi(\tilde{\alpha},\dot{\tilde{f}}(\tilde{\alpha})) = \frac{1 + \tilde{b}}{1 + \tilde{b}\tilde{\alpha}}\tilde{\alpha} = \tilde{\xi}_+(\tilde{\alpha})$$

and similarly, we would get $\theta_\gamma + \omega_\gamma = \xi(\tilde{\beta}, \dot{\tilde{f}}(\tilde{\beta})) = \frac{1-\tilde{c}}{1-\tilde{c}\tilde{\beta}}\tilde{\beta} = \tilde{\xi}_-(\tilde{\beta})$. □

Lemma 5.6 *Let* $\theta_\gamma \neq 0$ *and* $\kappa_\gamma^2 > 0$, *then* $\omega_\gamma^2(\theta_\gamma, \kappa_\gamma^2, \lambda) \to 0$ *as* $\lambda \to 0^+$, *where* $\omega_\gamma(\theta_\gamma, \kappa_\gamma^2, \lambda) \in (0, |\theta_\gamma|)$ *is defined by the equation*

$$v(\theta_\gamma, \kappa_\gamma^2, \lambda) = \frac{1 - \gamma}{2}\{\hat{\theta}_\gamma'\hat{\Sigma}\hat{\theta}_\gamma + \tilde{\sigma}^2(\theta_\gamma^2 - \omega_\gamma^2(\theta_\gamma, \kappa_\kappa^2, \lambda))\}.$$

Further, $\alpha(\theta_\gamma, \kappa_\gamma^2, \lambda)$, $\beta(\theta_\gamma, \kappa_\gamma^2, \lambda) \to \theta_\gamma$ *as* $\lambda \to 0^+$.

Proof: By lemma 5.4 and lemma 5.5, $v(\theta_\gamma, \kappa_\gamma^2, \lambda)$ is non-increasing in λ and therefore $\omega_\gamma^2(\theta_\gamma, \kappa_\gamma^2, \lambda)$ is non-decreasing in λ. Hence, there exists $\tilde{\omega}_\gamma \geq 0$ such that $\tilde{\omega}_\gamma^2 = \lim_{\lambda\to0^+} \omega_\gamma^2(\theta, \kappa_\gamma^2, \lambda)$. Let us consider $b_\lambda = e^{\lambda/2} - 1 > 0$ and $c_\lambda = 1 - e^{-\lambda/2} > 0$, for example, in order to get that $\lambda = \ln\frac{1+b_\lambda}{1-c_\lambda} > 0$. Then b_λ, $c_\lambda \to 0$ as $\lambda \to 0^+$. Let $f_\lambda, \alpha_\lambda, \beta_\lambda, \omega_{\gamma,\lambda}, v_\lambda$ be as in theorem 4.24, then $v_\lambda = v(\theta_\gamma, \kappa_\gamma^2, \lambda)$, $\omega_{\gamma,\lambda}^2 = \omega_\gamma^2(\theta_\gamma, \kappa_\gamma^2, \lambda)$ and

$$\alpha_\lambda \frac{1 + b_\lambda}{1 + b_\lambda \alpha_\lambda} = \theta_\gamma - \omega_{\gamma,\lambda}, \quad \beta_\lambda \frac{1 - c_\lambda}{1 - c_\lambda \beta_\lambda} = \theta_\gamma + \omega_{\gamma,\lambda}$$

and therefore $\beta_\lambda := \frac{\theta_\gamma + \omega_{\gamma,\lambda}}{1 - c_\lambda + c_\lambda(\theta_\gamma + \omega_{\gamma,\lambda})} \to \theta_\gamma + \tilde{\omega}_\gamma$ and similarly $\alpha_\lambda \to \theta_\gamma - \tilde{\omega}_\gamma$ as $\lambda \to 0^+$. Since $\dot{f}_\lambda$ is an increasing function, which changes the sign on $[\alpha, \beta]$, we obtain that

$$\sup_{[\alpha,\beta]} |\dot{f}_\lambda| \le \dot{f}_\lambda(\beta) - \dot{f}_\lambda(\alpha) = \frac{c_\lambda}{1 - c_\lambda \beta_\lambda} + \frac{b_\lambda}{1 + b_\lambda \alpha_\lambda} \to 0$$

as $\lambda \to 0^+$. Further, by Fatou's lemma

$$0 = \lim_{\lambda \to 0^+} \frac{\dot{f}_\lambda(\beta_\lambda) - \dot{f}_\lambda(\alpha_\lambda)}{1 - \gamma} = \liminf_{\lambda \to 0^+} \int \frac{\ddot{f}_\lambda(x) I\{\alpha_\lambda < x < \beta_\lambda\}}{1 - \gamma} dx \ge$$

$$\ge \int \liminf_{\lambda \to 0^+} \frac{\eta_{\dot{f}_\lambda}}{x^2} I\{\alpha_\lambda < x < \beta_\lambda\} dx = \int_{\theta_\gamma - \tilde{\omega}_\gamma}^{\theta_\gamma + \tilde{\omega}_\gamma} \liminf_{\lambda \to 0^+} \eta_{\dot{f}_\lambda}(x) \frac{dx}{x^2}.$$

Since $\theta_\gamma - \tilde{\omega}_\gamma = \lim_{\lambda \to 0^+} \theta_\gamma - \omega_{\gamma,\lambda} \ge 0$ if $\theta_\gamma > 0$ and similarly $\theta_\gamma + \tilde{\omega}_\gamma \le 0$ if $\theta_\gamma < 0$, we obtain that $0 \notin (\theta_\gamma - \tilde{\omega}_\gamma, \theta_\gamma + \tilde{\omega}_\gamma)$. Let $x \in (\theta_\gamma - \tilde{\omega}_\gamma, \theta_\gamma + \tilde{\omega}_\gamma)$, then $x \neq 0$. Moreover, if $\liminf_{\lambda \to 0^+} \eta_{\dot{f}_\lambda}(x) = 0$, then there exists a sequence $\lambda_m \to 0^+$ such that $\eta_{\dot{f}_{\lambda_m}}(x) \to 0$ as $m \to \infty$ and we obtain a contradiction

$$\tilde{\omega}_\gamma^2 > (\theta_\gamma - x)^2 \leftarrow G(x, \dot{f}_{\lambda_m}(x), \eta_{\dot{f}_{\lambda_m}}(x)) = \omega_{\gamma,\lambda_m}^2 \to \tilde{\omega}_\gamma^2,$$

since $1 + x\dot{f}_{\lambda_m}(x) \to 1 > 0$ as $m \to \infty$. Hence, we obtain that $0 < l(x) := := \liminf_{\lambda \to 0^+} \eta_{\dot{f}_\lambda}(x)/x^2$ holds for every $x \in (\theta_\gamma - \tilde{\omega}_\gamma, \theta_\gamma + \tilde{\omega}_\gamma)$. We have obtained from Fatou's lemma that $\int_{\theta_\gamma - \tilde{\omega}_\gamma}^{\theta_\gamma + \tilde{\omega}_\gamma} l(\xi)\, \delta\xi = 0$ and therefore we get that $\tilde{\omega}_\gamma = 0$. □

Lemma 5.7 *Let $\theta_\gamma \neq 0$ and $\kappa_\gamma^2 > 0$, then*

$$\lambda = \frac{4}{3} \frac{(1 - \gamma)\, \omega_\gamma^3(\theta_\gamma, \kappa_\gamma^2, \lambda)}{\theta_\gamma^2 [(1 - \theta_\gamma)^2 + \kappa_\gamma^2]} + o(\omega_\gamma^3(\theta_\gamma, \kappa_\gamma^2, \lambda))$$

as $\lambda \to 0^+$.

Proof: If $\lambda > 0$, we consider $b = e^{\lambda/2} - 1$, $c = 1 - e^{-\lambda/2}$ and $f, \alpha, \beta, v, \omega_\gamma$ given by theorem 4.24 without emphasizing the dependence on λ. By lemma 5.6, $\alpha, \beta \to \theta_\gamma$ and $0 < \omega_\gamma \to 0$ as $\lambda \to 0^+$. Further,

$$(58) \quad \beta - \theta_\gamma = \xi\ (\beta) - \theta\ + \beta - \xi_-(\beta) = \omega_\gamma + \beta c \frac{1 - \beta}{1 - c\beta} = \omega_\gamma + O(\|\dot{f}\|),$$

and similarly, we obtain that

$$(59) \qquad \theta_\gamma - \alpha = \omega_\gamma + b\alpha \frac{1 - \alpha}{1 + b\alpha} = \omega_\gamma + O(\|\dot{f}\|), \qquad \text{where}$$

$$(60) \qquad \|\dot{f}\| := \sup_{[\alpha,\beta]} |\dot{f}(x)| = \dot{f}(\beta) \wedge (-\dot{f}(\alpha)) = \frac{c}{1 - c\beta} \vee \frac{b}{1 + b\alpha} \to 0$$

as $\lambda \to 0^+$. Since $0 = (\mathscr{D}_e f)(x) + v$ holds for every $x \in [\alpha, \beta]$, we get from remark 4.17 that

$$(61) \quad \tilde{\sigma}^{-2} S^2(x) \frac{\ddot{f}(x) - \dot{f}(x)^2}{1 - \gamma} + (\theta_\gamma - \xi(x, \dot{f}(x)))^2 + \kappa_\gamma^2 \zeta(x, e_x, \dot{f}(x))^2 = \omega_\gamma^2.$$

Since the left-hand side is a sum of three non-negative terms, each of them is bounded by the value ω_γ^2. Obviously

$$1 - e_x = \frac{1 + x\dot{f}(x)}{\eta_f(x) + [1 + x\dot{f}(x)]^2} > 0$$

holds for every $x \in [\alpha, \beta]$. Further, we obtain from the inequality $\kappa_\gamma^2 \zeta(x, e_x, \dot{f}(x))^2 \leq \leq \omega_\gamma^2$ and the definition of $\xi(x, e, h) = e - x(1 - e)h$ that

$$\liminf_{\lambda \to 0^+} \inf_{x \in [\alpha, \beta]} 1 - e_x > 0 \quad \text{and therefore} \quad \liminf_{\lambda \to 0^+} \inf_{x \in [\alpha, \beta]} S^2(x) > 0.$$

Then

$$(62) \qquad \|\dot{f}\| \leq \dot{f}(\beta) - \dot{f}(\alpha) = \int_\alpha^\beta \ddot{f}(x)\,dx \leq (\beta - \alpha)[O(\omega_\gamma^2) + \|\dot{f}\|^2]$$

and we get from (58)–(60) and (62) that $\|\dot{f}\| = O(\omega_\gamma^3)$ as $\lambda \to 0^+$. Further, we obtain that

$$\theta_\gamma - \xi(x, \dot{f}(x)) = \theta_\gamma - x[1 - (1 - x)\dot{f}(x)] = \theta_\gamma - x + O(\omega_\gamma^3)$$

uniformly in $x \in [\alpha, \beta]$ as $\lambda \to 0^+$ and therefore $(\theta_\gamma - \xi(x, \dot{f}(x)))^2 = (\theta_\gamma - x)^2 + + O(\omega_\gamma^4)$ uniformly in $x \in [\alpha, \beta]$ as $\lambda \to 0^+$. From (61), we get that $\eta_f(x) = O(\omega_\gamma^2)$ uniformly in $x \in [\alpha, \beta]$ as $\lambda \to 0^+$. Then

$$e_x = \frac{\eta_f(x) + x\dot{f}(x)[1 + x\dot{f}(x)]}{\eta_f(x) + [1 + x\dot{f}(x)]^2} = O(\omega_\gamma^2)$$

and therefore $\zeta(x, e_x, \dot{f}(x))^2 = (e_x - x(1 - e_x)\dot{f}(x))^2 = O(\omega_\gamma^4)$ all uniformly in $x \in [\alpha, \beta]$ as $\lambda \to 0^+$. Hence, we obtain that

$$\ddot{f}(x) = \frac{\omega_\gamma^2 - (\theta_\gamma - x)^2}{\tilde{\sigma}^{-2} S^2(x)/(1 - \gamma)} + O(\omega_\gamma^4) = (1 - \gamma)\frac{\omega_\gamma^2 - (\theta_\gamma - x)^2}{\theta_\gamma^2[(1 - \theta_\gamma)^2 + \kappa_\gamma^2]} + o(\omega_\gamma^2)$$

uniformly in $x \in [\alpha, \beta]$ as $\lambda \to 0^+$. Further,

$$\frac{\dot{f}(\beta) - \dot{f}(\alpha)}{\omega_\gamma^3} = \int_{\frac{\alpha - \theta_\gamma}{\omega_\gamma}}^{\frac{\beta - \theta_\gamma}{\omega_\gamma}} \frac{\ddot{f}(\theta_\gamma + y\omega_\gamma)}{\omega_\gamma^2}\,dy \to \int_{-1}^1 \frac{1 - y^2}{\Lambda}\,dy = \frac{4}{3\Lambda}$$

as $\lambda \to 0^+$, where $\Lambda := \theta_\gamma^2[(1 - \theta_\gamma)^2 + \kappa_\gamma^2]/(1 - \gamma)$, and

$$\frac{\dot{f}(\beta) - \dot{f}(\alpha)}{\lambda} = \left[\frac{c}{1 - c\beta} + \frac{b}{1 + b\alpha}\right] / \ln\frac{1 + b}{1 - c} \to 1$$

as $\lambda \to 0^+$. □

6. Existence of strategies

This section is in a certain sense independent with the previous ones and it contains only statements of lemmas and theorem necessary in order to "prove" existence of optimal strategies, see corollary 7.5. The complete version can be sent to the reader if he or she is interested.

In this section, we fix a probability space $(\Omega, \mathscr{F}, P)$ with a complete filtration $\mathscr{F}_t$ and an n-dimensional $\mathscr{F}_t$-Wiener process W. If f is a continuous function on $\mathbb{R}$, we denote $f|_s^t := f(t) - f(s)$ and $\|f\|_t := \sup_{s\le t}|f(s)|$. Further, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Lemma 6.1 *Let $F(t)$ be a continuous $\mathscr{F}_t$-adapted process and τ an $\mathscr{F}_t$-stopping time. If $F(\tau) \ge \alpha$ or $F(\tau) \le \beta$ holds on $[\tau < \infty]$, then $\mathscr{L}_\alpha(F, \tau)$ or $\mathscr{L}^\beta(F, \tau)$ is a continuous $\mathscr{F}_t$-adapted process, respectively, where*

$$\mathbb{L}_a(t) := \mathscr{L}_\alpha(F, \tau)(t) := F(t) - \inf_{\tau\le s\le t}[F(s) - \alpha] \wedge 0,$$

$$\mathbb{L}_\beta(t) := \mathscr{L}^\beta(F, \tau)(t) := F(t) - \sup_{\tau\le s\le t}[F(s) - \beta] \vee 0.$$

Moreover,

$$\int_\tau^\infty I\{\mathbb{L}_\alpha > \alpha\} d(\mathbb{L}_\alpha - F) = 0, \qquad \int_\tau^\infty I\{\mathbb{L}_\beta > \beta\} d(F - \mathbb{L}_\beta) = 0.$$

Lemma 6.2 *Let $B : \mathbb{R} \to \mathbb{R}$, $S : \mathbb{R} \to \mathbb{R}^n$ be Lipschitz mappings. Then there exist $a, b \in (0, \infty)$ such that*

$$E \| \mathscr{L}_\alpha(I_W^\tau(F_1), \tau) - \mathscr{L}_\alpha(I_W^\tau(F_2), \tau) |_t^2 \le (a + bt) \int_0^t E \| F_1 - F_2 \|_s^2 \, ds$$

holds for every $t \ge 0$ whenever F_1, F_2 are $\mathscr{F}_t$-adapted continuous processes and τ an $\mathscr{F}_t$-stopping time such that $F_1 = F_2$ holds on $[0, \tau)$ and $F_1(\tau) = F_2(\tau) \in [\alpha, \beta]$ holds on $[\tau < \infty]$, where $I_W^\tau(F)$ stands for the following $\mathscr{F}_t$-adapted continuous process

$$t \mapsto F(t \wedge \tau) + \int_{t\wedge\tau}^t B(F(s))\, ds + \int_{t\wedge\tau}^t S(F(s))\, dW(s).$$

The same statement holds when $\mathscr{L}_\alpha$ is replaced by $\mathscr{L}^\beta$.

Lemma 6.3 *Let G be a continuous $\mathscr{F}_t$-adapted process and τ an $\mathscr{F}_t$-stopping time such that $G(t \wedge \tau) \in [\alpha, \beta]$ for every $t \ge 0$. Let $B : \mathbb{R} \to \mathbb{R}$, $S : \mathbb{R} \to \mathbb{R}^n$ be Lipschitz mappings. Then there exist continuous $\mathscr{F}_t$-adapted processes $G_\alpha \ge \alpha$, $G_\beta \le \beta$ such that $G_\alpha(t \wedge \tau) = G_\beta(t \wedge \tau) = G(t \wedge \tau)$ holds for every $t \ge 0$ and such that*

$$G_\alpha = \mathscr{L}_\alpha(I_W^\tau(G_\alpha), \tau), \quad G_\beta = \mathscr{L}^\beta(I_W^\tau(G_\beta), \tau)$$

hold almost surely.

Theorem 6.4 *Let $B:[\alpha,\beta]\to\mathbb{R}$ and $S:[\alpha,\beta]\to\mathbb{R}^n$ be a Lipschitz mappings, where $-\infty<\alpha<\beta<\infty$, and $g_0\in[\alpha,\beta]$. Then there exist $\mathcal{F}_t$-adapted continuous processes G, G_+, G_- such that G starts from g_0 and it does not leave the interval $[\alpha,\beta]$, G_+, G_- are non-decreasing processes starting from zero such that*

$$dG(t)=B(G(t))dt+S(G(t))dW(t)+dG_+(t)-dG_-(t) \tag{63}$$

and

$$\int_0^\infty I\{G(t)>\alpha\}dG_+(t)=0,\quad \int_0^\infty I\{G(t)<\beta\}dG_-(t)=0. \tag{64}$$

Corollary 6.5 Let $\theta_\gamma\neq 0$, $\kappa_\gamma^2>0$ and $b,c>0$. Let $f,\alpha,\beta,\omega_\gamma$ and ν be as in theorem 4.24 with $g_0\in[\alpha,\beta]$. Then the statement of theorem 6.4 holds with $B(x):=\mathrm{e}_1'\mathbb{B}(\mathrm{x})$ and $S(x)=\mathrm{e}_1'\mathbb{S}(\mathrm{x})$, where $\mathrm{x}=(x,\hat{x}')'$ and $\hat{x}$ is given by (55) with e_x given by (39).

Proof: Obviously, the function $x\in[\alpha,\beta]\mapsto e_x$ given by (39) is infinitely differentiable on $[\alpha,\beta]$. Then the mapping $x\in[\alpha,\beta]\mapsto\mathrm{x}=(x,\hat{x}')'$ is also infinitely differentiable, where $\hat{x}$ is given by (55). Hence, $x\in[\alpha,\beta]\mapsto\mathrm{e}_1'\mathbb{S}(\mathrm{x})$ and $x\in[\alpha,\beta]\mapsto\mathrm{e}_1'\mathbb{B}(\mathrm{x})$ are also infinitely differentiable mappings and we obtain from theorem 7.4 that there exist processes G, G_+, G_- such that (63) and (64) hold. □

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