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A GENERALIZATION OF SEMIFLOWS ON MONOMIALS

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Abstract. Let K be a field, $A = K[X_1, \ldots, X_n]$ and $\mathbb{M}$ the set of monomials of A. It is well known that the set of monomial ideals of A is in a bijective correspondence with the set of all subsemiflows of the $\mathbb{M}$-semiflow $\mathbb{M}$. We generalize this to the case of term ideals of $A = R[X_1, \ldots, X_n]$, where R is a commutative Noetherian ring. A term ideal of A is an ideal of A generated by a family of terms $cX_1^{\mu_1} \ldots X_n^{\mu_n}$, where $c \in R$ and $\mu_1, \ldots, \mu_n$ are integers $\geqslant 0$.

Keywords: monomial ideal, term ideal, Dickson's lemma, semiflow

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1. Notation and preliminaries

In our notation $\mathbb{N} = \{0, 1, 2, \ldots\}$. Inclusion is denoted by $\subset$ and strict inclusion by $\subsetneq$.

We always assume that R is a commutative Noetherian ring and $A = R[X_1, \ldots, X_n]$ the ring of polynomials in $n \geqslant 1$ variables $X = (X_1, \ldots, X_n)$ over R. All the notions or facts from Commutative Algebra that we use but not define or state in this paper can be found in [1]. An ideal generated by a subset S of R is denoted by $\langle S \rangle$. All terminology with respect to the semiflows will be given in the paper. (The reader can also consult [2].)

The elements of $\mathbb{N}^n$ $(n \geqslant 1)$ will be denoted by μ, ν, σ, etc. We will assume that on $\mathbb{N}^n$ we have the partial order $\leqslant$. It is defined by $(x_1, \ldots, x_n) \leqslant (y_1, \ldots, y_n)$ if for every $i = 1, 2, \ldots, n$, $x_i \leqslant y_i$ in the standard total order on $\mathbb{N}$. For $\mu(1), \ldots, \mu(m) \in \mathbb{N}^n$ we denote by $\sup(\mu(1), \ldots, \mu(m))$ the smallest element of $\mathbb{N}^n$ which is $\geqslant$ than any of the elements $\mu(i)$, $i = 1, 2, \ldots, n$.

Two elements of a partially ordered set, that are in relation, are said to be *comparable*, otherwise they are *incomparable*.

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$. We denote $X^\mu = X_1^{\mu_1} X_2^{\mu_2} \dots X_n^{\mu_n}$ the monomials in the variables $X_1, \dots, X_n$ over K (a field) or R. With the standard multiplication the monomials form a unital semigroup. (In the literature the unital semigroups are more often called monoids.) The identity element is the monomial $X^0 = 1$, where $0 = (0, \dots, 0) \in \mathbb{N}^n$.

The following proposition (called Dickson's lemma) is well-known:

Proposition 1.1 ([1, page 71]). *Let K be a field and let $I = \langle X^\nu | \nu \in N \subset \mathbb{N}^n \rangle \subset K[X_1, \dots, X_n]$ be a monomial ideal. Then I is generated by finitely many monomials $X^{\nu(1)}, \dots, X^{\nu(s)}$, where $\nu(1), \dots, \nu(s) \in N$.*

The next proposition is an equivalent form of Dickson's lemma.

Proposition 1.2 ([1, page 74]). *Given a subset $N \subset \mathbb{N}^n$, there are finitely many elements $\nu(1), \dots, \nu(s) \in N$ such that for every $\nu \in N$ there exists some $i \in \{1, 2, \dots, s\}$ such that $\nu \geqslant \nu(i)$.*

A *semiflow* (or an *S-semiflow*) is a triple (X, S, π), where X is a set, S is a unital semigroup and π is a unital semigroup action of S on X. This means that for all $s_1, s_2 \in S$, $x \in X$ we have

$$\pi(s_1, \pi(s_2, x)) = \pi(s_1 s_2, x),$$
$$\pi(e, x) = x.$$

Here e is the unital element of S. The element $\pi(s, x)$ is usually denoted by $s.x$ or sx. If $x \in X$, the set $Sx = \{s.x\colon\ s \in S\}$ is called the *orbit* of x. A subset $Y \subset X$ is *invariant* if for every $s \in S$ and $y \in Y$, $s.y \in Y$. If Y is invariant, then $(Y, S, \pi|_{S \times Y})$ is called a *subsemiflow* of (X, S, π).

The set $\mathbb{M}$ of monomials of $K[X_1, \dots, X_n]$ is partially ordered by the relation $X^\mu \leqslant X^\nu$ if $X^\mu | X^\nu$ (which, in turn, means that $X^\mu . X^\sigma = X^\nu$ for some $\sigma \in \mathbb{N}^n$, or, equivalently, $\mu \leqslant \nu$). The set $\mathbb{M}$ can be considered as an $\mathbb{M}$-semiflow under the action defined by $X^\mu . X^\nu = X^{\mu + \nu}$. Let M be a subsemiflow of the $\mathbb{M}$-semiflow $\mathbb{M}$. A subset N of M *generates* M if M is the union of orbits of elements of N. In other words, for every element X^μ of M there is an element X^ν of N such that $X^\mu \geqslant X^\nu$. We write $M = \langle N \rangle$.

Proposition 1.2 can now be reformulated (and made more precise) in the following way:

Proposition 1.3. *Every subsemiflow M of the $\mathbb{M}$-semiflow $\mathbb{M}$ is generated by a unique finite minimal (for inclusion) set of generators. This minimal set can be obtained by taking the minimal elements of any set of generators for M.*

The relation between the monomial ideals of $K[X_1, \dots, X_n]$ and the subsemiflows of the $\mathbb{M}$-semiflow $\mathbb{M}$ is further emphasized by the following

Proposition 1.4. *There is a bijective correspondence between the set of monomial ideals of $K[X_1, \dots, X_n]$ and the set of subsemiflows of the $\mathbb{M}$-semiflow $\mathbb{M}$.*

This follows either as a corollary of our Theorem 1, or as a consequence of the next

Proposition 1.5 ([1, page 74])**.** *Every monomial ideal of $K[X_1, \dots, X_n]$ has a unique finite minimal (for inclusion) generating set.*

Our goal is to find an analogue of Proposition 1.4 when, more generally, we replace $K[X_1, \dots, X_n]$ by $A = R[X_1, \dots, X_n]$, where R is a commutative Noetherian ring, and monomials by terms. A *term* in A is any expression $c_\mu X^\mu$, where $c_\mu \in R$. It turns out that we have to replace monomials as objects on which semiflows are considered by some more general objects. We obtain a bijective correspondence between the finitely generated term ideals of A and certain semiflows on those new objects. A possible use of that correspondence could be in exploiting the dynamical properties of semiflows in order to obtain some useful algebraic properties of term ideals of A. That will be the topic of our next article.

A reader interested in Constructive Mathematics can find the article [3] very interesting. It discusses some of the above ideas (for monomials of $K[X_1, \dots, X_n]$) in the context of Constructive Mathematics. Sections 1.1 and 1.2 are especially interesting in relation to this paper.

2. The set of "terms" and the function σ

Let $\mathcal{I}(R)$ be the set of the ideals of R, partially ordered by the relation $I \leqslant J$ if $I \supset J$. Consider the set

$$M_n(R) = \{IX^\mu \colon\ I \in \mathcal{I}(R),\ \mu \in \mathbb{N}^n\}.$$

We call its elements "*terms*" over $\mathcal{I}(R)$. The quotation marks are used to make a distinction between the terms from A and the "terms" from $M_n(R)$, even though it is always clear from the context which terms are used. We define multiplication on $M_n(R)$ by

$$IX^\mu \,.\, JX^\nu = (I \cap J)X^{\mu+\nu}. \tag{1}$$

This multiplication is associative and has an identity element, namely the element $R\,.\,1 = RX^0$. So $M_n(R)$ is a commutative unital semigroup. We say that $IX^\mu | JX^\nu$ if there is KX^σ such that $IX^\mu \,.\, KX^\sigma = JX^\nu$. We define a partial order on $M_n(R)$ in the following way: $IX^\mu \leqslant JX^\nu$ if $I \leqslant J$ and $X^\mu \leqslant X^\nu$.

Lemma 2.1. *$IX^\mu \leqslant JX^\nu$ if and only if $IX^\mu | JX^\nu$.*

Proof. Easy, left to the reader. □

We consider $M_n(R)$ also as an $M_n(R)$-semiflow with the action defined by (1). Let $\mathcal{M}_n(R)$ denote the set of all subsemiflows of $M_n(R)$.

For every nonempty finite subset $T = \{t_1, \ldots, t_s\}$ of $M_n(R)$, where $n \geqslant 1$ and $t_i = I_i X^{\mu(i)}$ for $i = 1, 2, \ldots, s$, we define $\sigma(T)$ as

$$\sigma(T) = (I_1 + \ldots + I_s)X^{\sup(\mu(1),\ldots,\mu(s))}.$$

Thus σ is a function from the set of nonempty finite subsets of $M_n(R)$ to $M_n(R)$.

Lemma 2.2. *Let T', T'' be two finite subsets of $M_n(R)$. Then $\sigma(\sigma(T'), \sigma(T'')) = \sigma(T' \cup T'')$.*

Proof. Let $T' = \{t'_1, \ldots, t'_r\}$ and $T'' = \{t''_1, \ldots, t''_s\}$, where $t'_i = I_i X^{\mu(i)}$, $i = 1, 2, \ldots, r$ and $t''_j = J_j X^{\nu(j)}$, $j = 1, 2, \ldots, s$. Then

$$\begin{aligned}
\sigma(\sigma(T'), \sigma(T'')) &= \sigma(I_1 + \ldots + I_r X^{\sup(\mu(1),\ldots,\mu(r))}, J_1 + \ldots + J_s X^{\sup(\nu(1),\ldots,\nu(s))}) \\
&= (I_1 + \ldots + I_r + J_1 + \ldots + J_s)X^{\sup(\sup(\ldots),\, \sup(\ldots))} \\
&= (I_1 + \ldots + I_r + J_1 + \ldots + J_s)X^{\sup(\mu(1),\ldots,\mu(r),\nu(1),\ldots,\nu(s))} \\
&= \sigma(T' \cup T'').
\end{aligned}$$

□

Lemma 2.3. *Let T', T'' be two finite subsets of $M_n(R)$. Let $s = \sigma(T')$ and $t' = \sigma(T'')$ for some $t' \in T'$. Then $s = \sigma((T' \setminus \{t'\}) \cup T'')$.*

Proof. Let $T' = \{t'_1, \ldots, t'_r\}$ and $T'' = \{t''_1, \ldots, t''_s\}$, where $t'_i = I_i X^{\mu(i)}$, $i = 1, 2, \ldots, r$ and $t''_j = J_j X^{\nu(j)}$, $j = 1, 2, \ldots, s$. We may assume that $t' = t'_r$. Denote $s = I_0 X^{\mu(0)}$. Then $I_r = J_1 + \ldots + J_s$ and $I_0 = I_1 + \ldots + I_{r-1} + I_r = I_1 + \ldots + I_{r-1} + J_1 + \ldots + J_s$. Also $X^{\mu(r)} = X^{\sup(\nu(1),\ldots,\nu(s))}$ and $X^{\mu(0)} = X^{\sup(\mu(1),\ldots,\mu(r-1),\mu(r))} = X^{\sup(\mu(1),\ldots,\mu(r-1),\sup(\nu(1),\ldots,\nu(s)))} = X^{\sup(\mu(1),\ldots,\mu(r-1),\nu(1),\ldots,\nu(s))}$. □

Lemma 2.4. *Let $t_1, \dots, t_k$ and $s_1, \dots, s_k$ be elements of $M_n(R)$ such that $s_i \geqslant t_i$ for $i = 1, 2, \dots, k$. Then $\sigma(t_1, \dots, t_k) \geqslant \sigma(s_1, \dots, s_k)$.*

P r o o f. Let $t_i = I_i X^{\mu(i)}$, $s_i = J_i X^{\nu(i)}$, $i = 1, 2, \dots, k$. Then $J_i \subset I_i$ and $\mu(i) \leqslant \nu(i)$ for $i = 1, 2, \dots, k$. Hence $J_1 + \ldots + J_k \subset I_1 + \ldots + I_k$ and $\sup(\mu(1), \dots, \mu(k)) \leqslant \sup(\nu(1), \dots, \nu(k))$. Hence $\sigma(t_1, \dots, t_k) \geqslant \sigma(s_1, \dots, s_k)$. □

Definition 2.5. If T is a nonempty subset of $M_n(R)$, the smallest subsemiflow of the $M_n(R)$-semiflow $M_n(R)$, containing T, is said to be *generated by* T and is denoted by $\langle T \rangle$.

Definition 2.6. A nonempty subset T of $M_n(R)$ is said to be *acceptable* if it is finite and for any nonempty subset $T' \subset T$ there is a $t \in T$ such that $\sigma(T') \geqslant t$. A subsemiflow M of $M_n(R)$ is said to be *acceptable* if it is generated by an acceptable subset of $M_n(R)$. The set of all acceptable subsemiflows of $M_n(R)$ is denoted by $\mathcal{M}'_n(R)$.

E x a m p l e 2.7. Here is an example of an acceptable set:

$$T = \{t_1 = 30\mathbb{Z}X^2Y,\ t_2 = 5\mathbb{Z}X^2Y^2,\ t_3 = 20\mathbb{Z}XY^2\} \subset M_2(\mathbb{Z}).$$

Here $\sigma(t_1, t_2) = \sigma(t_2, t_3) = t_2$, $\sigma(t_1, t_3) = 10\mathbb{Z}X^2Y^2 > t_2$, $\sigma(t_1, t_2, t_3) = \sigma(\sigma(t_1, t_2), \sigma(t_2, t_3)) = \sigma(t_2) = t_2$.

E x a m p l e 2.8. Here is an example of a nonacceptable subset T of $M_n(R)$: $T = \{t_1 = I_1 X^{\mu(1)}, t_2 = I_2 X^{\mu(2)}\}$, where I_1 and I_2 are incomparable and $\mu(1), \mu(2)$ are incomparable. Then $\sigma(t_1, t_2)$ is incomparable with t_1 and t_2 and so T is nonacceptable. The set $\{t_1, t_2, t_3 = \sigma(t_1, t_2)\}$ is then an acceptable set (see the next remark).

R e m a r k 2.9. Let T be any finite subset of $M_n(R)$ and let $\widehat{T}$ be the set of all $\sigma(T')$, where T' is a nonempty subset of T. Then the set $\widehat{T}$ is acceptable.

Indeed, each element of $\widehat{T}$ is of the form $\sigma(T')$ for some nonempty subset of T. This includes the elements $t \in T$ as $\sigma(\{t\}) = t$. Now for any elements $\sigma(T'_1), \sigma(T'_2), \dots, \sigma(T'_l)$ of $\widehat{T}$ we have (by Lemma 2.2) $\sigma(\sigma(T'_1), \sigma(T'_2), \dots, \sigma(T'_l)) = \sigma(T'_1 \cup T'_2 \cup \ldots \cup T'_l) \in \widehat{T}$ as $T'_1 \cup T'_2 \cup \ldots \cup T'_l$ is also a subset of T.

Lemma 2.10. *Let M be a subsemiflow of $M_n(R)$ which is generated by an acceptable subset of $M_n(R)$. Then every finite generating set of M is acceptable.*

P r o o f. Let $T = \{t_1, \dots, t_p\}$ be an acceptable set which generates M and let $S = \{s_1, \dots, s_q\}$ be another finite generating set. Let $S' = \{s_{j_1}, \dots, s_{j_k}\}$ be a nonempty subset of S. Then $s_{j_1} \geqslant t_{i_1}, \dots, s_{j_k} \geqslant t_{i_k}$ for some elements $t_{i_1}, \dots, t_{i_k}$ of T. Hence $\sigma(s_{j_1}, \dots, s_{j_k}) \geqslant \sigma(t_{i_1}, \dots, t_{i_k}) \geqslant t_l \geqslant s_m$. The first inequality holds by Lemma 2.4, the second one since T is acceptable, the third one since S is a generating set. Thus S is acceptable. □

Lemma 2.11. *If a subsemiflow M of $M_n(R)$ has a finite generating set, then every generating set of M contains a finite generating set.*

P r o o f. Suppose to the contrary. Let S be an infinite generating set of M which does not contain a finite generating set. Since M is the union of finitely many orbits (of the elements of T), at least one of the orbits contains an infinite generating set of that orbit, a subset of S, which does not contain a finite generating set of that orbit. So it is enough to assume that T consists of a single element, say $T = \{t\}$. Now t must be in the orbit of some element $s \in S$. Hence $t > s$ (if $t = s$ we have a contradiction). But now s is not in the orbit of t, a contradiction. □

Lemma 2.12. *Let M be a finitely generated subsemiflow of $M_n(R)$. Then:*

(i) *M has a unique minimal (for inclusion) finite generating set.*

(ii) *Every generating set of M contains the unique (finite) minimal generating set.*

P r o o f. (i) Let T be a finite generating set for M with the partial order induced from $M_n(R)$. Let T' be the set of minimal elements of T. Then T' still generates M and is a minimal set with that property.

Let now $T = \{t_1, \dots, t_p\}$ and $S = \{s_1, \dots, s_q\}$ be two minimal finite generating sets of M. Then for every $i \in \{1, 2, \dots, p\}$ there are a $j \in \{1, 2, \dots, q\}$ and a $k \in \{1, 2, \dots, p\}$ such that $t_i \geqslant s_j \geqslant t_k$. Since T is minimal, we get $t_i = s_j = t_k$. Hence $T \subset S$. By symmetry $S \subset T$. Thus $S = T$.

(ii) Let S be a generating set of M. By Lemma 2.11, S contains a finite generating set T. The set of minimal elements of T is the unique (by (i)) minimal generating set of M. □

In the next example we give a subsemiflow of $M_n(R)$ which does not have a finite generating set.

E x a m p l e 2.13. Let $R = \mathbb{Z}$ and $n = 1$. Denote the only variable by X. Then consider the subsemiflow M of $M_1(\mathbb{Z})$ generated by the set $\{2\mathbb{Z}X, 3\mathbb{Z}X, 5\mathbb{Z}X, \dots\}$. Any two elements of this set are incomparable. M is the union of orbits of these elements. The orbits start at their minimal elements, which are $2\mathbb{Z}X$, $3\mathbb{Z}X$, $5\mathbb{Z}X \dots$. Hence M does not have a finite generating set.

Definition 2.14. Let

$$G = \{c_{1,1}X^{\mu(1)}, \dots, c_{1,k_1}X^{\mu(1)}, \dots, c_{s,1}X^{\mu(s)}, \dots, c_{s,k_s}X^{\mu(s)}\}$$

be a (finite) generating set of a term ideal I of A. Then we say that the set

$$T = \{I_1X^{\mu(1)}, \dots, I_sX^{\mu(s)}\},$$

where, for $i = 1, 2, \ldots, s$,

$$I_i = \langle c_{i,1}, c_{i,2}, \ldots, c_{i,k_i} \rangle,$$

of "terms" from $M_n(R)$ is *associated* to G.

Lemma 2.15. *Let I be a finitely generated term ideal of A. There is a finite generating set G of I such that the set T of "terms" associated to G satisfies the following conditions:*

(a) *all "terms" in T have distinct monomials;*

(b) *if two "terms" $t_i = I_i X^{\mu(i)}$ and $t_j = I_j X^{\mu(j)}$ of T satisfy $X^{\mu(i)} < X^{\mu(j)}$, then $I_j < I_i$.*

(c) *for $t \in T$ and $T' \subset T$ a nonempty subset not containing t, it is never true that $t \geqslant \sigma(T')$.*

Proof. We will start with an arbitrary finite generating set G of I and the set T of "terms" associated to G. We will then modify G in a finitely many steps and to each new (modified) G we will associate (a new, modified) T. We end up with a finite generating set G whose associated set T satisfies the conditions (a), (b) and (c).

So let G be a finite generating set of I and let T be the set of "terms" associated to G. The condition (a) is satisfied by T because we form each "term" by forming the ideal generated by the coefficients by the same monomial and then use that ideal as the coefficient by that monomial.

Let us explain how we can adjust the generating set G so that the set T associated to it satisfies (b). Among the monomials $X^{\mu(i)}$ for which there is a monomial $X^{\mu(j)}$ with $X^{\mu(i)} < X^{\mu(j)}$ select a minimal one, say $t_i = I_i X^{\mu(i)}$. Let $t_j = I_j X^{\mu(j)}$ be such that $X^{\mu(i)} < X^{\mu(j)}$. If $t_i < t_j$, then we delete from G all the terms corresponding to t_j and replace T by $T \setminus \{t_j\}$. The new set G still generates I. If $I_j < I_i$, we do not change anything and we are done with the pair $t_i < t_j$. If I_i and I_j are incomparable, we delete from G the terms with $X^{\mu(j)}$ and replace them by the new terms, also with $X^{\mu(j)}$, but now with the coefficients generating $I_i + I_j$. We also replace the "term" t_j from T by the "term" $(I_i + I_j)X^{\mu(j)}$. Then G still generates I and we are done with the pair $t_i < t_j$. Next we consider a new pair $t_i < t_j$, with a minimal $X^{\mu(i)}$ and do the same change for it, and so on. After finitely many steps we finish adjustment of G and T and the condition (b) holds.

Let us now explain how we can adjust the generating set G so that the set T associated to it satisfies (c). We do that by induction. Suppose that $t_{j_1} \geqslant \sigma(S_{1,1})$ for two disjoint subsets $\{t_{j_1}\}$ and $S_{1,1}$ of T. Then we delete all the terms from G that correspond to t_{j_1}. The new generating set G_1 still generates the same ideal I and corresponding to it is the new set $T_1 = T \setminus \{t_{j_1}\}$.

Suppose now that we have a generating set G_k, which generates the same ideal I. Corresponding to it is the set $T_k = T \setminus \{t_{j_1}, \dots, t_{j_k}\}$. Each of t_{j_i} is $\geqslant \sigma(S_{k,i})$ for some subset $S_{k,i}$ of G_k ($i = 1, 2, \dots, k$). If now we have some $t_{j_{k+1}} \geqslant \sigma(S_{k+1,k+1})$, then we delete all the terms from G_k that correspond to $t_{j_{k+1}}$. We get the new set of generating terms, G_{k+1}, and corresponding to it a new set of "terms", $T_{k+1} = T_k \setminus \{t_{j_{k+1}}\}$. The set G_{k+1} still generates I. Indeed, if, for $i = 1, 2, \dots, k$, $t_{j_{k+1}} \in S_{k,i}$, then we put $S_{k+1,i} = (S_{k,i} \setminus \{t_{j_{k+1}}\}) \cup S_{k+1,k+1}$ and if $t_{j_{k+1}} \notin S_{k,i}$, we keep $S_{k+1,i} = S_{k,i}$. Then we have $t_{j_i} \geqslant \sigma(S_{k+1,i})$, $i = 1, 2, \dots, k$.

This inductive procedure stops after finitely many steps since we cannot delete all the generators of I. The resulting set T will still satisfy the conditions (a) and (b) since they are not affected by the changes needed for (c). Thus we end up with a set G, generating I, such that all three conditions (a), (b) and (c) hold for the set T associated to it. □

Lemma 2.16. *Let $M \in \mathcal{M}'_n(R)$. Then the unique minimal finite generating set of M is acceptable.*

Proof. Let $T = \{t_1, \dots, t_p\}$ be a finite acceptable generating set. Let $T' \subset T$ be the unique finite minimal generating set of M. We can assume that $T' = \{t_1, \dots, t_q\}$, where $q \leqslant p$. Consider $\sigma(t_{i_1}, \dots, t_{i_m})$, where $\{i_1, \dots, i_m\} \subset \{t_1, \dots, t_q\}$. If we consider $\{i_1, \dots, i_m\}$ as a subset of T, then there is a $k \in \{1, 2, \dots, p\}$ such that $\sigma(t_{i_1}, \dots, t_{i_m}) \geqslant t_k$. But there is also an element $t_j \in T'$ such that $t_k \geqslant t_j$. Hence $\sigma(t_{i_1}, \dots, t_{i_m}) \geqslant t_j$. Hence T' is acceptable. □

Lemma 2.17. *Let $M \in \mathcal{M}'_n(R)$ and let $T = \{t_1, \dots, t_s\}$ be the unique finite minimal generating set of M. (It is acceptable by Lemma 2.16.) Then T satisfies the following conditions:*

(a′) *all monomials appearing in T are distinct;*

(b′) *if two monomials appearing in T satisfy $X^{\mu(i)} < X^{\mu(j)}$, then their "coefficients" satisfy $I_j < I_i$;*

(c′) *for $t_j \in T$ and $T' \subset T$ a nonempty subset not containing t_j, it is never true that $t_j > \sigma(T')$.*

Proof. The condition (a′): Suppose $t_i = I_i X^\mu$, $t_j = I_j X^\mu$, $i \neq j$. Then I_i and I_j are incomparable. We have $\sigma(t_i, t_j) = (I_i + I_j)X^\mu$, which is $< I_i X^\mu$ and $< I_j X^\mu$. Since T is acceptable (by Lemma 2.10) there is a t_k such that $t_k \leqslant (I_i + I_j)X^\mu < t_i$. So $t_k < t_i$, contradicting the minimality of T.

The condition (b′): Suppose that t_i and t_j satisfy $X^{\mu(i)} < X^{\mu(j)}$. Then $\sigma(t_i, t_j) = (I_i + I_j)X^{\mu(j)}$. Since t_i and t_j are incomparable, we must have either $I_i > I_j$ or I_i, I_j incomparable. If they are incomparable, $I_i + I_j < I_j$, hence $\sigma(t_i, t_j) < t_j$, hence, since T is acceptable, $t_k \leqslant \sigma(t_i, t_j) < t_j$ for some k, hence $t_j > t_k$, contradicting the minimality of T. Thus $I_i > I_j$.

The condition (c′): If (c′) is not satisfied, we would have (since T is acceptable) $t_j > \sigma(T') \geqslant t_k$ for some k, hence $t_j > t_k$, contradicting the minimality of T. $\square$

3. Order preserving bijections between $\mathcal{M}'_n(R)$ and $\mathcal{T}'_n(R)$

Recall that $\mathcal{M}'_n(R)$ denotes the set of all acceptable subsemiflows of $M_n(R)$. Let $\mathcal{T}'_n(R)$ denote the set of all term ideals of A which are generated by finitely many terms. On both of these sets we assume partial order by inclusion.

Theorem 3.1. *There are two bijections $\varphi\colon \mathcal{M}'_n(R) \to \mathcal{T}'_n(R)$ and $\psi\colon \mathcal{T}'_n(R) \to \mathcal{M}'_n(R)$, inverse to each other, that preserve the partial orders.*

Proof. Definition of φ: Let $M \in \mathcal{M}'_n(R)$ and let

$$T = \{t_1, \dots, t_s\},$$

where $t_i = I_i X^{\mu(i)}$, $i = 1, 2, \dots, s$, be the unique finite minimal generating set of M. It is acceptable by Lemma 2.16. For $i = 1, 2, \dots, s$, let

$$\{c_{i,1}, c_{i,2}, \dots, c_{i,k_i}\}$$

be a generating set of I_i. Then we define $I = \varphi(M)$ in the following way:

$$I = \langle G \rangle,$$

where

$$G = \{c_{1,1}X^{\mu(1)}, \dots, c_{1,k_1}X^{\mu(1)}, \dots, c_{s,1}X^{\mu(s)}, \dots, c_{s,k_s}X^{\mu(s)}\}.$$

φ is well-defined: Suppose that for $i = 1, 2, \dots, r$ the ideal I_i is generated by the set

$$\{c'_{i,1}, c'_{i,2}, \dots, c'_{i,l_i}\}.$$

Let

$$I' = \langle G' \rangle,$$

where

$$G' = \{c'_{1,1}X^{\mu(1)}, \dots, c'_{1,l_1}X^{\mu(1)}, \dots, c'_{s,1}X^{\mu(s)}, \dots, c'_{s,l_s}X^{\mu(s)}\}.$$

We need to show that $I = I'$. It is enough to show that an arbitrary term $c_{i,j}X^{\mu(i)}$ from G belongs to I'. (Then $I \subset I'$, and, by symmetry, $I' \subset I$, so $I = I'$.) Since

$$c_{i,j} = r_{i,1}c'_{i,1} + \ldots + r_{i,l_i}c'_{i,l_i}$$

for some elements $r_{i,1}, \ldots, r_{i,l_i} \in R$, we have

$$c_{i,j}X^{\mu(i)} = r_{i,1}c'_{i,1}X^{\mu(i)} + \ldots + r_{i,l_i}c'_{i,l_i}X^{\mu(i)}.$$

Hence $c_{i,j}X^{\mu(i)} \in I'$. Thus φ is well-defined.

Definition of ψ: Let $I \in \mathcal{T}'_n(R)$ and let

$$G = \{c_{1,1}X^{\mu(1)}, \ldots, c_{1,k_1}X^{\mu(1)}, \ldots, c_{s,1}X^{\mu(s)}, \ldots, c_{s,k_s}X^{\mu(s)}\}$$

be a generating set of I. For $i = 1, 2, \ldots, s$ let

$$I_i = \langle c_{i,1}, c_{i,2}, \ldots, c_{i,k_i}\rangle.$$

The set

$$T = \{I_1X^{\mu(1)}, \ldots, I_sX^{\mu(s)}\}$$

is said to be associated to G. According to Lemma 2.15 we can adjust the set G (and hence the set T associated to it) so that we get the set G_a and the associated set T_a with G_a still generating I and T_a satisfying the conditions (a), (b), (c) from that lemma. Let $\widehat{T_a}$ be the set of all $\sigma(T')$ with T' a nonempty subset of T_a. Then $\widehat{T_a}$ is acceptable by Remark 2.9. We define $M = \psi(I) = \langle\widehat{T_a}\rangle$.

ψ is well-defined: Let H be another generating set of terms of I and

$$S = \{s_1, \ldots, s_r\},$$

where $s_i = J_iX^{\nu(i)}$, $i = 1, 2, \ldots, r$, its associated set of "terms". Suppose that S satisfies the conditions (a), (b) and (c) from Lemma 2.15. Let $\widehat{S}$ be the set of all $\sigma(S')$, where S' is a nonempty subset of S, and let $N = \langle\widehat{S}\rangle$. We need to show that $M = N$.

All the terms in I are described by the elements of $\widehat{S}$, namely every $\sigma(s_{j_1}, \ldots, s_{j_l}) = (J_{j_1} + \ldots + J_{j_l})X^{\sup(\nu(j_1), \ldots, \nu(j_l))}$ describes all the terms $c_\nu X^\nu$ with $c_\nu \in J_{j_1} + \ldots + J_{j_l}$ and $\nu \geqslant \sup(\nu(j_1), \ldots, \nu(j_l))$. Hence, since T_a and S generate the same term ideal, the sets of terms described by $\widehat{T_a}$ and $\widehat{S}$ are the same. Hence for every $S' \subset S$ there is some $T' \subset T_a$ such that $\sigma(S') \geqslant \sigma(T')$. Hence $\langle\widehat{S}\rangle \subset \langle\widehat{T}\rangle$. By symmetry $\langle\widehat{T}\rangle \subset \langle\widehat{S}\rangle$ and so $\langle\widehat{S}\rangle = \langle\widehat{T}\rangle$. Hence $M = N$ and thus ψ is well-defined.

$\psi \circ \varphi$ is the identity on $\mathcal{M}'_n(R)$: Let $M \in \mathcal{M}'_n(R)$ and let $T^* = \{t_1, \ldots, t_s\}$ be the unique finite minimal generating set of M, where $t_i = I_i X^{\mu(i)}$, $i = 1, 2, \ldots, s$. (T^* is acceptable by Lemma 2.16.) Then T^* satisfies the conditions (a′), (b′) and (c′) from Lemma 2.17. For $i = 1, 2, \ldots, s$ let $\{c_{i,1}, c_{i,2}, \ldots, c_{i,k_i}\}$ be a generating set of I_i. Then

$$I = \varphi(M) = \langle G \rangle,$$

where

$$G = \{c_{1,1}X^{\mu(1)}, \ldots, c_{1,k_1}X^{\mu(1)}, \ldots, c_{s,1}X^{\mu(s)}, \ldots, c_{s,k_s}X^{\mu(s)}\}.$$

Now we want to calculate $\psi(I)$. First we adjust the generating set G so that the set T, associated to G, satisfies (a), (b), (c). Initially $T = T^*$ and the conditions (a) and (b) hold for T since (a′) and (b′) hold for T^*. The condition (c) is not implied by the condition (c′) since we can have some cases where $t_j = \sigma(T')$. So we make adjustments to G (and T) as described in Lemma 2.15 for the condition (c). Each inequality of the type $t_j \geqslant \sigma(T')$ from that description will here be equality since initially (because of (c′)) we can have only equalities for $T = T^*$ and in each step of the inductive procedure we will be keeping equality. After finitely many steps G (and T) will be appropriately adjusted. Denote those adjusted sets by G_a and T_a. Then we form $\widehat{T_a}$ and put $\psi(I) = \langle\widehat{T_a}\rangle$. Note that in $\widehat{T_a}$ we have some of the t_i's , say $t_{i_1}, \ldots, t_{i_m}$, and some σ's of these t_i's. Some of those σ's are equal to the remaining t_i's from the original $T = T^*$ (i.e., all of the elements from $T^* \setminus \{t_{i_1}, \ldots, t_{i_m}\}$). Thus $\widehat{T_a} \supset T^*$. Since T^* is acceptable, any other of the σ's of the elements $t_{i_1}, \ldots, t_{i_m}$, say $\sigma(T'')$ where $T'' \subset \{t_{i_1}, \ldots, t_{i_m}\}$, is $> t_l$ for some l. If $t_l \in \{t_{i_1}, \ldots, t_{i_m}\}$, then $\sigma(T'')$ is not among minimal elements of $\widehat{T_a}$. If $t_l \notin \{t_{i_1}, \ldots, t_{i_m}\}$, then $t_l = \sigma(T''')$ for some $T''' \subset t_l \in \{t_{i_1}, \ldots, t_{i_m}\}$ (this follows from the construction of T_a), hence $\sigma(T'') > \sigma(T''')$ and so again $\sigma(T'')$ is not among minimal elements of $\widehat{T_a}$. This holds for any $T'' \subset \{t_{i_1}, \ldots, t_{i_m}\}$. Thus the set of minimal elements of $\widehat{T_a}$ is precisely T^*. Hence $\psi(\varphi(M)) = M$.

$\varphi \circ \psi$ is the identity on $\mathcal{T}'_n(R)$: Let $I = \langle G \rangle$ be a term ideal of A, where

$$G = \{c_{1,1}X^{\mu(1)}, \ldots, c_{1,k_1}X^{\mu(1)}, \ldots, c_{s,1}X^{\mu(s)}, \ldots, c_{s,k_s}X^{\mu(s)}\}$$

is a generating set of I satisfying the conditions (a), (b), (c) from Lemma 2.15. (We can assume that G satisfies these conditions because of Lemma 2.15.) We associate to G the set

$$T = \{t_1, \ldots, t_s\},$$

where, for $i = 1, 2, \ldots, s$,

$$I_i = \langle c_{i,1}, c_{i,2}, \ldots, c_{i,k_i} \rangle$$

and $t_i = I_i X^{\mu(i)}$. Let $\widehat{T}$ be the set of all $\sigma(T')$, where T' is a nonempty subset of T. Then

$$M = \psi(I) = \langle \widehat{T} \rangle.$$

Since $\widehat{T}$ is acceptable, the unique finite minimal generating set T^* of M is acceptable. It is obtained by selecting minimal elements of $\widehat{T}$. Because of the condition (c), which holds for T, the set T^* contains all $t_1, \ldots, t_s$. It may contain some $\sigma(T')$, $T' \subset T$, in addition to the t_i's. Suppose

$$T^* = \{t_1, \ldots, t_s, \sigma(T'_1), \ldots, \sigma(T'_r)\},$$

where $T'_1, \ldots, T'_r$ are subsets of T. Now $\varphi(\psi(I)) = \varphi(M) = \langle G^* \rangle$, where G^* is the set of terms associated to T^*. Note that G is a set of terms associated to $T \subset T^*$ and that $\langle G^* \rangle = \langle G \rangle$ since the difference of these sets consists of the terms that correspond to σ's. Hence $\varphi(\psi(I)) = I$. $\square$

In the next example we follow the steps from the proof that $\psi \circ \varphi$ is the identity on $\mathcal{M}'_n(R)$.

Example 3.2. Let M be an acceptable semiflow, generated by the acceptable set $T^* = \{t_1 = 30\mathbb{Z}X^2Y, t_2 = 5\mathbb{Z}X^2Y^2, t_3 = 20\mathbb{Z}XY^2\}$ from Example 2.7. The set T^* is the unique finite minimal generating set of M by Lemma 2.12. Let $G = \{30X^2Y, 5X^2Y^2, 20XY^2\}$ and $\varphi(M) = I = \langle G \rangle$. The set $T = T^*$ is associated to G and it satisfies the conditions (a), (b), (c). Hence $G_a = G$, $T_a = T = T^*$. Then $\widehat{T_a} = \{t_1, t_2, t_3, \sigma(t_1, t_3) = 10\mathbb{Z}X^2Y^2\}$ and $\psi(I) = \langle \widehat{T_a} \rangle$. Since $\sigma(t_1, t_3) > t_2$, we have $\psi(\varphi(M)) = M$.

Example 3.3. Let M be an acceptable semiflow generated by the acceptable set $T^* = \{t_1 = 10\mathbb{Z}X^2Y, t_2 = 5\mathbb{Z}X^2Y^2, t_3 = 15\mathbb{Z}XY^2\}$. The set T^* is the unique finite minimal generating set of M by Lemma 2.12. Let $G = \{10X^2Y, 5X^2Y^2, 15XY^2\}$ and let $T = T^*$. Let $I = \langle G \rangle$. Since $\sigma(t_1, t_3) = t_2$, we adjust G to $G_a = \{10X^2Y, 15XY^2\}$ and so $T_a = \{10\mathbb{Z}X^2Y, 15\mathbb{Z}XY^2\}$. Then $\widehat{T_a} = \{10\mathbb{Z}X^2Y, 15\mathbb{Z}XY^2, 5\mathbb{Z}X^2Y^2\}$ and so $\psi(\varphi(M)) = \langle \widehat{T_a} \rangle = M$.

We now give an example where we start with a nonacceptable semiflow M and get $\psi(\varphi(M)) \supsetneq M$ when we follow the steps from the proof that $\psi \circ \varphi$ is the identity on $\mathcal{M}'_n(R)$.

Example 3.4. Let M be the semiflow generated by the set $T^* = \{t_1 = 4\mathbb{Z}X, t_2 = 6\mathbb{Z}Y\} \subset M_2(\mathbb{Z})$. Here T^* is not acceptable since $\sigma(t_1, t_2) = 2\mathbb{Z}XY$ is not comparable with neither t_1 nor t_2. Let $G = \{4X, 6Y\}$ and $I = \varphi(M) = \langle G \rangle$. Then $G_a = G$ and $T_a = T^*$. But $\widehat{T_a} = \{t_1, t_2, \sigma(t_1, t_2)\}$ and so $\psi(\varphi(M)) = \langle t_1, t_2, \sigma(t_1, t_2) \rangle \supsetneq M$.

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