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FRACTAL CONSTRUCTION OF AN ATOMIC ARCHIMEDEAN EFFECT ALGEBRA WITH NON-ATOMIC SUBALGEBRA OF SHARP ELEMENTS

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Does there exist an atomic Archimedean lattice effect algebra with non-atomic subalgebra of sharp elements? An affirmative answer to this question is given.

Keywords: atomic Archimedean lattice effect algebra, sharp element

Classification: 06D35, 06C15, 03G12, 81P10

1. INTRODUCTION

A set E equipped with a partial commutative and associative operation $\oplus$, containing elements $\mathbf{0}$ and $\mathbf{1}$, in which the existence of a unique inverse element x' to any $x \in E$ is guaranteed, and $a \oplus \mathbf{1}$ is admitted only if $a = \mathbf{0}$ is well known as *effect algebra*. It was introduced by Foulis and Bennet [2] and simultaneously by Kôpka and Chovanec as a D-poset [4]. It can be equipped with partial order $\leqslant$ as follows: For any $x, y \in E, x \leq y$ iff there exists $z \in E$ such that $x \oplus z = y$. The effect algebra E with $\leq$ can form a lattice and in such a case it is called *lattice effect algebra* [1]. This structure generalizes both orthomodular lattices, i. e. the effect algebras in which $x \oplus x$ is not defined for any $x \in E$, and MV effect algebras, i. e. effect algebras with all pairs of elements being compatible [1], and is applied as a carrier of probability of unsharp or fuzzy events.

In connection with existence of states on lattice effect algebras properties of the subalgebra of sharp elements are studied. The following definitions are consistent with the ones in [1].

Definition 1.1. An element $x \in E$ is called a *sharp* element, if $x \wedge x' = \mathbf{0}$.

Let us denote $\mathcal{S}(E)$ the sets of all sharp elements of E. It is known that $\mathcal{S}(E)$ is an orthomodular lattice [3].

Definition 1.2. An effect algebra is called *Archimedean effect algebra*, if for every $x \in E$ there is a positive integer n_x such that $\underbrace{x \oplus \cdots \oplus x}_{n_x \text{ times}}$ is defined and $\underbrace{x \oplus \cdots \oplus x}_{n_x + 1 \text{ times}}$ is not defined. The integer n_x is called the *isotropic index* of x.

Definition 1.3. A non-zero element $x \in E$ is called an *atom* if for every $y \in E$ such that $y \leq x$ we have $y = \mathbf{0}$ or $y = x$. An effect algebra E is called *atomic* if for every non-zero element $x \in E$ there is an atom $a \in E$ such that $a \leq x$. An effect algebra E is called *non-atomic* if there is no atom in E.

Z. Riečanová studied atomicity of lattice effect algebras regarding the atomicity of $\mathcal{S}(E)$ and/or the atomicity of blocks. For instance, in [7] she proved that $\mathcal{S}(E)$ of every complete atomic Archimedean modular effect algebra is atomic.

In [8], as well as at several presentations, Z. Riečanová formulated the following open problem: Does there exist an atomic lattice effect algebra with non-atomic subalgebra $\mathcal{S}(E)$? In [5] we have found an affirmative answer to this question and even slightly more. We presented an atomic MV-effect algebra with non-atomic subalgebra $\mathcal{S}(E)$ of sharp elements. However, the example presented in [5] was a **non-Archimedean** lattice effect algebra so that the above open problem remained partially open. In particular, it remained unclear if there exists an atomic **Archimedean** lattice effect algebra with non-atomic subalgebra of $\mathcal{S}(E)$ of sharp elements. In this article we give an affirmative answer to this question.

2. ATOMIC ARCHIMEDEAN LATTICE EFFECT ALGEBRA WITH A NON-ATOMIC SUBALGEBRA OF SHARP ELEMENTS

Example 2.1. Denote $I = [0, 1)$ the real unit half open interval, $\mathbb{N} = \{0, 1, 2, \dots\}$ and

$$M = \{\mathbf{0}, a_0, a_1, a_2, \dots, a_n, \dots, \mathbf{1}\}$$

the zero-one pasting of infinitely many MV effect algebras $\{\mathbf{0}, a_i, \mathbf{1} = 2a_i\}$ for $i \in \mathbb{N}$. In particular, for every $x, y \in M$, $x \oplus y$ is defined iff

(a) $x = \mathbf{0}$ or $y = \mathbf{0}$. Then $x \oplus y = y$ or $x \oplus y = x$ respectively, or

(b) $x = y = a_i$ for some $i \in \mathbb{N}$. Then $x \oplus y = 2a_i = \mathbf{1}$.

The set E is the subset of the set M^I defined as the set of all functions $f : I \to M$ satisfying the following two conditions

(i) there exists $n \in \mathbb{N}$ such that f is constant on every half open interval of the binary division $\{0, 1/2^n, 2/2^n, \dots, i/2^n, \dots, 1\}$,

(ii) for every $x \in I$, if $f(x) = a_k$ and $j \in \{1, 2, \dots, 2^k\}$ is an integer with $(j-1)/2^k \leq x < j/2^k$ then for every $t \in [(j-1)/2^k, j/2^k)$, $f(t) = a_k$.

The effect algebra operations $\oplus$ and the complement $(\ )'$ are defined on E as the restricted ones from point-wise operations on M^I.

Proposition 2.2. The above defined set E equipped with the operations $\oplus$ and $(\ )'$ is an atomic Archimedean lattice effect algebra with non-atomic subalgebra $\mathcal{S}(E)$ of sharp elements.

P r o o f. Since the effect algebra operations $\oplus$ and the complement $(\ )'$ are defined on E as the restricted ones from point-wise operations on M^I, their properties are preserved. It also follows that the zero element $\mathbf{0}_E$ and the unit element $\mathbf{1}_E$ of E are the constant functions equal to $\mathbf{0}$ and $\mathbf{1}$ respectively on I.

We will prove that the order on E coincides with the point-wise order as well. Assume $f, g \in E$ and $f(x) \leq g(x)$ for all $x \in I$. If both functions take only the values $\mathbf{0}$ and $\mathbf{1}$ and satisfy the condition (i) then their difference satisfies the condition (i) as well and therefore $f \leq g$ with respect to the order derived from the $\oplus$ operation on E. Take an arbitrary $x_0 \in I$ and assume that $g(x_0) = a_k$ for some $k \in \mathbb{N}$. According to (ii) $g(x) = a_k$ for all x in an interval $J(j,k) = [(j-1)/2^k, j/2^k)$ for some $j \in \{1, 2, \ldots, 2^k\}$. Since $f(x_0) \leq g(x_0)$, $f(x_0) = \mathbf{0}$ or $f(x_0) = a_k$. If $f(x_0) = a_k$ then $f(x) = a_k$ for all $x \in J(j,k)$. If $f(x_0) = \mathbf{0}$ then from the assumption $f(x) \leq g(x)$ on I and from (ii) we have $f(x) = \mathbf{0}$ for all $x \in J(j,k)$. The case $f(x_0) = a_k$ can be inspected applying the duality principle interchanging $\leq$ with $\geq$ and $\mathbf{0}$ with $\mathbf{1}$. Summarizing, for any two elements $f, g \in E$ such that $f(x) \leq g(x)$ on I we obtain: if one of them takes a value a_k for some $k \in \mathbb{N}$ (and consequently it is constant on an interval $J(j,k)$) then the other one is also constant on that interval and their difference is a_k or $\mathbf{0}$. Thus, $f \leq g$ with respect to the order derived from the $\oplus$ operation on E.

It is not difficult to see that the lattice operations meet and join cannot be taken point-wise. Next, we will show what is the meet of two functions in E. For any $f, g \in E$ denote $m(x)$ the point-wise meet of f and g, i.e $m(x) = \min\{f(x), g(x)\}$ if $f(x) \leq g(x)$ or $f(x) \geq g(x)$ and $m(x) = \mathbf{0}$ otherwise. We modify the function $m(x)$ such that it becomes the greatest lower bound of f, g.

$$M(x) = \begin{cases} m(x) & \text{if } m(x) = \mathbf{0} \text{ or } \mathbf{1} \text{ or } m(x) = a_k \text{ for all } x \in J(j,k) \\ \mathbf{0} & \text{if } m(x) = a_k \text{ and } m(t) = \mathbf{0} \text{ for some } t \in J(j,k). \end{cases}$$

First, realize that $M \in E$ for any two functions $f, g \in E$. In fact, the second row of the definition of $M(x)$ secures that the condition (ii) is fulfilled. Suppose $h \in E$ is such that $h \leq f$ and $h \leq g$, i.e. $h(x) \leq f(x)$ and $h(x) \leq g(x)$ for all $x \in I$. Then $h(x) \leq m(x)$. It remains to show that $h(x) \leq M(x)$ in case $M(x) = \mathbf{0}$ and $M(x) < m(x)$, i.e. $m(x) = a_k$ and $m(t) = \mathbf{0}$ for some $t \in J(j,k)$. Let $x_0 \in I$ be such that $m(x_0) = a_k$ and $m(t) = \mathbf{0}$ for some $t \in J(j,k)$. Then at least one of the values $f(x_0), g(x_0)$ equals a_k. Thus $h(x) \leq a_k$ for all $x \in J(j,k)$. However, at the same time there is a value $t \in J(j,k)$ such that $m(t) = \mathbf{0}$ and consequently $h(t) = \mathbf{0}$. Summarizing, for $h \in E$ we have $h(x) \leq a_k$ for all $x \in J(j,k)$ and $h(t) = \mathbf{0}$ for some $t \in J(j,k)$. Hence $h(x) = \mathbf{0}$ for all $x \in J(j,k)$. The analogous property of the meet operation can be verified using the duality principle interchanging $\leq$ with $\geq$, $\wedge$ with $\vee$ and $\mathbf{0}$ with $\mathbf{1}$. Since every non-zero element $f \in E$ is non-zero on a whole interval $[(j-1)/2^n, j/2^n)$ for some $n \in \mathbb{N}$ and for some $j \in \{1, 2, \ldots, 2^n\}$, the function $\alpha(x) = a_n$ if $x \in [(j-1)/2^n, j/2^n)$ and $\alpha(x) = \mathbf{0}$ otherwise, is an atom in E with $\alpha \leq f$. Hence E is atomic.

Evidently, E is Archimedean with the isotropic index of every atom equal to 2.

It is not difficult to verify that the subalgebra $\mathcal{S}(E)$ of sharp elements of the lattice effect algebra E is the Boolean algebra consisting of all zero-one functions in E. Since the process of binary divisions of the interval I is infinite, the subalgebra $\mathcal{S}(E)$ of sharp elements is non-atomic. $\square$

Note 2.3. Denote $\chi_{i,k}$ the characteristic function of the interval $J_{i,k} = [(i-1)/2^k, i/2^k)$. The mappings $\Phi : [0, \chi_{i,k}] \to [0, \chi_{2i-1,k+1}]$ and $\Psi : [0, \chi_{i,k}] \to [0, \chi_{2i,k+1}]$ defined by $\Phi(f)(x) = f(x/2 + (i-1)/2^{k+1})$ and $\Psi(f)(x) = f(x/2 + i/2^{k+1})$ respectively, can be identified as homomorhisms of an interval of E onto its "halves". It justifies calling the example construction 'fractal'.

Note 2.4. The lattice effect algebra E defined in the Example can be generalized by replacing the MV effect algebras $\{\mathbf{0}, a_i, \mathbf{1} = 2a_i\}$ with chains of any length, i. e. $\{\mathbf{0}, a_i, 2a_i, \dots \mathbf{1} = n_i a_i\}$, where n_i represents the isotropic index of the atom a_i.

Note 2.5. The lattice effect algebra E defined in the Example is uniformly Archimedean [6], i. e. the isotropic indices of all elements are not only finite, they are bounded.

Note 2.6. To see better "inside" the above example let us describe the blocks of the lattice effect algebra E. A routine inspection shows that every block (a maximal set of mutually compatible elements) in E uniquely corresponds to a binary partition D of I into finitely many intervals $J(j_i, k_i) = [(j_i - 1)/2^k, j_i/2^k), i = 1, 2, \dots, n_D$, i. e. D is characterized by the set of boundaries of the intervals

$$\left\{ 0 = \frac{j_1 - 1}{2^{k_1}}, \frac{j_1}{2^{k_1}} = \frac{j_2 - 1}{2^{k_2}}, \frac{j_2}{2^{k_2}} = \frac{j_3 - 1}{2^{k_3}}, \dots, \frac{j_{n_D} - 1}{2^{k_{n_D - 1}}} = \frac{j_{n_D} - 1}{2^{k_{n_D}}}, \frac{j_{n_D}}{2^{k_{n_D}}} = 1 \right\}.$$

A function $f \in E$ belongs to the block corresponding to the partition D iff

$$f(x) = \sum_{i=1}^{n_D} \beta_i \alpha_{k_i}(x)$$

where

$$\alpha_{k_i}(x) = \begin{cases} \alpha_{k_i} & \text{if } x \in [(j_i - 1)/2^{k_i}, j_i/2^{k_i}) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$\beta_i \in \{0, 1, 2\}$ and $0a_{k_i} = \mathbf{0}$, $1a_{k_i} = a_{k_i}$ and $2a_{k_i} = \mathbf{1}$.

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REFERENCES

[1] A. Dvurečenskij and S. Pulmannová: New trends in Quantum Structures Theory. Kluwer Academic Publications, Dordrecht / Ister Science, Bratislava 2000.

[2] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. *24* (1994), 1331–1352.

[3] G. Jenča and Z. Riečanová: On sharp elements in lattice ordered effect algebras. BUSEFAL *80* (1999), 24–29.

[4] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.

[5] V. Olejček: An atomic MV-effect algebra with non-atomic center. Kybernetika *43* (2007), 3, 343–346.

[6] V. Olejček: Uniformly Archimedean MV-effect algebra is sharply dominating. Kybernetika *46* (2010), 6, 948–952.

[7] Z. Riečanová: Modular atomic effect algebras and the existence of subadditive states. Kybernetika *40* (2004), 4, 459–468.

[8] Z. Riečanová, I. Marinová, and M. Zajac: Some aspects of generalized prelattice effect algebras. In: Lecture Notes in Artificial Intelligence (LNAI) *4342* "Theory and Application of Relational Structures as Knowledge Instruments II" (H. de Swart et all, eds.), Springer 2006, pp. 290–317.

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