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INTERIOR REGULARITY OF WEAK SOLUTIONS TO THE PERTURBED NAVIER-STOKES EQUATIONS*

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Abstract. In this paper we establish interior regularity for weak solutions and partial regularity for suitable weak solutions of the perturbed Navier-Stokes system, which can be regarded as generalizations of the results in *L. Caffarelli*, *R. Kohn*, *L. Nirenberg*: Partial regularity of suitable weak solutions of the Navier-Stokes equations, Commun. Pure. Appl. Math. *35* (1982), 771–831, and *S. Takahashi*, On interior regularity criteria for weak solutions of the Navier-Stokes equations, Manuscr. Math. *69* (1990), 237–254.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be an open domain with smooth boundary $\partial\Omega$. The dynamical behavior of the viscous incompressible fluid flow in Ω is governed by the following perturbed Navier-Stokes system (see page 140 in [1]):

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla\pi_1 = -\partial_t v_0 - (\partial_t\omega_0) \times x - \omega_0 \times (\omega_0 \times x) - 2\omega_0 \times v, \\ \nabla \cdot v = 0, \end{cases}$$

where the translational velocity $v_0 = v_0(t)$ and the angular velocity $\omega_0 = \omega_0(t) = (0, 0, \ldots, 0, \theta(t))$ are N-vectors depending only on the time-variable t; $v = (v^1(x,t), v^2(x,t), \ldots, v^N(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure, respectively.

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Set $u = v + v_0 + \omega_0 \times x$, $\pi = \pi_1 - v_0 \cdot (\omega_0 \times x)$. Then the above equations can be rewritten as follows:

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\pi = v_0 \cdot \nabla u + (\omega_0 \times x) \cdot \nabla u - \omega_0 \times u, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$\nabla \cdot u = 0, \quad (x,t) \in \Omega \times (0,T). \tag{1.2}$$

Definition. The function u is called a *weak solution* of (1.1), (1.2) if $u \in L^\infty(0,T;L^2(\Omega))$ with $\nabla u \in L^2(0,T;L^2(\Omega))$ satisfies

$$\begin{aligned} -\int_0^T\int_\Omega u\partial_\tau v \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^T\int_\Omega (u \cdot \nabla)u \cdot v \,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^T\int_\Omega [v_0 \cdot \nabla u + (\omega_0 \times x) \cdot \nabla u - \omega_0 \times u] v \,\mathrm{d}x\,\mathrm{d}\tau \\ \text{for all } v \in C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega)). \end{aligned}$$

Furthermore, we say that the weak solution u is a *suitable weak solution* of (1.1), (1.2) if the associated pressure π is in $L^{N/2}(\Omega \times (0,T))$, and the following generalized energy inequality holds:

$$\begin{aligned} \int_\Omega |u(x,t)|^2\phi(x,t)\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant \int_0^t\int_\Omega [|u|^2(\phi_\tau + \Delta\phi) + (|u|^2 + 2\pi)u \cdot \nabla\phi]\,\mathrm{d}x\,\mathrm{d}\tau \\ - 2\int_0^t\int_\Omega |u|^2(v_0(\tau) + \omega_0(\tau) \times x) \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau \\ \text{for each } 0 \leqslant \phi \in C_0^\infty(\Omega \times (0,T)). \end{aligned}$$

In the absence of the terms on the right-hand side of (1.1), problem (1.1), (1.2) is reduced to the standard incompressible Navier-Stokes equation. In the three dimensional case, a large gap remains between the regularity available in the existence results and the additional regularity required in the sufficient conditions to guarantee the smoothness of weak solutions of the standard Navier-Stokes equations. This gap has been narrowed by the works of Iskauriaza-Seregin-Sverak [18], Ladayzhenskaya-Seregin [19], Scheffer [25], Serrin [27], Struwe [29], see also [2], [3], [4], [5], [6], [8], [9], [10], [13], [14], [15], [16], [20], [22], [23], [24], [26], [31], [32] and the references therein, which bring about a deeper understanding of the regularity. In particular, some local partial regularity results and Hausdorff dimension estimates on the possible singular set have been obtained for a class of suitable weak solutions defined and constructed in [7], where the principal tools are the so-called generalized energy inequality and a scaling argument.

The exterior problem of (1.1), (1.2) was studied in [11] from the point of view of mechanics. Hishida [17] considered the problem (1.1), (1.2) with $v_0(t) \equiv 0$, $\omega_0(t) \equiv \text{const}$, $0 < |\omega_0(t)| \ll 1$, and established an existence result in a three-dimensional exterior domain. Chen-Miyakawa [12] discussed the existence of a global weak solution in the whole space $\mathbb{R}^N$ ($N = 2, 3$) and the algebraic decay rates for the kinetic energy of the weak solutions constructed. The global existence of weak solutions of (1.1), (1.2) with inhomogeneous boundary values in smooth bounded domains can be constructed by using Galerkin methods and weak convergence theories, see [21] for example. In general domains, it is difficult to construct a global weak solution for (1.1), (1.2), because the perturbed terms contain the space variable x, which causes many difficulties at infinity (see Hishida's paper [17] for example). However, in this paper we mainly focus on the regularity of weak solutions, not the existence.

As far as we know, there are few results on the regularity of weak solutions of (1.1), (1.2). In the present paper, we try to establish regularity criteria for (1.1), (1.2), which are similar to Serrin's class in [27] and Takahashi's criteria in [30]. In addition, we also discuss partial regularity for suitable weak solutions of (1.1), (1.2), which can be regarded as an extension of results in [7]. The regularity of the solution at the point (x_0, t_0) seems to follow almost directly from the corresponding results for the non-perturbed Navier-Stokes equations, but we cannot find exact references explaining these questions. This is why we give strict and detailed proofs in the present paper.

In this paper, we always suppose that $v_0(t)$, $\omega_0(t)$ are smooth bounded on $[0, T)$, $0 < T < \infty$. Our main results read as follows:

Theorem 1.1. *Suppose that u is a weak solution of* (1.1), (1.2) *with $N = 2, 3$. Let one of the following conditions hold:*

$$\|u\|_{L^q(0,T;L^p(\Omega))} < \infty \quad \textit{for any } N < p \leqslant \infty \textit{ with } \frac{N}{p} + \frac{2}{q} \leqslant 1,$$

or

$$\|u\|_{L^\infty(0,T;L^N(\Omega))} \quad \textit{is suitable small.}$$

Then, for any Ω' with $\overline{\Omega'} \subset \Omega$ and any $0 < \sigma < T$,

$$u, \operatorname{curl} u \in L^\infty(\Omega' \times (\sigma, T)).$$

Now we recall the Lorentz space $L^{(q)}$ for $1 < q < \infty$,

$$L^{(q)}(0,T) = \Big\{ f \in L^1(0,T)\colon \sup_{\lambda>0} \lambda\{\mu\{t \in (0,T)\colon |f(t)| > \lambda\}\}^{1/q} \Big\},$$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

Define $g \in L^{(q),p}(Q)$, $Q = \Omega \times (0,T)$ if

$$\|g\|_{L^{(q),p}(Q)} = \{\|g(\cdot,t)\|_{L^p(\Omega)}\}_{L^{(q)}(0,T)} < \infty.$$

Set $Q_R(x,t) = \{(y,\tau) \in \Omega \times (0,T)\colon |y-x| < R,\ t - R^2 < \tau < t\}$.

Theorem 1.2. *Let* $1 \leqslant p,q \leqslant \infty$ *with* $N/p + 2/q \leqslant 1$ *and* $p > N$ $(N = 2,3)$. *Assume that* u *is a weak solution of* (1.1), (1.2) *in* $Q_R = Q_R(x_0,t_0) \subset \Omega \times (0,T)$. *If there exists a positive number* $\varepsilon = \varepsilon(N,p) \ll 1$ *such that*

$$\|u\|_{L^{(q),p}(Q_R)} \leqslant \varepsilon,$$

then

$$u, \operatorname{curl} u \in L^\infty(Q_{R/2}).$$

Next we deal with the partial regularity of suitable weak solutions of (1.1), (1.2). As in [7], we call a point (x,t) singular if u is not L^∞_{loc} in any neighborhood of (x,t); the remaining points, where u is locally essentially bounded, will be called regular points.

For any $X \subset \mathbb{R}^N \times \mathbb{R}$ and $k \geqslant 0$, define (see [7])

$$\mathcal{F}^k(X) = \lim_{\delta\to 0} \mathcal{F}^k_\delta(X), \quad \text{where } \mathcal{F}^k_\delta(X) = \inf\Big\{\sum_{i=1}^\infty r_i^k\colon\ X \subset \bigcup_{i=1}^\infty Q_{r_i},\ r_i < \delta\Big\}.$$

Then $\mathcal{F}^k$ is a Borel regular measure. Hausdorff measure $\mathcal{H}^k$ is defined in an entirely similar manner, but with Q_{r_i} replaced by an arbitrary closed subset in $\mathbb{R}^N \times \mathbb{R}$ with diameter at most r_i. Clearly, $\mathcal{H}^k \leqslant C\mathcal{F}^k$.

It could be easily verified that any weak solution u is in $L^q(0,T;L^p(\Omega))$ with $N/p + 2/q = 1$, $p > N = 2$. Therefore, based on Theorem 1.1, we always take the dimension $N = 3$ in the next arguments.

Theorem 1.3. *Let $\varepsilon_1 > 0$ be an absolute constant. Suppose that (u, π) is a suitable weak solution of* (1.1), (1.2) *on $Q_R(x_0, t_0) \subset \Omega \times (0, T)$ such that*

$$R^{-2} \iint_{Q_R(x_0,t_0)} (|u|^3 + |u||\pi|) \,\mathrm{d}x \,\mathrm{d}t + R^{-13/4} \int_{t_0-R^2}^{t_0} \left(\int_{|x-x_0|<R} |\pi| \,\mathrm{d}x \right)^{5/4} \mathrm{d}t \leqslant \varepsilon_1.$$

Then $|u(x,t)| \leqslant c(R)$ for almost all $(x,t) \in Q_{R/2}(x_0, t_0)$. In particular, $\mathcal{F}^{5/3}(S) = 0$, where S denotes the set of singular points.

Compared to Proposition 1 in [7], Theorem 1.3 shows that the perturbed terms in problem (1.1) have little effect on the regularity of the standard Navier-Stokes flows.

Theorem 1.4. *Let $\varepsilon_2 > 0$ be an absolute constant. Suppose that (u, π) is a suitable weak solution of* (1.1), (1.2) *on $Q_{R_1}(x_0, t_0) \subset \Omega \times (0, T)$ such that*

$$\limsup_{r \longrightarrow 0} \iint_{Q_r(x_0,t_0)} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant \varepsilon_2.$$

Then (x_0, t_0) is a regular point of u. Furthermore, $\mathcal{F}^1(S) = 0$, which implies that S has Hausdorff dimension at most 1.

Compared to Proposition 2 in [7], Theorem 1.4 reveals that the perturbed Navier-Stokes system has the same Hausdorff dimensional estimates at the singular points of suitable weak solutions of (1.1).

The local theory of partial regularity which we develop to prove Theorem 1.4 has other applications as well. We use it to study the initial value problem on $\mathbb{R}^3$ in case the initial velocity u_0 satisfies either

$$\frac{1}{2} \int_{\mathbb{R}^3} |x||u_0(x)|^2 \,\mathrm{d}x \triangleq G < \infty \tag{1.3}$$

or

$$\frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 |x|^{-1} \,\mathrm{d}x \triangleq L < L_0. \tag{1.4}$$

Condition (1.3) stipulates that u_0 should "decay sufficiently rapidly at ∞", and under such assumption we can show that u is regular for $|x|$ large enough.

Theorem 1.5. *Suppose that* (u,π) *is a suitable weak solution of* (1.1), (1.2) *on* $\mathbb{R}^3\times(0,T)$ *with initial data* $u_0\in L^2_\sigma(\mathbb{R}^3)$, *and suppose* (1.3) *holds. Then* u *is regular in the region* $\{(x,t)\in\mathbb{R}^3\times(0,T)\colon |x|^2t>K_1\}$, *where* $K_1=K_1(E,G)$ *is a constant depending only on* G *and* $E\triangleq\frac12\int_{\mathbb{R}^3}|u_0(x)|^2\,\mathrm{d}x$.

In the limit $t\to 0$, the region $\{(x,t)\in\mathbb{R}^3\times(0,T)\colon |x|^2t>K_1\}$ in Theorem 1.5 expands to include all space. However, if u_0 is regular enough (for example, $u_0\in H^1(\mathbb{R}^3)$), u is regular for a short time interval; in this case, Theorem 1.5 shows that the set of possible singular points is bounded.

Theorem 1.6. *Suppose that* (u,π) *is a suitable weak solution of* (1.1), (1.2) *on* $\mathbb{R}^3\times(0,T)$ *with initial data* $u_0\in L^2_\sigma(\mathbb{R}^3)$. *Let there exist an absolute constant* $L_0>0$ *such that* (1.4) *holds. Then* u *is regular in the region*

$$Y\triangleq\Big\{(a,s)\in\mathbb{R}^3\times(0,T)\colon |a|^2<\frac{s(L_0-L)}{2M^2}-\Big(4+\frac{M_0}{2M}\Big)s^2\Big\}.$$

Here $M_0\triangleq\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}$, $M\triangleq 2\sup\limits_{0\leqslant t<T}(|v_0(t)|+(1+t)|\omega_0(t)|)<\infty$.

R e m a r k. If $|a|>0$, $s>0$ in Theorem 1.6 are small enough, we know that the set $Y\neq\emptyset$. Condition (1.4) represents a restriction that u_0 "not be too singular" at the origin. As it turns out, the argument works only if L is sufficiently small. Theorem 1.6 shows, for an appropriate choice of the constant L_0, that a suitable weak solution with initial data satisfying (1.4) is regular in a parabolic domain above the origin.

Let $\Omega=\mathbb{R}^3$ or an exterior domain in $\mathbb{R}^3$. If the assumptions (1.3) and (1.4) are removed, then any suitable weak solution of (1.1), (1.2) is still regular if the space variable x is large enough and the time variable t is bounded below by a positive small number.

Theorem 1.7. *Let* $\Omega=\mathbb{R}^3$ *or an exterior domain in* $\mathbb{R}^3$, *and let* $\delta\in(0,T)$. *Suppose that* (u,π) *is a suitable weak solution of* (1.1), (1.2) *on* $\Omega\times(0,T)$. *Then there exist constants* $K=K(\delta,T,u,\pi)>0$, $C=C(\delta,T,u,\pi)>0$ *such that* $|u(x,t)|\leqslant C$ *for almost all* $(x,t)\in\Omega\times[\delta,T)$ *with* $|x|\geqslant K$.

R e m a r k. The results in Theorems 1.1–1.7 are similar to those obtained for the standard Navier-Stokes equations, and reveal that the perturbed terms (even if they include the velocity and its first derivatives) in problem (1.1) have little effects on the regularity of (suitable) weak solutions of (1.1), (1.2). This is what the present paper shows to the interested readers.

This paper is organized as follows: Section 2 is devoted to the proofs of Theorems 1.1, 1.2, and we establish regularity criteria for weak solutions of (1.1), (1.2). In Section 3, we focus on the partial regularity of weak solutions of (1.1), (1.2) in the 3D whole space.

2. Interior regularity for weak solutions of (1.1), (1.2)

Set

$$C_{0,\sigma}^{\infty}(\Omega) = \{u \in C_0^{\infty}(\Omega)\colon \operatorname{div} u = 0\},$$
$$L_{\sigma}^2(\Omega) = \text{the closure of } C_{0,\sigma}^{\infty}(\Omega) \text{ in } L^2(\Omega),$$
$$B_r(x_0) = \{x \in \mathbb{R}^N\colon |x - x_0| < r\},$$

and $\overline{u}_D = 1/|D| \int_D u(y)\,\mathrm{d}y$.

Consider the problem

$$\begin{cases} \partial_t v - \Delta v + \nabla b v = F \quad \text{in } \mathbb{R}^N \times (0,T), \\ v(x,0) = 0, \end{cases} \tag{2.1}$$

where $v = (v^1, v^2, \ldots, v^d)$, $d = N(N-1)/2$, $N \geqslant 2$, $F = \nabla g + h$, $\nabla g = \Big(\sum\limits_{j=1}^{N} \partial g^{ij}/\partial x_i\Big)_{i=1}^{d}$, $h = (h^1, h^2, \ldots, h^d)$, $b = (b_{jk}^i(x,t))_{i=1}^d$ for $1 \leqslant i,k \leqslant d$, $1 \leqslant j \leqslant N$, and $\nabla b v = \Big\{\sum\limits_{j=1}^{N}\sum\limits_{k=1}^{d} (\partial/\partial x_j)(b_{jk}^i(x,t)v^k(x,t))\Big\}_{i=1}^{d}$.

The function $v \in L^{s'}(0,T;L^s(\mathbb{R}^N))$ $(1 < s, s' \leqslant \infty)$ is said to be a *weak solution* of (2.1) if

$$-\int_0^T\int_{\mathbb{R}^N} (\partial_t\varphi + \Delta\varphi + b\nabla\varphi)v \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^N} F\varphi\,\mathrm{d}x\,\mathrm{d}t \quad \text{for any } \varphi \in C_0^{\infty}(\mathbb{R}^N \times (0,T)).$$

The following lemma can be found in [30].

Lemma 2.1. *Assume that* $1 \leqslant p,q \leqslant \infty$ *and* $N/p + 2/q = 1$ $(N \geqslant 2)$, l, l' *satisfy* $1/l = 1/p + 1/m$, $1/l' = 1/q + 1/m'$, *where* $2 \leqslant m \leqslant r \leqslant \infty$, $2 \leqslant m' \leqslant r' < \infty$ *satisfy* $2/m + 2/m' \leqslant N/r + 2/r' + 1$. *Suppose that* $v \in L^2(0,T;L^2(\mathbb{R}^N))$ *is a weak solution of* (2.1) *with* $F = \nabla g + h$ *and* $g, h \in L^2(0,T;L^2(\mathbb{R}^N))$. *There exists a positive constant* ε *such that*

(i) *if* $\|b\|_{L^q(0,T;L^p(\mathbb{R}^N))} \leqslant \varepsilon$, *then*

$$\|b\|_{L^{r'}(0,T;L^r(\mathbb{R}^N))} \leqslant C(\|g\|_{L^{m'}(0,T;L^m(\mathbb{R}^N))} + \|h\|_{L^{l'}(0,T;L^l(\mathbb{R}^N))});$$

or

(ii) *if* $\|b\|_{L^{(q)}(0,T;L^p(\mathbb{R}^N))} \leqslant \varepsilon$, *then*

$$\|b\|_{L^{(r')}(0,T;L^r(\mathbb{R}^N))} \leqslant C(\|g\|_{L^{(m')}(0,T;L^m(\mathbb{R}^N))} + \|h\|_{L^{(l')}(0,T;L^l(\mathbb{R}^N))}).$$

Here $C = C(N,d,T,r,r',m,m')$ *and* $\varepsilon = \varepsilon(N,d,p,r')$ *if* $p > N$, *and* $\varepsilon = \varepsilon(N,d,r,r')$ *if* $p = N$. *The exponent* r *should be*

(1) $1 < r < \infty$ *if* $p = N$, $q = \infty$, *or*

(2) $m' = r'$ *and* $1/m = 1/r + 1/N$.

Proof of Theorem 1.1. Set $\omega = \operatorname{curl} u$. Then $\omega = (\partial_2 u_3 - \partial_3 u_2, -\partial_1 u_3 + \partial_3 u_1, \partial_1 u_2 - \partial_2 u_1)$ for $N = 3$, and $\omega = \partial_1 u_2 - \partial_2 u_1$ for $N = 2$. Moreover,

$$\partial_t \omega - \Delta\omega + \operatorname{curl}((u\cdot\nabla)u) = \operatorname{curl} g(v_0(t)\cdot\nabla u + (\omega_0\times x)\cdot\nabla u - \omega_0\times u). \tag{2.2}$$

Note that for $N = 3$,

$$\operatorname{curl}(v_0(t)\cdot\nabla u) = \operatorname{curl}\Bigl(\sum_{i=1}^{3} v_0^i(t)\partial_i u\Bigr) = (v_0(t)\cdot\nabla)\omega, \tag{2.3}$$

$$\operatorname{curl}((u\cdot\nabla)u) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u, \tag{2.4}$$

and

$$\begin{aligned} \operatorname{curl}&[(\omega_0\times x)\cdot\nabla u - \omega_0\times u] \\ &= \operatorname{curl}[(-x_2\theta(t)\partial_1 + x_1\theta(t)\partial_2)u - (-u_2\theta(t), u_1\theta(t), 0)] \\ &= \bigl((\omega_0\times x)\cdot\nabla\omega_1 - \theta(t)\partial_1 u_3, (\omega_0\times x)\cdot\nabla\omega_2 - \theta(t)\partial_2 u_3, \\ &\quad (\omega_0\times x)\cdot\nabla\omega_3 + \theta(t)(\partial_2 u_2 + \partial_2 u_2)\bigr) \\ &\quad - (-\theta(t)\partial_3 u_1, -\theta(t)\partial_3 u_2, 0) \\ &= (\omega_0\times x)\cdot\nabla\omega - \omega_0\times\omega. \end{aligned} \tag{2.5}$$

Inserting (2.3)–(2.5) into (2.2), we obtain

$$\partial_t\omega - \Delta\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = (v_0(t)\cdot\nabla)\omega + (\omega_0\times x)\cdot\nabla\omega - \omega_0\times\omega, \tag{2.6}$$

which also holds for $N = 2$. For any $(x_0,t_0)\in\Omega\times(0,T)$ we have $\overline{Q_R(x_0,t_0)}\subset\Omega\times(0,T)$ with some small number $R > 0$. Set $v = \omega\psi$, where $\psi\in C_0^\infty(Q_R(x_0,t_0))$ with $\psi\equiv 1$ in $Q_{\theta R}(x_0,t_0)$, $\theta\in(0,1)$. Then $v = \omega$ in $Q_{\theta R}(x_0,t_0)$, and $v\in L^{2,2}(\mathbb{R}^N\times(-R^2+t_0,t_0))$ solves in the sense of distribution

$$\begin{cases} \partial_t v - \Delta v + \nabla b v = F(\omega) & \text{in } \mathbb{R}^N\times(t_0-R^2,t_0), \\ v(x,t_0-R^2) = 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $b = (b^i_{jk}(x,t))^d_{i=1}$ with $d = 1$ if $N = 2$, and $d = 3$ if $N = 3$, $b^i_{jk}(x,t) = u^j\delta_{ik} - u^i\delta_{jk}$, $F(\omega) = \nabla g + h$,

$$g = -2\omega \otimes (\nabla\psi) + \psi(v_0(t) + \omega_0(t) \times x) \otimes \omega,$$

and

$$h = -\omega \sum_{i=1}^{N} \partial_i(\psi v_0^i) - \psi\omega_0 \times \omega + \omega\psi_t + \omega\Delta\psi + \Bigl(\sum_{j,k=1}^{N} \omega^k b^i_{jk} \frac{\partial\psi}{\partial x_j}\Bigr)_{i=1}^{N}.$$

Applying Lemma 2.1 and using the assumptions in Theorem 1.1, we get

$$\begin{aligned} \|v\|_{L^{r'}(0,T;L^r(\mathbb{R}^N))} &\leqslant C(\|g\|_{L^{m'}(0,T;L^m(\mathbb{R}^N))} + \|h\|_{L^{l'}(0,T;L^l(\mathbb{R}^N))}) \\ &\leqslant C\|\omega\|_{L^{m',m}(Q_R)}. \end{aligned} \tag{2.7}$$

Here $L^{m',m}(Q_R) = L^{m'}(t_0 - R^2, t_0; L^m(B_R(x_0)))$. The norm $\|b\|_{L^q(0,T;L^p(\mathbb{R}^N))}$ in the above formula is sufficiently small, and the exponents $p, q, l, m, r, l', m', r'$ satisfy the relations in Lemma 2.1.

In particular, by the choice of the cut-off function ψ, one has from (2.7) that

$$\|\omega\|_{L^{r',r}(Q_{R/2})} \leqslant C\|\omega\|_{L^{m',m}(Q_R)}. \tag{2.8}$$

Step 1. Take $m = r = m' = 2$, $r' = \beta$ in (2.8), $\beta \in [2, \infty)$. Since $\omega \in L^{2,2}(Q_R)$, we deduce $\omega \in L^{\beta,2}(Q_{R/2})$ for any $\beta \in [2, \infty)$.

Step 2. Take $m = 2$, $m' = r' = \beta$ in (2.8). Then it follows from (2.8) that $\omega \in L^{\beta,r_0}(Q_{R/4})$, where $r_0 = N/(1 - 2\varepsilon_0/\beta(\beta + \varepsilon_0))$, and $\varepsilon_0 > 0$ is a small number. Moreover, $1 + N/r_0 = 2 - 2\varepsilon_0/\beta(\beta + \varepsilon_0) > N/2$ ($N = 2, 3$) if $\varepsilon_0 > 0$ is small enough.

Step 3. Take $r = \infty$, $m = r_0$, $m' = \beta$, $r' = \beta + \varepsilon_0 > m'$ in (2.8). Then

$$\frac{N}{m} + \frac{2}{m'} - \Bigl(\frac{N}{r} + \frac{2}{r'} + 1\Bigr) = \frac{2}{\beta} - \frac{2}{(\beta + \varepsilon_0)} - \frac{2}{\beta(\beta + \varepsilon_0)} = 0.$$

Hence, from (2.8) we obtain $\omega \in L^{\beta,\infty}(Q_{R/8})$ for any $\beta \in [2, \infty)$.

For any open region G with compact closure in Ω we have (see [27])

$$u(x,t) = \int_G (\nabla_x H(x - \xi)) \times \omega(\xi, t)\, \mathrm{d}\xi + A(x,t), \tag{2.9}$$

where $H(x)$ is the fundamental solution of the Laplace equation, and $A(x,t)$ is harmonic in G. Moreover, in any region $S = G \times (T_1, T_2)$ with compact closure in $\Omega \times (0,T)$, we also have

$$\omega(x,t) = \nabla K * g + B. \tag{2.10}$$

Here K is the fundamental solution of the heat equation, which is given by $K = K(x,t) = (4\pi t)^{-N/2}\mathrm{e}^{-|x|^2/(4t)}$ for $t > 0$; and $K = K(x,t) = 0$ for $t \leqslant 0$. The function $B = B(x,t)$ is a solution of $\partial_t B - \Delta B = 0$, and $|g| \leqslant C(|\omega||u| + |\nabla u| + |u|)$.

Since $\omega \in L^{\beta,\infty}(Q_{R/8})$ for any $\beta \in [2,\infty)$, we infer from (2.9) that $u, \nabla u \in L^{\beta,\infty}(Q_{R_1})$ for some $R_1 \in (0, R/8)$, and $g \in L^{q',q}(Q_{R_1})$ with $q, q' \in [2,\infty)$. As in the proof of the main theorem with non-conservative force in [27], we, together with (2.9), (2.10), obtain $u, \omega \in L^{\infty,\infty}(Q_{R_2})$ for some $R_2 \in (0, R_1)$. Using a standard covering argument, we complete the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. As in the proof of (2.8), we also have

$$\|v\|_{L^{(r')}(0,T;L^r(\mathbb{R}^N))} \leqslant \|\omega\|_{L^{m',m}(Q_R)}.$$

Note that $\omega \in L^{2,2}(Q_R)$ yields $\omega \in L^{(2),2}(Q_{R/2})$. Repeating the proof of Theorem 1.1, we complete the proof of Theorem 1.2. $\square$

3. Proofs of Theorems 1.3–1.7

Proof of Theorem 1.3. We rewrite our hypothesis as follows:

$$R^{-2}\iint_{Q_R(x_0,t_0)} (|u|^3 + |u||\pi|)\,\mathrm{d}x\,\mathrm{d}t + R^{-13/4}\int_{t_0-R^2}^{t_0}\Big(\int_{|x-x_0|<R} |\pi|\,\mathrm{d}x\Big)^{5/4}\mathrm{d}t \leqslant \varepsilon_1. \tag{3.1}$$

Note that $Q_{R/2}(a,s) \subset Q_R(x_0,t_0)$ for any $(a,s) \in Q_{R/2}(x_0,t_0)$. From (3.1) we have

$$R^{-2}\iint_{Q_{R/2}(a,s)} (|u|^3 + |u||\pi|)\,\mathrm{d}x\,\mathrm{d}t + R^{-13/4}\int_{-R^2/4+s}^{s}\Big(\int_{B_{R/2}(a)} |\pi|\,\mathrm{d}x\Big)^{5/4}\mathrm{d}t \leqslant \varepsilon_1. \tag{3.2}$$

Set $r_n = 2^{-n}R$, $Q^n = Q_{r_n}(a,s)$. Now we verify by induction that inequalities $(3.3)_n$ and $(3.4)_n$ below hold:

$$\Big(\oint_{Q^n} |u|^3\,\mathrm{d}x\,\mathrm{d}t\Big)^{4/3} + r_n^{3/5}\oint_{Q^n} |u||\pi - \overline{\pi}_n|\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon_1^{2/3}, \tag{$3.3_{n\,(n\geqslant 3)}$}$$

and

$$\sup_{-r^2+s<t\leqslant s}\oint_{|x-a|<r_n} |u(x,t)|^2\,\mathrm{d}x + r_n^{-3}\iint_{Q^n} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_0\varepsilon_1^{2/3}. \tag{$3.4_{n\,(n\geqslant 2)}$}$$

Here $\fint_{Q^n} |f| \,\mathrm{d}x\,\mathrm{d}t = |Q^n|^{-1} \iint_{Q^n} |f| \,\mathrm{d}x\,\mathrm{d}t$, and $\overline{g}_n = \overline{g}_n(t) = \fint_{|x-a|<r_n} g \,\mathrm{d}x = 1/|B_{r_n}(a)| \int_{|x-a|<r_n} g \,\mathrm{d}x$.

Step 1: $(3.3)_k$ $(3 \leqslant k \leqslant n)$ implies $(3.4)_{n+1}$. Set $\phi_n = \chi\psi_n$, where $\chi \in C^\infty$ with $0 \leqslant \chi \leqslant 1$ and $\chi \equiv 1$ on $Q^2 = Q_{r_2}(a,s)$; $\chi \equiv 0$ out of $Q_{R/3}(a,s)$; $\psi_n(x,t) = (r_n^2 + s - t)^{-N/2} \mathrm{e}^{-|x-a|^2/4(r_n^2+s-t)}$ for $t < s + r_n^2$.

Note that

$$\begin{gathered}
\partial_t \phi_n + \Delta\phi_n = 0 \quad \text{on } Q^2, \\
|\partial_t \phi_n + \Delta\phi_n| \leqslant C \quad \text{on } Q^1, \\
C_1 r_n^{-3} \leqslant \phi_n(x,t) \leqslant C_2 r_n^{-3} \quad \text{and} \quad |\nabla\phi_n(x,t)| \leqslant C r_n^{-4}, \quad \forall\,(x,t) \in Q^n, \\
|\phi_n(x,t)| \leqslant C r_k^{-3} \quad \text{and} \quad |\nabla\phi_n(x,t)| \leqslant C r_k^{-4} \quad \text{for any } (x,t) \in Q^{k-1} \setminus Q^k.
\end{gathered}$$

Taking $\phi = \phi_n$ in the generalized energy inequality, we obtain

$$\sup_{s-r^2<t\leqslant s} \fint_{|x-a|<r_{n+1}} |u(x,t)|^2 \,\mathrm{d}x + r_{n+1}^{-3} \iint_{Q^n} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C(I + II + III),$$

where

$$\begin{aligned}
I &= \left| \iint_{Q^1} |u|^2 (\partial_t\phi_n + \Delta\phi_n) \,\mathrm{d}x\,\mathrm{d}t - 2 \iint_{Q^1} |u|^2 (v_0(t) + \omega_0(t) \times x) \cdot \nabla\phi_n \,\mathrm{d}x\,\mathrm{d}t \right|, \\
II &= \iint_{Q^1} |u|^3 |\nabla\phi_n| \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

and

$$III = \left| \iint_{Q^1} \pi u \cdot \nabla\phi_n \,\mathrm{d}x\,\mathrm{d}t \right|.$$

Therefore,

$$\begin{aligned}
I &\leqslant C \iint_{Q^1} |u|^2 \,\mathrm{d}x\,\mathrm{d}t + C \sum_{k=1}^{n} \left[(r_k^{-4} + r_k^{-3}) \iint_{Q^k} |u|^2 \,\mathrm{d}x\,\mathrm{d}t \right] \\
&\leqslant C\varepsilon_1^{2/3} + C \sum_{k=1}^{n} \left[(r_k^{-4} + r_k^{-3}) r_k^{5/3} \left(\iint_{Q^k} |u|^3 \,\mathrm{d}x\,\mathrm{d}t \right)^{2/3} \right] \\
&\leqslant C\varepsilon_1^{2/3} + C \sum_{k=1}^{n} [(r_k^{-4} + r_k^{-3}) r_k^{5/3} (\varepsilon_1^{1/2} r_k^5)^{2/3} \\
&\leqslant C\varepsilon_1^{2/3} + C\varepsilon_1^{1/3} \sum_{k=1}^{n} (r_k + r_k^2) \leqslant C\varepsilon_1^{2/3}.
\end{aligned} \tag{3.5}$$

The estimates of II, III are given in [7], namely

$$II + III \leqslant C\varepsilon_1^{2/3}. \tag{3.6}$$

Combining (3.5) and (3.6), we infer that $(3.4)_{n+1}$ $(n \geqslant 3)$ holds.

Step 2: $(3.4)_k$ $(2 \leqslant k \leqslant n)$ implies $(3.3)_{n+1}$ $(n \geqslant 2)$. As in [7], we have

$$\int_{Q^{n+1}} |u|^3 \,\mathrm{d}x\,\mathrm{d}t \leqslant C^* \varepsilon_1,$$

where C^* is independent of n.

Hence,

$$\left(\int_{Q^{n+1}} |u|^3 \,\mathrm{d}x\,\mathrm{d}t\right)^{4/3} \leqslant (C^*\varepsilon_1)^{4/3} \leqslant \frac{1}{2}\varepsilon_1^{2/3},$$

provided that $(C^*\varepsilon_1)^{4/3} \leqslant \frac{1}{2}$, which is possible if ε_1 is small enough. The rest of the proof is the same as that in [7] due to the following two facts:

(i) The generalized energy inequality is not used in Step 2.

(ii) It is not difficult to verify that

$$\operatorname{div}\{(v_0(t) + \omega_0(t) \times x) \cdot \nabla u - \omega_0(t) \times u\} = 0 \quad \text{in the sense of distribution.}$$

Therefore, $-\Delta\pi = \operatorname{div}\operatorname{div}(u \otimes u)$ holds in the sense of distribution, and then the pressure π has the same formula as in [7].

From the above arguments, we conclude that for any $n \geqslant 2$

$$\fint_{|x-a|<r_n} |u(x,t)|^2 \,\mathrm{d}x \leqslant C_0\varepsilon_1^{2/3} \quad \text{for each } (a,s) \in Q_{R/2}(x_0,t_0).$$

Then $|u(a,s)|^2 \leqslant C\varepsilon_1^{2/3}$ provided that (a,s) is a Lebesgue point for u, hence almost everywhere on $Q_{R/2}(x_0,t_0)$. The proof of $\mathcal{F}^{5/3}(S) = 0$ is the same as that in [7], so we omit the details here. □

Proof of Theorem 1.4. For any $0 < r \leqslant \frac{1}{2}\varrho$ with $\varrho \leqslant R_1$, a test function ϕ in the generalized energy inequality will be chosen satisfying $\phi \in C_0^\infty(Q_\varrho(x_0,t_0))$, $0 \leqslant \phi \leqslant 1$, and $\phi \equiv 1$ on $Q_r(x_0,t_0)$, $|\nabla\phi| \leqslant C\varrho^{-1}$, $|\partial_t\phi| + |\Delta\phi| \leqslant C\varrho^{-2}$. Then

$$\begin{aligned}\left|-2\iint_{Q_\varrho(x_0,t_0)} |u|^2(v_0(t) + \omega_0(t) \times x) \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t\right| \\ \leqslant C(1+\varrho)\varrho^{-1} \iint_{Q_\varrho(x_0,t_0)} |u|^2 \,\mathrm{d}x\,\mathrm{d}t \\ \leqslant C\varrho\left(\varrho^{-2}\iint_{Q_\varrho(x_0,t_0)} |u|^3 \,\mathrm{d}x\,\mathrm{d}t\right)^{2/3}.\end{aligned}$$

The estimates of the other terms on the right-hand side of the generalized energy inequality can be obtained by following the proof of Lemma 5.5 in [7]. So Lemma 5.5 in [7] also holds for the suitable weak solution of (1.1), (1.2). Since the next arguments are the same as those of Proposition 3 in [7], we omit the details here. $\square$

P r o o f of Theorem 1.5. Let (u,π) be a suitable weak solution of (1.1), (1.2) with $G \triangleq \frac{1}{2}\int_{\mathbb{R}^3} |x||u_0(x)|^2\,\mathrm{d}x < \infty$. Using a sequence of test functions converging to $\phi \equiv 1$ in the generalized energy inequality, we conclude that for each $t > 0$

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}y\,\mathrm{d}s \leqslant \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2\,\mathrm{d}x \triangleq E. \tag{3.7}$$

As in the proofs of Lemmas 8.1, 8.2 in [7], one has for almost all $t > 0$

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^3} |x||u(x,t)|^2\,\mathrm{d}x &+ \int_0^t\int_{\mathbb{R}^3} |x||\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant A(t) + \frac{C}{2}\int_0^t\int_{\mathbb{R}^3} (1+|x|)|u|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3.8}$$

where $A(t) = G + CEt^{1/2} + Ct^{1/4}E^{3/2}$.

Set $g(t) = \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}(1+|x|)|u|^2\,\mathrm{d}x\,\mathrm{d}s$. Then (3.7), (3.8) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\,g(t) \leqslant B(t) + Cg(t), \quad \text{where } B(t) = A(t) + E.$$

After a direct calculation, we get for each $t > 0$

$$g(t) \leqslant \mathrm{e}^{Ct}\int_0^t \mathrm{e}^{-Cs}B(s)\,\mathrm{d}s. \tag{3.9}$$

Inserting (3.9) into (3.8), we conclude that

$$\frac{1}{2}\int_{\mathbb{R}^3} |x||u(x,t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3} |x||\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant A(t) + C\mathrm{e}^{Ct}\int_0^t \mathrm{e}^{-Cs}B(s)\,\mathrm{d}s.$$

The rest of the proof is similar to that of Theorem C in [7], so we omit the details here. $\square$

P r o o f of Theorem 1.6. Define $v(y,t) = u(x,t)$, where $y = x - t\xi$. Then for any $\xi \in \mathbb{R}^3$,

$$\partial_t v - \Delta v + (v-\xi)\cdot\nabla v + \nabla\pi = (v_0(t) + \omega_0(t)\times(y+t\xi))\cdot\nabla v - \omega_0(t)\times v.$$

Moreover, as in the proof of Lemma 8.3 in [7], one has for each $t > 0$

$$
\begin{aligned}
(3.10)\quad \psi(t)a(t) - a(0) + \frac{1}{k}(1 - \mathrm{e}^{-kB_0(t)}) \\
\leqslant -k\int_0^t \psi a\partial_\tau B_0 \,\mathrm{d}\tau + \widetilde{C}\int_0^t \psi a\partial_\tau B_0 \,\mathrm{d}\tau + \theta^2\int_0^t \psi(\tau)\,\mathrm{d}\tau + I(t),
\end{aligned}
$$

where $a(t) = \int_{\mathbb{R}^3} \sigma_\varepsilon |v|^2 \,\mathrm{d}x$, $B_0(t) = \int_0^t \int_{\mathbb{R}^3} \sigma_\varepsilon |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}\tau$, $\psi(t) = \mathrm{e}^{-kB_0(t)}$, $\theta = |\xi|$, $k > 0$ is an absolute constant to be chosen later, $\sigma_\varepsilon(x) = (\varepsilon + |x|^2)^{-1/2}$, $\varepsilon > 0$, and

$$
I(t) = -2\int_0^t \int_{\mathbb{R}^3} \psi |v|^2 (v_0(\tau) + \omega_0(\tau) \times (y + \tau\xi)) \cdot \nabla\sigma_\varepsilon \,\mathrm{d}y\,\mathrm{d}\tau.
$$

Set $M \triangleq \sup_{0\leqslant t<T} 2\{|v_0(t)| + (1+t)|\omega_0(t)|\} < \infty$. Then

$$
\begin{aligned}
(3.11)\qquad |I(t)| &\leqslant 2\int_0^t \int_{\mathbb{R}^3} \psi|v|^2(|v_0(\tau)| + |\omega_0(\tau)|(|y| + \tau|\xi|))|\nabla\sigma_\varepsilon| \,\mathrm{d}y\,\mathrm{d}\tau \\
&\leqslant M\int_0^t \int_{\mathbb{R}^3} \psi|v|^2(1 + |y| + |\xi|)\sigma_\varepsilon^2 \,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(3.12)\quad &\int_0^t \int_{\mathbb{R}^3} \psi|v|^2|y|\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&= \int_0^t \int_{\{|y|\geqslant 1\}} \psi|v|^2|y|\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_{\{|y|<1\}} \psi|v|^2|y|\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&\leqslant \int_0^t \int_{\mathbb{R}^3} \psi|v|^2 \,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_{\mathbb{R}^3} \psi|v|^2\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau \\
&\leqslant \|v\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}\int_0^t \psi\,\mathrm{d}\tau + \int_0^t \int_{\mathbb{R}^3} \psi|v|^2\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

Inserting (3.12) into (3.11), we conclude that

$$
\begin{aligned}
(3.13)\quad |I(t)| &\leqslant M(2+\theta)\int_0^t \int_{\mathbb{R}^3} \psi|v|^2\sigma_\varepsilon^2(y)\,\mathrm{d}y\,\mathrm{d}\tau + M\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}\int_0^t \psi\,\mathrm{d}\tau \\
&\leqslant CM(2+\theta)\int_0^t \psi(a\partial_\tau B)^{1/2}\,\mathrm{d}\tau + Mt\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\leqslant C^2\int_0^t \psi a\partial_\tau B\,\mathrm{d}\tau + [2M^2(4+\theta^2) + M\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}]t,
\end{aligned}
$$

where we have used the fact (see [7]) that $\int_{\mathbb{R}^3} \sigma_\varepsilon^2|v|^2\,\mathrm{d}x \leqslant C(a\partial_\tau B)^{1/2}$.

Combining (3.10) and (3.13), we obtain

$$\psi(t)a(t)-a(0)+\frac{1}{k}(1-\mathrm{e}^{-kB(t)}) \leqslant (-k+\widetilde{C}+C^2)\int_0^t \psi a\partial_\tau B\,\mathrm{d}\tau+D(\theta,u)t = D(\theta,u)t.$$

Here we choose $k=\widetilde{C}+C^2$ and $D(\theta,u)=2M^2(4+\theta^2)+M\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3)))}$.

Hence,

$$a(t)-\frac{1}{k}+\Big(\frac{1}{k}-a(0)-D(\theta,u)t\Big)\mathrm{e}^{kB(t)} \leqslant 0. \tag{3.14}$$

Passing to the limit $\varepsilon\to 0$ in (3.14), replacing the variable y by $x-t\xi$, and taking $L_0=k^{-1}$, we obtain

$$\int_{\mathbb{R}^3\times\{t\}}\frac{|u(x,t)|^2}{|x-t\xi|^2}\,\mathrm{d}x+(L_0-L-D(\theta,u)t)\mathrm{e}^{1/L\int_0^t\int_{\mathbb{R}^3}|\nabla u(x,\tau)|^2/|x-t\xi|^2\,\mathrm{d}x\,\mathrm{d}\tau} \leqslant L_0.$$

In particular,

$$\int_0^t\int_{\mathbb{R}^3}\frac{|\nabla u(x,\tau)|^2}{|x-t\xi|^2}\,\mathrm{d}x\,\mathrm{d}\tau<\infty, \tag{3.15}$$

provided that

$$0<t<\frac{L_0-L}{D(\theta,u)}=\frac{L_0-L}{2M^2(4+|\xi|^2)+\widetilde{M}(u)}, \tag{3.16}$$
$$\widetilde{M}(u)=M\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}.$$

Note that

$$|a|^2<\frac{(L_0-L)s}{2M^2}-\Big(4+\frac{1}{2M}\|u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big)s^2$$

is equivalent to

$$\frac{L_0-L}{D(|\xi|,u)}>s,\quad \xi=s^{-1}a.$$

Thus one can choose $t\in(s-r^2,s)$ such that (3.16) holds if $r>0$ is sufficiently small. Consequently,

$$|x-t\xi|\leqslant|x-a|+|a-t\xi|=|x-a|+|\xi||s-t|\leqslant r+|\xi|r^2\leqslant 2r\quad\text{if } |\xi|r\leqslant 1,$$

and so

$$\limsup_{r\longrightarrow 0} r^{-1}\iint_{Q_r(a,s)}|\nabla u(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau\leqslant\frac{1}{4}\limsup_{r\longrightarrow 0}\iint_{Q_r(a,s)}\frac{|\nabla u(x,\tau)|^2}{|x-t\xi|^2}\,\mathrm{d}x\,\mathrm{d}\tau=0.$$

From Theorem 1.4 we infer that u is regular at (a,s). The proof of $\mathcal{F}^1(S)=0$ is the same as that in [7], so we omit the details here. □

P r o o f of Theorem 1.7. Let $\delta > 0$ and $\Omega = \mathbb{R}^3$. For any $(x,t) \in \mathbb{R}^3 \times (\delta, T)$ we can find a small number $R > 0$ such that $Q_R(x,t) \subset \mathbb{R}^3 \times (\delta/2, T)$. Thus,

$$\begin{aligned}
J(x,t) &= R^{-2} \iint_{Q_R(x,t)} (|u|^3 + |u||\pi|)\,\mathrm{d}y\,\mathrm{d}\tau + R^{-13/4} \int_{t-R^2}^{t} \Big(\int_{B_R(x)} |\pi|\,\mathrm{d}y\Big)^{5/4}\,\mathrm{d}\tau \\
&\leqslant C(R) \Big(\iint_{Q_R(x,t)} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{3/4} \Big(\int_{t-R^2}^{t} \Big(\int_{B_R(x)} |u|^2\,\mathrm{d}y\Big)^3\,\mathrm{d}\tau\Big)^{1/4} \\
&\quad + C(R) \int_{t-R^2}^{t} \Big(\int_{B_R(x)} |u|^2\,\mathrm{d}y\Big)^{3/2}\,\mathrm{d}\tau + C(R) \iint_{Q_R(x,t)} |\pi|^{3/2}\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad + C(R) \Big(\iint_{Q_R(x,t)} |\pi|^{3/2}\Big)^{5/6}\,\mathrm{d}\tau.
\end{aligned}$$

Since (u,π) is a suitable weak solution of (1.1), (1.2), we deduce that for a fixed $R > 0$, $\lim\limits_{|x|\to\infty} J(x,t) = 0$ for any $t \in [\delta, T)$. Therefore, there exists $K = K(\delta, T, R, u, \pi) > 0$ such that for any $t \in [\delta, T)$, $J(x,t) \leqslant \varepsilon_1$ for every $x \in \mathbb{R}^3$ with $|x| \geqslant K$. Using Theorem 1.3, one can find $K_1 > K$ such that $|u(x,t)| \leqslant C$ for almost all $(x,t) \in \mathbb{R}^3 \times [\delta, T)$ with $|x| \geqslant K_1$. If Ω is an exterior domain in $\mathbb{R}^3$, the proof is similar to the above arguments, so we omit the details here. $\square$

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