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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF AN SEIR MODEL WITH PERIODIC COEFFICIENTS*

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Abstract. An SEIR model with periodic coefficients in epidemiology is considered. The global existence of periodic solutions with strictly positive components for this model is established by using the method of coincidence degree. Furthermore, a sufficient condition for the global stability of this model is obtained. An example based on the transmission of respiratory syncytial virus (RSV) is included.

Keywords: epidemic model coincidence degree, Fredholm mapping

MSC 2010: 34C25, 54H25, 92D30

1. Introduction

Epidemic model dynamics, due to its theoretical and practical significance, has been studied extensively [2], [3], [5]–[7], [13], [19]. There have been many good results on the existence of the threshold values which determine whether the infectious disease will die out, the local and global stability of the disease-free equilibrium and the endemic equilibrium, the existence of periodic solutions through Hopf bifurcation [23], [24], the persistence and extinction of the disease etc. In most of the epidemic models by far, all the parameters in the models are constants. Since periodicity is ubiquitous in nature, one can easily find many different periodic population models [9], [15], [20], [21]. Many diseases also show seasonal behavior, such as flu [8], measles, chickenpox, mumps [17], mucormycosis [1] etc. This is the case for the contagious diseases spread by mosquitos, where most of the mosquitos die out in winter but they reproduce hugely in summer, hence the spread of the disease

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is seasonal. Thus under the periodic environment, it is more realistic to investigate the corresponding epidemic models with periodic coefficients.

Compartments with labels such as S, E, I, and R are often used for the epidemiological classes. SEIR is the abbreviation of "susceptible-exposed-infectious-recovered". Acronyms for epidemiology models are often based on the flow patterns between the compartments such as SEIR model. In an SEIR model, when there is an adequate contact of a susceptible with an infective so that transmission occurs, then the susceptible enters the exposed class E of those in the latent period, who are infected but not yet infectious. After the latent period ends, the individual enters the class I of infectives, who are infectious in the sense that they are capable of transmitting the infection. When the infectious period ends, the individual enters the recovered class R consisting of those with permanent infection-acquired immunity. In 1995, Michael Y. Li and James S. Muldowney studied an SEIR model [16] in epidemiology. After that, there have been many works about the epidemic models with latent period [12], [18], [25], [26]. Relying on the above-mentioned statements, we will confine ourselves here to the case that the biological or environmental parameters are periodic with some common period. For autonomous epidemic models, the existence and stability of the positive equilibrium play an important role. A periodic solution with strictly positive components in the periodic model will play the same role as a positive equilibrium in the autonomous model does [4], [14]. Its global existence and stability are very basic and important epidemiologic problems and can be used to interpret the periodic phenomenon for some diseases. It is natural to ask for conditions under which the resulting periodic model would have a periodic solution.

The purpose of this paper is to consider the SEIR epidemic model with periodic coefficients of the form

$$\begin{cases} S'(t) = \Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t), \\ E'(t) = \beta(t)S(t)I(t) - (\mu(t) + \varepsilon(t))E(t), \\ I'(t) = \varepsilon(t)E(t) - (\mu(t) + \alpha(t) + \gamma(t))I(t), \\ R'(t) = \gamma(t)I(t) - \mu(t)R(t), \end{cases} \tag{1.1}$$

where $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\varepsilon(t)$, $\alpha(t)$, and $\gamma(t)$ are positive periodic continuous functions with the common period T. The total population N is divided into four classes: S, E, I, and R which are susceptible, exposed, infective and recovered, respectively. $\Lambda(t)$ is the recruitment rate (including newborns, immigrant et al.) at time t. The function $\beta(t)$ is the disease transmission coefficient of the disease, $\mu(t)$ is the instantaneous death rate. The instantaneous per capita rates of leaving the exposed class and infective class are denoted by $\varepsilon(t)$ and $\gamma(t)$, respectively. The function $\alpha(t)$ is the mortality induced by the disease. Because $R(t)$ does not appear in the first three equations of (1.1), system (1.1) reduces to the following 3-dimensional system:

$$\tag{1.2} \begin{cases} S'(t) = \Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t), \\ E'(t) = \beta(t)S(t)I(t) - (\mu(t) + \varepsilon(t))E(t), \\ I'(t) = \varepsilon(t)E(t) - (\mu(t) + \alpha(t) + \gamma(t))I(t). \end{cases}$$

Obviously, the global existence of positive periodic solutions of (1.2) implies that of (1.1).

The plan is the following: In Section 2, we give some preliminaries. In Section 3, coincidence degree theory proposed by Gaines and Mawhin [10] is used to establish the existence of positive periodic solutions of (1.2). In Section 4, by constructing a Lyapunov function, we establish a sufficient condition for the global stability of model (1.2).

2. Preliminaries

In this section, we will give the positivity and global existence for the solutions of (1.1). For the sake of convenience in the presentation of the results, we define the real numbers $\overline{f}$, f^l, and f^u by

$$\tag{2.1} \overline{f} = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t, \quad f^l = \min_{t\in[0,T]} f(t), \quad f^u = \max_{t\in[0,T]} f(t)$$

for a continuous T-periodic function $f(t)$.

Theorem 2.1. *The solution* $(S(t), E(t), I(t), R(t))$ *of system* (1.1) *with positive initial condition is positive and ultimately uniformly bounded on* $[0,\infty)$.

Proof. Assume the solution $(S(t), E(t), I(t), R(t))$ with a positive initial condition exists and is unique on $[0, b)$, where $0 < b \leqslant \infty$ (see [11]). Since

$$S'(t) = \Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t) \geqslant -(\beta(t)I(t) + \mu(t))S(t),$$

we have

$$S(t) \geqslant S(0)\exp\Big(-\int_0^t \beta(\theta)I(\theta) + \mu(\theta)\,\mathrm{d}\theta\Big) > 0$$

for all $t \in [0, b)$. Hence, one must have $E(t) > 0$ for all $t \in [0, b)$. Otherwise, there will exist a $t_1 \in (0, b)$ such that $E(t_1) = 0$ and $E(t) > 0$ in $(0, t_1)$. Thus for any $t \in [0, t_1)$,

$$I'(t) = \varepsilon(t)E(t) - (\mu(t) + \alpha(t) + \gamma(t))I(t) \geqslant -(\mu(t) + \alpha(t) + \gamma(t))I(t).$$

Integrating the above inequality from 0 to t yields

$$I(t) \geqslant I(0)\exp\Bigl(-\int_0^t (\mu(\theta)+\alpha(\theta)+\gamma(\theta))\,\mathrm{d}\theta\Bigr) > 0$$

for all $t \in (0, t_1)$. For $t \in [0, t_1]$,

$$E'(t) = \beta(t)S(t)I(t) - (\mu(t)+\varepsilon(t))E(t) \geqslant -(\mu(t)+\varepsilon(t))E(t).$$

Integrating this inequality from 0 to t_1, we have

$$E(t_1) \geqslant E(0)\exp\Bigl(-\int_0^{t_1} (\mu(\theta)+\varepsilon(\theta))\,\mathrm{d}\theta\Bigr) > 0,$$

a contradiction to $E(t_1) = 0$. So, $E(t) > 0$ for all $t \in [0, b)$. Using the same method, we can show that $I(t) > 0$ for all $t \in [0, b)$. From the fourth equation of system (1.1) we have $R(t) \geqslant R(0)\exp\bigl(-\int_0^t \mu(\theta)\,\mathrm{d}\theta\bigr) > 0$ on $[0, b)$. Therefore, we obtain $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for all $t \in [0, b)$. Furthermore, the total number of population $N(t)$ satisfies the equation

$$N'(t) = \Lambda(t) - \mu(t)N(t) - \alpha(t)I(t) \leqslant \Lambda^u - \mu^l N(t) \tag{2.2}$$

which implies that

$$N(t) \leqslant N(0)\mathrm{e}^{-\mu^l t} + \frac{\Lambda^u}{\mu^l}(1-\mathrm{e}^{-\mu^l t}) \leqslant N(0) + \frac{\Lambda^u}{\mu^l}.$$

Thus $(S(t), E(t), I(t), R(t))$ is bounded on $[0, b)$. Therefore, we have $b = \infty$. Again by (2.2), we get

$$\limsup_{t\to\infty} N(t) \leqslant \frac{\Lambda^u}{\mu^l}. \tag{2.3}$$

Hence $S(t)$, $E(t)$, $I(t)$, and $R(t)$ are all ultimately uniformly bounded with common upper bound Λ^u/μ^l. The proof is complete. $\square$

3. Existence of positive periodic solutions

In this section we begin by recalling the notion of Mawhin's continuation theorem, which will be used to prove the existence of positive periodic solutions. The proof is based on the following observation.

Let X and Z be normed vector spaces, let $L\colon \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and let $N\colon X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}\colon (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N\colon \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$.

For convenience of the reader, we introduce the continuation theorem [10] as follows.

Lemma 3.1. *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Assume*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda N x$,*
(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$,*
(iii) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

Consider the change of variables

$$S(t) = \exp\{u_1(t)\}, \quad E(t) = \exp\{u_2(t)\}, \quad I(t) = \exp\{u_3(t)\}.$$

Then the system (1.2) can be transformed into

$$\begin{cases} u_1'(t) = \Lambda(t)\mathrm{e}^{-u_1(t)} - \beta(t)\mathrm{e}^{u_3(t)} - \mu(t), \\ u_2'(t) = \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)} - (\mu(t)+\varepsilon(t)), \\ u_3'(t) = \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)} - (\mu(t)+\alpha(t)+\gamma(t)). \end{cases} \tag{3.1}$$

It is obvious that if equation (3.1) admits a T-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^{\mathrm{T}}$, then $(\exp u_1^*(t), \exp u_2^*(t), \exp u_3^*(t))^{\mathrm{T}}$ is a positive T-periodic solution of (1.2). Therefore, we first study equation (3.1). In order to use the continuation theorem, we first define

$$X = Z = \{u(t) = (u_1(t), u_2(t), u_3(t))^{\mathrm{T}} \in C(\mathbb{R}, \mathbb{R}^3) : u(t) = u(t+T)\}$$

with the norm

$$\|u\| = \max_{t\in[0,T]} |u_1(t)| + \max_{t\in[0,T]} |u_2(t)| + \max_{t\in[0,T]} |u_3(t)|.$$

Then $(X, \|\cdot\|)$ and $(Z, \|\cdot\|)$ are Banach spaces. Define

$$\begin{aligned}
Nu(t) &= \begin{bmatrix} \Lambda(t)\mathrm{e}^{-u_1(t)} - \beta(t)\mathrm{e}^{u_3(t)} - \mu(t) \\ \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)} - (\mu(t)+\varepsilon(t)) \\ \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)} - (\mu(t)+\alpha(t)+\gamma(t)) \end{bmatrix}, \quad u \in X, \\
Lu(t) &= \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \quad Pu = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t, \quad u \in X, \\
Qz &= \frac{1}{T}\int_0^T z(t)\,\mathrm{d}t, \quad z \in Z.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\operatorname{Dom} L &= \{u(t) = (u_1(t), u_2(t), u_3(t))^{\mathrm{T}} \in C^1(\mathbb{R}, \mathbb{R}^3)\colon\ u(t) = u(t+T)\}, \\
\operatorname{Im} P &= \operatorname{Ker} L = \mathbb{R}^3, \\
\operatorname{Im} L &= \operatorname{Ker} Q = \operatorname{Im}(I-Q) = \left\{u \in X\colon\ \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t = 0\right\}.
\end{aligned}$$

Since $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 3$, L is a Fredholm mapping of index zero. Furthermore, $\operatorname{Im} L$ is closed in X. Also, the generalized inverse K_p of L, $K_p\colon \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ admits the expression

$$K_p z = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T\int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t, \quad t \in [0,T].$$

Thus

$$QNu(t) = \begin{bmatrix} \frac{1}{T}\int_0^T \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t - \frac{1}{T}\int_0^T (\beta(t)\mathrm{e}^{u_3(t)} + \mu(t))\,\mathrm{d}t \\ \frac{1}{T}\int_0^T \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t - \frac{1}{T}\int_0^T (\mu(t)+\varepsilon(t))\,\mathrm{d}t \\ \frac{1}{T}\int_0^T \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t - \frac{1}{T}\int_0^T (\mu(t)+\alpha(t)+\gamma(t))\,\mathrm{d}t \end{bmatrix},$$

and

$$\begin{aligned}
&K_p(I-Q)Nu(t) \\
&\quad= \begin{bmatrix} \int_0^t \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t - \int_0^t (\beta(t)\mathrm{e}^{u_3(t)} + \mu(t))\,\mathrm{d}t \\ \int_0^t \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t - \int_0^t (\mu(t)+\varepsilon(t))\,\mathrm{d}t \\ \int_0^t \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t - \int_0^t (\mu(t)+\alpha(t)+\gamma(t))\,\mathrm{d}t \end{bmatrix}
\end{aligned}$$

$$
-\begin{bmatrix}
\frac{1}{T}\int_0^T\int_0^t \Lambda(s)\mathrm{e}^{-u_1(s)}\,\mathrm{d}s\,\mathrm{d}t-\frac{1}{T}\int_0^T\int_0^t(\beta(s)\mathrm{e}^{u_3(s)}+\mu(s))\,\mathrm{d}s\,\mathrm{d}t\\
\frac{1}{T}\int_0^T\int_0^t \beta(s)\mathrm{e}^{u_1(s)+u_3(s)-u_2(s)}\,\mathrm{d}s\,\mathrm{d}t-\frac{1}{T}\int_0^T\int_0^t(\mu(s)+\varepsilon(s))\,\mathrm{d}s\,\mathrm{d}t\\
\frac{1}{T}\int_0^T\int_0^t \varepsilon(s)\mathrm{e}^{u_2(s)-u_3(s)}\,\mathrm{d}s\,\mathrm{d}t-\frac{1}{T}\int_0^T\int_0^t(\mu(s)+\alpha(s)+\gamma(s))\,\mathrm{d}s\,\mathrm{d}t
\end{bmatrix}
$$

$$
-\Big(\frac{t}{T}-\frac{1}{2}\Big)\begin{bmatrix}
\int_0^T \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t-\int_0^T(\beta(t)\mathrm{e}^{u_3(t)}+\mu(t))\,\mathrm{d}t\\
\int_0^T \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t-\int_0^T(\mu(t)+\varepsilon(t))\,\mathrm{d}t\\
\int_0^T \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t-\int_0^T(\mu(t)+\alpha(t)+\gamma(t))\,\mathrm{d}t
\end{bmatrix}.
$$

It is easily seen that QN and $K_p(I-Q)N$ are continuous. By using the Arzela-Ascoli theorem, it is not difficult to show that $K_p(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$.

Now we reach the position where we search for an appropriate open bounded subset for the application of the continuation theorem. Corresponding to the operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, we have

$$
\begin{cases}
u_1'(t)=\lambda\big(\Lambda(t)\mathrm{e}^{-u_1(t)}-\beta(t)\mathrm{e}^{u_3(t)}-\mu(t)\big),\\
u_2'(t)=\lambda\big(\beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}-(\mu(t)+\varepsilon(t))\big),\\
u_3'(t)=\lambda\big(\varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}-(\mu(t)+\alpha(t)+\gamma(t))\big).
\end{cases}\tag{3.2}
$$

Assume that $u(t)=(u_1(t),u_2(t),u_3(t))^{\mathrm{T}}\in X$ is a solution of equation (3.2) for a certain $\lambda\in(0,1)$. Let $\xi_i,\eta_i\in[0,T]$ for $i=1,2,3$, be defined as

$$
u_1(\xi_1)=\min_{t\in[0,T]}u_1(t),\quad u_2(\xi_2)=\min_{t\in[0,T]}u_2(t),\quad u_3(\xi_3)=\min_{t\in[0,T]}u_3(t)\tag{3.3}
$$

and

$$
u_1(\eta_1)=\max_{t\in[0,T]}u_1(t),\quad u_2(\eta_2)=\max_{t\in[0,T]}u_2(t),\quad u_3(\eta_3)=\max_{t\in[0,T]}u_3(t).\tag{3.4}
$$

By their periodicity, we have $\dot u_i(\xi_i)=0$ and $\dot u_i(\eta_i)=0$ for $i=1,2,3$. From the third equation of (3.2) we obtain

$$
\varepsilon(\xi_3)\mathrm{e}^{u_2(\xi_3)-u_3(\xi_3)}=\mu(\xi_3)+\alpha(\xi_3)+\gamma(\xi_3)\tag{3.5}
$$

which implies

$$
\mathrm{e}^{u_2(\xi_2)-u_3(\xi_3)}\leqslant\frac{(\mu+\alpha+\gamma)^u}{\varepsilon^l}.\tag{3.6}
$$

From the second equation of (3.2) we obtain

$$\beta(\xi_2)\mathrm{e}^{u_1(\xi_2)+u_3(\xi_2)-u_2(\xi_2)} = \mu(\xi_2) + \varepsilon(\xi_2). \tag{3.7}$$

Combining (3.6) with (3.7) yields

$$\mathrm{e}^{u_1(\xi_1)} \leqslant \frac{(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u}{\beta^l\varepsilon^l}. \tag{3.8}$$

On the other hand, by the third equation of (3.2) we get

$$\varepsilon(\eta_3)\mathrm{e}^{u_2(\eta_3)-u_3(\eta_3)} = \mu(\eta_3) + \alpha(\eta_3) + \gamma(\eta_3) \tag{3.9}$$

which implies

$$\mathrm{e}^{u_2(\eta_2)-u_3(\eta_3)} \geqslant \frac{(\mu+\alpha+\gamma)^l}{\varepsilon^u}. \tag{3.10}$$

By the second equation of (3.2) we conclude that

$$\beta(\eta_2)\mathrm{e}^{u_1(\eta_2)+u_3(\eta_2)-u_2(\eta_2)} = \mu(\eta_2) + \varepsilon(\eta_2)$$

which together with (3.10) implies that

$$\mathrm{e}^{u_1(\eta_1)} \geqslant \frac{(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}{\beta^u\varepsilon^u}. \tag{3.11}$$

From the first equation of (3.2) one obtains

$$\Lambda(\xi_1)\mathrm{e}^{-u_1(\xi_1)} = \mu(\xi_1) + \beta(\xi_1)\mathrm{e}^{u_3(\xi_1)}$$

which together with (3.8) implies that

$$\mathrm{e}^{u_3(\eta_3)} \geqslant \frac{\mu^u}{\beta^u}\left[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u} - 1\right]. \tag{3.12}$$

From (3.10) and (3.12) we also have

$$\mathrm{e}^{u_2(\eta_2)} \geqslant \frac{\mu^u(\mu+\alpha+\gamma)^l}{\varepsilon^u\beta^u}\left[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u} - 1\right]. \tag{3.13}$$

By the first equation of (3.2) we get

$$\Lambda(\eta_1)\mathrm{e}^{-u_1(\eta_1)} = \mu(\eta_1) + \beta(\eta_1)\mathrm{e}^{u_3(\eta_1)}$$

which implies

$$\mathrm{e}^{u_3(\xi_3)} \leqslant \frac{\mu^l}{\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}-1\Big]. \tag{3.14}$$

Combining (3.5) with (3.14) gives

$$\mathrm{e}^{u_2(\xi_2)} \leqslant \frac{\mu^l(\mu+\alpha+\gamma)^u}{\varepsilon^l\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}-1\Big]. \tag{3.15}$$

Assume the following inequality holds:

$$\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u} > 1.$$

Then it follows that

$$\begin{aligned}
u_1(\xi_1) &\leqslant \ln\Big\{\frac{(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u}{\beta^l\varepsilon^l}\Big\},\\
u_2(\xi_2) &\leqslant \ln\Big\{\frac{\mu^l(\mu+\alpha+\gamma)^u}{\varepsilon^l\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}-1\Big]\Big\},\\
u_3(\xi_3) &\leqslant \ln\Big\{\frac{\mu^l}{\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}-1\Big]\Big\},\\
u_1(\eta_1) &\geqslant \ln\Big\{\frac{(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}{\beta^u\varepsilon^u}\Big\},\\
u_2(\eta_2) &\geqslant \ln\Big\{\frac{\mu^u(\mu+\alpha+\gamma)^l}{\varepsilon^u\beta^u}\Big[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u}-1\Big]\Big\},\\
u_3(\eta_3) &\geqslant \ln\Big\{\frac{\mu^u}{\beta^u}\Big[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u}-1\Big]\Big\}.
\end{aligned} \tag{3.16}$$

Integrating the last two equations of (3.2) over the interval $[0,T]$, we obtain

$$\int_0^T \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t = \overline{(\mu+\varepsilon)}T$$

and

$$\int_0^T \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t = \overline{(\mu+\alpha+\gamma)}T.$$

It follows from (3.16) that

$$
\begin{aligned}
(3.17)\quad u_2(t) &\leqslant u_2(\xi_2) + \int_0^T |\dot{u}_2(t)|\,\mathrm{d}t \\
&= u_2(\xi_2) + \lambda \int_0^T |\beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)} - (\mu(t)+\varepsilon(t))|\,\mathrm{d}t \\
&\leqslant u_2(\xi_2) + 2\int_0^T \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t \\
&\leqslant \ln\Big\{\frac{\mu^l(\mu+\alpha+\gamma)^u}{\varepsilon^l\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l} - 1\Big]\Big\} + 2\overline{(\mu+\varepsilon)}T \triangleq \varrho_3, \\
u_2(t) &\geqslant u_2(\eta_2) - \int_0^T |\dot{u}_2(t)|\,\mathrm{d}t \\
&= u_2(\eta_2) - \lambda \int_0^T |\beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)} - (\mu(t)+\varepsilon(t))|\,\mathrm{d}t \\
&\geqslant u_2(\eta_2) - 2\int_0^T \beta(t)\mathrm{e}^{u_1(t)+u_3(t)-u_2(t)}\,\mathrm{d}t \\
&\geqslant \ln\Big\{\frac{\mu^u(\mu+\alpha+\gamma)^l}{\varepsilon^u\beta^u}\Big[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u} - 1\Big]\Big\} - 2\overline{(\mu+\varepsilon)}T \triangleq \varrho_4, \\
u_3(t) &\leqslant u_3(\xi_3) + \int_0^T |\dot{u}_3(t)|\,\mathrm{d}t \\
&= u_3(\xi_3) + \lambda \int_0^T |\varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)} - (\mu(t)+\alpha(t)+\gamma(t))|\,\mathrm{d}t \\
&\leqslant u_3(\xi_3) + 2\int_0^T \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t \\
&\leqslant \ln\Big\{\frac{\mu^l}{\beta^l}\Big[\frac{\beta^u\Lambda^u\varepsilon^u}{\mu^l(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l} - 1\Big]\Big\} + 2\overline{(\mu+\alpha+\gamma)}T \triangleq \varrho_5, \\
u_3(t) &\geqslant u_3(\eta_3) - \int_0^T |\dot{u}_3(t)|\,\mathrm{d}t \\
&= u_3(\eta_3) - \lambda \int_0^T |\varepsilon(t)e^{u_2(t)-u_3(t)} - (\mu(t)+\alpha(t)+\gamma(t))|\,\mathrm{d}t \\
&\geqslant u_3(\eta_3) - 2\int_0^T \varepsilon(t)\mathrm{e}^{u_2(t)-u_3(t)}\,\mathrm{d}t \\
&\geqslant \ln\Big\{\frac{\mu^u}{\beta^u}\Big[\frac{\beta^l\Lambda^l\varepsilon^l}{\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u} - 1\Big]\Big\} - 2\overline{(\mu+\alpha+\gamma)}T \triangleq \varrho_6.
\end{aligned}
$$

On the other hand, integrating the first equation of (3.2) yields

$$\int_0^T \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t = \int_0^T (\mu(t) + \beta(t)\mathrm{e}^{u_3(t)})\,\mathrm{d}t \leqslant \overline{(\mu+\beta\mathrm{e}^{\varrho_5})}T.$$

This implies

$$
\begin{aligned}
(3.18)\qquad u_1(t) &\leqslant u_1(\xi_1)+\int_0^T |\dot u_1(t)|\,\mathrm{d}t\\
&= u_1(\xi_1)+\lambda\int_0^T |\Lambda(t)\mathrm{e}^{-u_1(t)}-(\mu(t)+\beta(t)\mathrm{e}^{u_3(t)})|\,\mathrm{d}t\\
&\leqslant u_1(\xi_1)+2\int_0^T \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t\\
&\leqslant \ln\Big\{\frac{(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u}{\beta^l\varepsilon^l}\Big\}+2\overline{(\mu+\beta\mathrm{e}^{\varrho_5})}T \triangleq \varrho_1,\\
u_1(t) &\geqslant u_1(\eta_1)-\int_0^T |\dot u_1(t)|\,\mathrm{d}t\\
&= u_1(\eta_1)-\lambda\int_0^T |\Lambda(t)\mathrm{e}^{-u_1(t)}-(\mu(t)+\beta(t)\mathrm{e}^{u_3(t)})|\,\mathrm{d}t\\
&\geqslant u_1(\eta_1)-2\int_0^T \Lambda(t)\mathrm{e}^{-u_1(t)}\,\mathrm{d}t\\
&\geqslant \ln\Big\{\frac{(\mu+\varepsilon)^l(\mu+\alpha+\gamma)^l}{\beta^u\varepsilon^u}\Big\}-2\overline{(\mu+\beta\mathrm{e}^{\varrho_5})}T \triangleq \varrho_2.
\end{aligned}
$$

Let $M_1=\max\{|\varrho_1|,|\varrho_2|\}$, $M_2=\max\{|\varrho_3|,|\varrho_4|\}$, and $M_3=\max\{|\varrho_5|,|\varrho_6|\}$. Note that M_1, M_2, and M_3 are independent of λ. Let us take M_0 large enough such that the only solution $(p,q,r)\in\mathbb{R}^3$ of the algebraic equation

$$
(3.19)\qquad \begin{cases}
\overline{\Lambda}-\overline{\mu}\mathrm{e}^p-\overline{\beta}\mathrm{e}^{p+r}=0,\\
\overline{\beta}\mathrm{e}^{p+r}-\overline{(\mu+\varepsilon)}\mathrm{e}^q=0,\\
\overline{\varepsilon}\mathrm{e}^q-\overline{(\mu+\alpha+\gamma)}\mathrm{e}^r=0
\end{cases}
$$

with

$$
\begin{cases}
\mathrm{e}^p=\dfrac{\overline{(\mu+\varepsilon)}\,\overline{(\mu+\alpha+\gamma)}}{\overline{\varepsilon}\overline{\beta}},\\[2mm]
\mathrm{e}^q=\dfrac{\overline{\varepsilon}\overline{\beta}\overline{\Lambda}-\overline{\mu}\,\overline{(\mu+\varepsilon)}\,\overline{(\mu+\alpha+\gamma)}}{\overline{\varepsilon}\overline{\beta}},\\[2mm]
\mathrm{e}^r=\dfrac{\overline{\varepsilon}\overline{\beta}\overline{\Lambda}-\overline{\mu}\,\overline{(\mu+\varepsilon)}\,\overline{(\mu+\alpha+\gamma)}}{\overline{\varepsilon}\overline{\beta}\,\overline{(\mu+\varepsilon)}}
\end{cases}
$$

satisfies $\|(p,q,r)^{\mathrm{T}}\|=|p|+|q|+|r|<M_0$. Let $M=M_0+M_1+M_2+M_3$ and let Ω be the open set given by

$$
\Omega=\{u(t)=(u_1(t),u_2(t),u_3(t))^{\mathrm{T}}\in X\colon\ \|u\|<M\}.
$$

It is clear that Ω satisfies the requirement (i) in the continuation theorem. When $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^3$, u is a constant vector in $\mathbb{R}^3$ with $\|u\| = M$. Then

$$QNu = \begin{bmatrix} \overline{\Lambda}\mathrm{e}^{-u_1} - \overline{\beta}\mathrm{e}^{u_3} - \overline{\mu} \\ \overline{\beta}\mathrm{e}^{u_1+u_3-u_2} - (\overline{\mu+\varepsilon}) \\ \overline{\varepsilon}\mathrm{e}^{u_2-u_3} - (\overline{\mu+\alpha+\gamma}) \end{bmatrix} \neq 0.$$

Furthermore, direct calculation produces

$$\begin{aligned} &\deg(JQN(u_1,u_2,u_3)^{\mathrm{T}}, \partial\Omega \cap \operatorname{Ker} L, (0,0,0)^{\mathrm{T}}) \\ &= \operatorname{sign} \begin{vmatrix} \overline{\Lambda}\mathrm{e}^{u_1} & 0 & -\overline{\beta}\mathrm{e}^{u_3} \\ \overline{\beta}\mathrm{e}^{u_1+u_3-u_2} & -\overline{\beta}\mathrm{e}^{u_1+u_3-u_2} & \overline{\beta}\mathrm{e}^{u_1+u_3-u_2} \\ 0 & \overline{\varepsilon}\mathrm{e}^{u_2-u_3} & -\overline{\varepsilon}\mathrm{e}^{u_2-u_3} \end{vmatrix}_{(p,q,r)} \\ &= \operatorname{sign}(-\overline{\varepsilon}\overline{\beta}^2\mathrm{e}^{p+r}) = -1 \neq 0, \end{aligned}$$

where J can be the identity mapping and (p,q,r) is the unique solution to equation (3.19). By the continuation theorem, system (3.1) admits at least one T-periodic solution. Now, we are able to state our main result.

Theorem 3.1. *If* $\beta^l\Lambda^l\varepsilon^l/\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u > 1$, *then system* (1.2) *has at least one* T*-periodic solution with strictly positive components.*

Remark 3.1. Clearly, when $\beta^l\Lambda^l\varepsilon^l/\mu^u(\mu+\varepsilon)^u(\mu+\alpha+\gamma)^u > 1$, system (1.1) has at least one positive T-periodic solution. When all coefficients $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\varepsilon(t)$, $\alpha(t)$, and $\gamma(t)$ are positive constants, then system (1.1) degenerates into an autonomous system. The basic reproductive number can be given by $\mathcal{R}_0 = \beta\Lambda\varepsilon/\mu(\mu+\varepsilon)(\mu+\alpha+\gamma)$. By Theorem 3.1, $\mathcal{R}_0 > 1$ implies the existence of a positive equilibrium.

4. Global stability

In this section we will use the method of Lyapunov functions to study the global stability for system (1.2).

Definition 4.1. System (1.2) is said to be globally asymptotically stable (or globally attractive) if, for any two solutions $(S_1(t), E_1(t), I_1(t))$ and $(S_2(t), E_2(t), I_2(t))$ with positive initial values, the following equation holds:

$$\lim_{t\to\infty}(|S_1(t)-S_2(t)| + |E_1(t)-E_2(t)| + |I_1(t)-I_2(t)|) = 0.$$

Theorem 4.1. *If $\overline{\mu} > 2\overline{\beta}\Lambda^u/\mu^l$, then system (1.2) is globally asymptotically stable.*

Proof. Let $(S_1(t), E_1(t), I_1(t))$ and $(S_2(t), E_2(t), I_2(t))$ be any two solutions of (1.2) with positive initial values. Since $\overline{\mu} > 2\overline{\beta}\Lambda^u/\mu^l$, we can find an $\varepsilon > 0$ small enough such that

$$\overline{\mu} > 2\overline{\beta}\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big). \tag{4.1}$$

By (2.3), one can ensure that $N(t) \leqslant \Lambda^u/\mu^l + \varepsilon$ for sufficiently large t. Without loss of generality, we assume

$$S_i(t) + E_i(t) + I_i(t) \leqslant \frac{\Lambda^u}{\mu^l} + \varepsilon, \quad i = 1, 2.$$

We construct

$$V(t) = |S_1(t) - S_2(t)| + |E_1(t) - E_2(t)| + |I_1(t) - I_2(t)|.$$

The upper right derivative of $V(t)$ along (1.2) is given by

$$\begin{aligned}
D^+V(t) = {} & \operatorname{sign}\{S_1(t) - S_2(t)\}(-\beta(t)S_1(t)I_1(t) + \beta(t)S_2(t)I_2(t) \\
& - \mu(t)S_1(t) + \mu(t)S_2(t)) + \operatorname{sign}\{E_1(t) - E_2(t)\} \\
& \times (\beta(t)S_1(t)I_1(t) - \beta(t)S_2(t)I_2(t) - (\mu(t) + \varepsilon(t))(E_1(t) - E_2(t))) \\
& + \operatorname{sign}\{I_1(t) - I_2(t)\}(\varepsilon(t)(E_1(t) - E_2(t)) \\
& - (\mu(t) + \alpha(t) + \gamma(t))(I_1(t) - I_2(t))) \\
\leqslant {} & 2\beta(t)|S_1(t)I_1(t) - S_2(t)I_2(t)| - \mu(t)|S_1(t) - S_2(t)| \\
& - \mu(t)|E_1(t) - E_2(t)| - (\mu(t) + \alpha(t) + \gamma(t))|I_1(t) - I_2(t)| \\
\leqslant {} & -\Big(\mu(t) - 2\beta(t)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)|S_1(t) - S_2(t)| - \mu(t)|E_1(t) - E_2(t)| \\
& - \Big(\mu(t) + \alpha(t) + \gamma(t) - 2\beta(t)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)|I_1(t) - I_2(t)| \\
\leqslant {} & -\Big(\mu(t) - 2\beta(t)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)V(t).
\end{aligned} \tag{4.2}$$

For any $t \geqslant 0$, there exists an $n \in \mathbb{N}$ satisfying $nT \leqslant t < (n+1)T$, which implies $n \to \infty$ as $t \to \infty$. Applying (4.1) and (4.2) yields

$$\begin{aligned}
V(t) &\leqslant V(0)\exp\Big(-\int_0^t \Big(\mu(s) - 2\beta(s)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)\,\mathrm{d}s\Big)\\
&\leqslant V(0)\exp\Big(-\int_0^{nT} \Big(\mu(s) - 2\beta(s)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)\,\mathrm{d}s\Big)\\
&\quad \times \exp\Big(\int_{nT}^t \Big|\mu(s) - 2\beta(s)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big|\,\mathrm{d}s\Big)\\
&= V(0)\exp\Big(-\Big(\overline{\mu} - 2\overline{\beta}\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)nT\Big)\\
&\quad \times \exp\Big(\int_{nT}^t \Big|\mu(s) - 2\beta(s)\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big|\,\mathrm{d}s\Big)\\
&\leqslant V(0)\exp\Big(\mathcal{M}T - \Big(\overline{\mu} - 2\overline{\beta}\Big(\frac{\Lambda^u}{\mu^l} + \varepsilon\Big)\Big)nT\Big),
\end{aligned}$$

where $\mathcal{M} = \max\limits_{s\in[0,T]} |\mu(s) - 2\beta(s)(\Lambda^u/\mu^l + \varepsilon)|$. The above inequality implies that $V(t) \to 0$ as $t \to \infty$. Therefore, the system (1.2) is globally asymptotically stable. This completes the proof. $\square$

Remark 4.1. Here we need to show that the conditions in Theorem 3.1 and Theorem 4.1 cannot hold simultaneously. In fact, the conditions in both these theorems are sufficient but not necessary. The stability of the positive periodic solution is a very interesting open problem, which can be considered in our future research.

Example 4.1. We choose parameters in (1.2) similar to those in [22] for respiratory syncytial virus (RSV), one kind of childhood disease. We fix $\mu = 0.041$; $\beta(t) = 256(1 + 0.2\cos(2\pi t + 0.26))$; $\varepsilon = 91$; $\alpha = 0$; $\gamma = 36$. We also assume that the annual recruitment rate is periodic due to opening and closing of schools, set $\Lambda(t) = 1 + 0.2\cos(2\pi t)$. Then we obtain

$$\begin{cases}
S'(t) = 1 + 0.2\cos(2\pi t) - 0.041S(t) - 256(1 + 0.2\cos(2\pi t + 0.26))S(t)I(t),\\
E'(t) = 256(1 + 0.2\cos(2\pi t + 0.26))S(t)I(t) - 0.041E(t) - 91E(t),\\
I'(t) = 91E(t) - 0.041I(t) - 36I(t).
\end{cases}$$

Clearly, the condition in Theorem 3.1 holds for this system. Numerical simulation is depicted in Fig. 1.

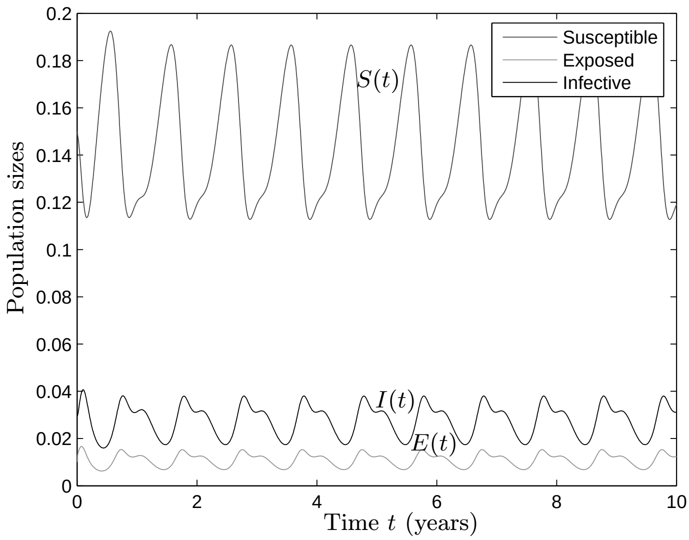

Figure 1. The figure shows the movement paths of S, E, and I as functions of time t. The 1-periodic positive solution exists with initial value $(0.15, 0.01, 0.03)$.

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