Applications of Mathematics

Lubomír Kubáček

Linear error propagation law and plug-in estimators

Applications of Mathematics, Vol. 57 (2012), No. 6, 655–666

Persistent URL: http://dml.cz/dmlcz/143009

57 (2012) APPLICATIONS OF MATHEMATICS No. 6, 655–666

LINEAR ERROR PROPAGATION LAW AND PLUG-IN ESTIMATORS*

Lubomír Kubáček, Olomouc

(Received October 29, 2009)

Abstract. In mixed linear statistical models the best linear unbiased estimators need a known covariance matrix. However, the variance components must be usually estimated. Thus a problem arises what is the covariance matrix of the plug-in estimators.

Keywords: mixed linear model, variance components, plug-in estimator

MSC 2010: 62J05

Introduction

In mixed linear statistical models [7] the estimated parameters of the covariance matrix (variance components) which must be used for the estimation of some parameters of the mean value of the observation vector make it necessary to use the plug-in estimator instead of the BLUE (best linear unbiased estimator). This enlarges the variances and the problem is to determine this enlargement in comparison to the variances of the BLUE.

The aim of the paper is to contribute to the solution of the problem.

1. Notation and preliminaries

The notation $\mathbf{Y} \sim N_n\Bigl(\mathbf{X}\boldsymbol{\beta}, \sum\limits_{i=1}^{p} \vartheta_i \mathbf{V}_i\Bigr)$ means that $\mathbf{Y}$ is an n-dimensional normally distributed random vector (observation vector) with the mean value $E(\mathbf{Y})$ equal to $\mathbf{X}\boldsymbol{\beta}$. The $n \times k$ matrix $\mathbf{X}$ is given and its rank $r(\mathbf{X})$ is equal to $k < n$. The k-dimensional vector $\boldsymbol{\beta}$ is an unknown vector parameter which must be estimated.

* Supported by the Czech Government under research project MSM 6198959214.

The vector $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ of the variance components must be estimated as well. It is assumed that $\vartheta_i > 0$, $i = 1, \ldots, p$, and the symmetric matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are positive semidefinite and known. In the following text it is assumed that $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is positive definite in a neighbourhood of a chosen point $\boldsymbol{\vartheta}_0$.

The BLUE of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = \left[\mathbf{X}'\Big(\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\Big)^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'\Big(\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\Big)^{-1}\mathbf{Y}$$

(in more detail see [5]). If the vector $\boldsymbol{\vartheta}$ is estimated, then the plug-in estimator is

$$\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}}) = \left[\mathbf{X}'\Big(\sum_{i=1}^{p} \hat{\vartheta}_i \mathbf{V}_i\Big)^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'\Big(\sum_{i=1}^{p} \hat{\vartheta}_i \mathbf{V}_i\Big)^{-1}\mathbf{Y},$$

where $\hat{\boldsymbol{\vartheta}} = (\hat{\vartheta}_1, \ldots, \hat{\vartheta}_p)'$ is an estimator of $\boldsymbol{\vartheta}$.

The $\boldsymbol{\vartheta}_0$-MINQUE (minimum norm quadratic unbiased estimator) of $\boldsymbol{\vartheta}$ is (in more detail see [7])

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_X\Sigma_0 M_X)^+}\begin{pmatrix} \mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y} \\ \vdots \\ \mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_p(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y} \end{pmatrix}, \tag{1}$$

where $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})'$ is an approximate value of the vector $\boldsymbol{\vartheta}$, $\mathbf{M}_X = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is the projection matrix on the Euclidean complement of the subspace $\mathcal{M}\{\mathbf{Xu}\colon\ \mathbf{u} \in \mathbb{R}^k\}$ and $\mathbb{R}^k$ is the k-dimensional real linear vector space. The (i,j)th entry of the matrix $\boldsymbol{\Sigma}_{(M_X\Sigma_0 M_X)^+}$ is

$$\{\mathbf{S}_{(M_X\Sigma_0 M_X)^+}\}_{i,j} = \operatorname{Tr}[\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+], \quad i,j = 1, \ldots, p.$$

The symbol $(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+$ means the Moore-Penrose generalized inverse of the matrix $\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X$ and under our assumption it can be expressed as

$$(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+ = \boldsymbol{\Sigma}_0^{-1} - \boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1} \tag{2}$$

(in more detail see [6]).

The symbol $*$ means the Hadamard multiplication [6]. It is defined as follows. Let $\mathbf{A}_{n,m}$ and $\mathbf{B}_{n,m}$ be $n \times m$ matrices. Then $\{\mathbf{A}_{n,m} * \mathbf{B}_{n,m}\}_{i,j} = a_{i,j}b_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where $a_{i,j} = \{\mathbf{A}_{n,m}\}_{i,j}$, $b_{i,j} = \{\mathbf{B}_{n,m}\}_{i,j}$. (It is used in Lemma 2.3.)

In the case of normality, i.e. $\mathbf{Y} \sim N_n\Big(\mathbf{X}\boldsymbol{\beta}, \sum\limits_{i=1}^{p} \vartheta_i \mathbf{V}_i\Big)$, an important property of the estimation (1) is that its covariance matrix $\operatorname{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}})$ can be expressed as

$$\operatorname*{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}_{(M_X\Sigma_0 M_X)^+}. \tag{3}$$

The symbol $\otimes$ means the tensor (Kronecker) multiplication [6], i.e.

$$\mathbf{A}\otimes\mathbf{B} = \begin{pmatrix} a_{1,1}\mathbf{B}, & \dots, & a_{1,m}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n,1}\mathbf{B}, & \dots, & a_{n,m}\mathbf{B} \end{pmatrix},$$

$\mathbf{1}_n = (1,\dots,1)' \in \mathbb{R}^n$ and $\mathbf{e}_i^{(n)} \in \mathbb{R}^n$ is the vector with the ith entry equal to 1 and the other entries are zero.

The problem is to find $\operatorname{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\vartheta}})]$ at least approximately.

In the following text it is assumed that $\boldsymbol{\vartheta}_0$ is obtained by the iterative procedure, i.e.

$$\boldsymbol{\vartheta}_{s+1} = \mathbf{S}^{-1}_{[M_x\Sigma(\vartheta_s)M_X]^+} \begin{pmatrix} \mathbf{Y}'[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_s)\mathbf{M}_X]^+\mathbf{V}_1[\mathbf{M}_x\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_s)\mathbf{M}_X]^+\mathbf{Y} \\ \vdots \\ \mathbf{Y}'[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_s)\mathbf{M}_X]^+\mathbf{V}_p[\mathbf{M}_x\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_s)\mathbf{M}_X]^+\mathbf{Y} \end{pmatrix}.$$

Lemma 1.1. *The random variable* $\mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y}$ *can be expressed as* $\mathbf{v}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}$, *where* $\mathbf{v} = \mathbf{Y} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{Y}$.

Proof. It is a direct consequence of (2). □

Lemma 1.2. *Let* $\mathbf{v} \sim N_n(\mathbf{0},\mathbf{T})$ *and* $t_{i,j} = \{\mathbf{T}\}_{i,j}$, $i,j = 1,\dots,n$. *Then*

$$E(v_iv_jv_kv_lv_rv_s) = \begin{cases} t_{i,j}t_{k,l}t_{r,s} + t_{i,j}t_{k,r}t_{l,s} + t_{i,j}t_{k,s}t_{l,r} \\ +t_{i,k}t_{j,l}t_{r,s} + t_{i,k}t_{j,r}t_{l,s} + t_{i,k}t_{j,s}t_{l,r} \\ +t_{i,l}t_{j,k}t_{r,s} + t_{i,l}t_{j,r}t_{k,s} + t_{i,l}t_{j,s}t_{k,r} \\ +t_{i,r}t_{j,k}t_{l,s} + t_{i,r}t_{j,l}t_{k,s} + t_{i,r}t_{j,s}t_{k,l} \\ +t_{i,s}t_{j,k}t_{l,r} + t_{i,s}t_{j,l}t_{k,r} + t_{i,s}t_{j,r}t_{k,l}, \end{cases}$$

$i,j,k,l,r,s = 1,\dots,n$.

Proof. The characteristic function of $\mathbf{v}$ is

$$\varphi_v(\mathbf{u}) = E[\exp(\mathrm{i}\mathbf{u}'\mathbf{v})] = \exp\Big(-\frac{1}{2}\mathbf{u}'\mathbf{T}\mathbf{u}\Big).$$

Thus

$$\frac{\partial^6(\varphi_v(\mathbf{u}))}{\partial u_i\partial u_j\partial u_k\partial u_l\partial u_r\partial u_s}\Big|_{u=0} = -E(v_iv_jv_kv_lv_rv_s).$$

After simple however rather tedious calculation we obtain the statement of the lemma. □

R e m a r k 1.3. If the $n^3 \times n^3$ matrix

$$\Phi_6 = E[(\mathbf{v}\mathbf{v}') \otimes (\mathbf{v}\mathbf{v}') \otimes (\mathbf{v}\mathbf{v}')]$$

is used, then $E(v_i v_j v_k v_l v_r v_s)$ is its entry in the position given by the matrix

$$\mathbf{e}_i^{(n)} \otimes \big(\mathbf{e}_j^{(n)}\big)' \otimes \mathbf{e}_k^{(n)} \otimes \big(\mathbf{e}_l^{(n)}\big)' \otimes \mathbf{e}_r^{(n)} \otimes \big(\mathbf{e}_s^{(n)}\big)'.$$

2. Linear error propagation law

Lemma 2.1. *Let $\boldsymbol{\vartheta}_0$ be an approximate value of $\boldsymbol{\vartheta}$,*

$$\mathbf{v} = \mathbf{Y} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{Y},$$

and let the matrix $\mathbf{S}_{(M_x\Sigma_0 M_X)^+}$ be regular. Here

$$\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{V}_i, \quad \boldsymbol{\vartheta}_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})'.$$

Then

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}}) \approx \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) - \sum_{i=1}^{p} \mathbf{C}_0\mathbf{X}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}\widehat{\delta\boldsymbol{\vartheta}}_i,$$

where $\mathbf{C}_0 = \mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X}$, and

$$\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_X\Sigma_0 M_X)^+} \begin{pmatrix} \mathbf{v}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}\mathbf{v} \\ \vdots \\ \mathbf{v}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_p\boldsymbol{\Sigma}_0^{-1}\mathbf{v} \end{pmatrix}.$$

P r o o f. We have

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}}) \approx \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) + \frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}'}\widehat{\delta\boldsymbol{\vartheta}},$$

where

$$\begin{aligned}\frac{\partial\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)}{\partial\vartheta_i} &= \frac{\partial}{\partial\vartheta_i}[(\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)]\mathbf{Y} \\ &= \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Y} \\ &\quad - \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Y} = -\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}.\end{aligned}$$

The equality $(\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X)^{+}\mathbf{Y} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}$ is obvious (see Lemma 1.1). □

Let

$$\{\mathbf{S}^{-1}_{(M_X\Sigma_0 M_X)^+}\}_{i,j} = s^{i,j}, \quad i,j = 1,\dots,p,$$

$$\mathbf{D} = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{C}_0^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) = \mathbf{C}_0^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0),$$

$$\mathbf{U}_j = \sum_{i=1}^{p} s^{i,j}\mathbf{C}_0^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0).$$

Then

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}}) \approx \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) - \sum_{j=1}^{p} \mathbf{U}_j\mathbf{v}\mathbf{v}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v},$$

since

$$\mathbf{D}\mathbf{v} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\{\mathbf{Y} - \mathbf{X}[\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Y}\} = \mathbf{0}.$$

In the case of normality the vectors $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)$ and $\mathbf{v}$ are stochastically independent and

$$E_{\vartheta_0}(\mathbf{U}_j\mathbf{v}\mathbf{v}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}) = \mathbf{0}.$$

Thus the following statement is valid.

Lemma 2.2. *The covariance matrix of the plug-in estimator $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})$ is approximately given by the expression*

$$\mathbf{C}_0^{-1} + E_{\vartheta_0}\Bigg(\sum_{j=1}^{p}\sum_{l=1}^{p} \mathbf{U}_j\mathbf{v}\mathbf{v}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}\mathbf{v}'\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_l\mathbf{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{v}\mathbf{v}'\mathbf{U}_l'\Bigg).$$

Lemma 2.3. *Let*

$$\mathbf{A}_{j,l} = E_{\vartheta_0}(\mathbf{v}\mathbf{v}'\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\mathbf{v}\mathbf{v}'\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\mathbf{v}\mathbf{v}').$$

Then

$$\{\mathbf{A}_{j,l}\}_{r,s} = \mathbf{1}'_{n^2}\{\mathbf{K}_{r,s} * [(\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}) \otimes (\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1})]\}\mathbf{1}_{n^2},$$

where

$$\mathbf{K}_{r,s} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n}\big[\mathbf{e}_\alpha^{(n)} \otimes \big(\mathbf{e}_\beta^{(n)}\big)' \otimes \mathbf{e}_\gamma^{(n)} \otimes \big(\mathbf{e}_\delta^{(n)}\big)'\big] E_{\vartheta_0}(v_r v_s v_\alpha v_\beta v_\gamma v_\delta).$$

Here $\mathbf{v} \sim N_n(\mathbf{0}, \mathbf{\Sigma}_0 - \mathbf{X}\mathbf{C}_0^{-1}\mathbf{X}')$ (see Lemma 1.2).

P r o o f. The (r,s)-entry of the matrix $\mathbf{A}_{j,l}$ is

$$
\begin{aligned}
(4)\quad & \{\mathbf{A}_{j,l}\}_{r,s} \\
&= E_{\vartheta_0}\Bigg(v_r v_s \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n} v_\alpha \big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big\}_{\alpha,\beta} v_\beta v_\gamma \big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big\}_{\gamma,\delta} v_\delta\Bigg) \\
&= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n} E_{\vartheta_0}(v_r v_s v_\alpha v_\beta v_\gamma v_\delta)\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big\}_{\alpha,\beta}\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big\}_{\gamma,\delta}.
\end{aligned}
$$

The definition of the Kronecker multiplication implies the equality

$$
\begin{aligned}
&\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n}\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big\}_{\alpha,\beta}\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big\}_{\gamma,\delta} \\
&\qquad = \mathbf{1}'_{n^2}\big[\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big)\otimes\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big)\big]\mathbf{1}_{n^2}.
\end{aligned}
$$

Since in (4) each term

$$
\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big\}_{\alpha,\beta}\big\{\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big\}_{\gamma,\delta}
$$

is multiplied by $E_{\vartheta_0}(v_r v_s v_\alpha v_\beta v_\gamma v_\delta)$ it is sufficient to use the Hadamard multiplication, i.e.

$$
\{\mathbf{A}_{j,l}\}_{r,s} = \mathbf{1}'_{n^2}\big\{\mathbf{K}_{r,s} * \big[\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big)\otimes\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big)\big]\big\}\mathbf{1}_{n^2}.
$$

□

The full expression for $\operatorname{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0+\widehat{\delta\boldsymbol\vartheta})]$ is

$$
\begin{aligned}
&\operatorname*{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0+\widehat{\delta\boldsymbol\vartheta})] \\
&\quad\approx \mathbf{C}_0^{-1} + \sum_{j=1}^{n}\sum_{l=1}^{n}\Bigg\{\Bigg(\sum_{i=1}^{p}\mathbf{C}_0^{-1}\mathbf{X}'\mathbf{\Sigma}_0^{-1}s^{i,j}\mathbf{V}_i\mathbf{\Sigma}_0^{-1}\Bigg)\sum_{r=1}^{n}\sum_{s=1}^{n}\big[\mathbf{e}_r^{(n)}\otimes\big(\mathbf{e}_s^{(n)}\big)'\big] \\
&\qquad\times \mathbf{1}'_{n^2}\Bigg(\Bigg\{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n}\big[\mathbf{e}_\alpha^{(n)}\otimes\big(\mathbf{e}_\beta^{(n)}\big)'\otimes\mathbf{e}_\gamma^{(n)}\otimes\big(\mathbf{e}_\delta^{(n)}\big)'\big]E_{\vartheta_0}(v_r v_s v_\alpha v_\beta v_\gamma v_\delta\Bigg\} \\
&\qquad * \big[\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_j\mathbf{\Sigma}_0^{-1}\big)\otimes\big(\mathbf{\Sigma}_0^{-1}\mathbf{V}_l\mathbf{\Sigma}_0^{-1}\big)\big]\Bigg)\mathbf{1}_{n^2}\Bigg(\sum_{k=1}^{n}\mathbf{\Sigma}_0^{-1}\mathbf{V}_k s^{k,l}\mathbf{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}\Bigg)'\Bigg\}.
\end{aligned}
$$

Thus the enlargement of $\operatorname{Var}_{\vartheta_0}[\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)]$ caused by the use of $\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0+\widehat{\delta\boldsymbol\vartheta})$ instead of $\hat{\boldsymbol\beta}(\boldsymbol\vartheta_0)$ is relatively complicated for numerical calculation even for an approximation only.

R e m a r k 2.4. The linear error propagation law gives a good results in such a case only when the standard deviation of the random argument is relatively small as compared with its mean value. If $\eta = f(\xi)$, where $\xi \sim_1 (\mu, \sigma^2)$, then the approximation $\mathrm{Var}(\eta) \approx (\mathrm{d}f(\mu)/\mathrm{d}\mu)^2\sigma^2$ is admissible if σ/μ is sufficiently small, e.g. $\sigma/\mu < 0.1$ (in more detail see [1]).

If instead of ξ the estimation of the variance components $\hat{\vartheta}_1, \ldots, \hat{\vartheta}_p$ is considered in the case of normality, i.e. $\mathbf{Y} \sim N_n\Big(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\Big)$, the values (see (3))

$$\frac{\sqrt{\{2\mathbf{S}^{-1}_{(M_X\Sigma_0 M_X)^+}\}_{i,i}}}{\vartheta_i}, \quad i = 1, \ldots, p,$$

must be sufficiently small.

3. Observation vector $\mathbf{Y}$ and the estimator $\hat{\boldsymbol{\vartheta}}$ of the vector $\boldsymbol{\vartheta}$ are independent

The expression for $\mathrm{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})]$ given in the preceding section needs a relatively tedious numerical calculation.

The situation seems to be simpler if the experiment is replicated. It is not so rare, since in experimental science a rule "one measurement is no measurement" governs. In a replicated experiment the residual vector $\mathbf{v}$ and the estimator $\hat{\boldsymbol{\vartheta}}$ can be determined in such a way that they are, in the case of normality, stochastically independent.

The situation is also favourable in the case that $\boldsymbol{\vartheta}$ is estimated from another experiment and thus also the vectors $\mathbf{Y}$ and $\hat{\boldsymbol{\vartheta}}$ are independent.

Lemma 3.1. *In an m-times replicated experiment the model considered is*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_m \end{pmatrix} \sim N_{mn}\Big[(\mathbf{1}_m \otimes \mathbf{X}_{n,k})\boldsymbol{\beta}, \Big(\mathbf{I}_{m,m} \otimes \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\Big)\Big].$$

Let

$$\mathbf{S} = \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{Y}_i - \overline{\mathbf{Y}}(\mathbf{Y}_i - \overline{\mathbf{Y}}))', \quad \overline{\mathbf{Y}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{Y}_i$$

and $\mathbf{S}_{\Sigma_0^{-1}}$, where

$$\{\mathbf{S}_{\Sigma_0^{-1}}\}_{i,j} = \mathrm{Tr}\big(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\big), \quad i,j = 1, \ldots, p,$$

be regular. Then one of the unbiased estimators of $\boldsymbol{\vartheta}$ *is*

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{\Sigma_0^{-1}} \begin{pmatrix} \mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}) \\ \vdots \\ \mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_p\boldsymbol{\Sigma}_0^{-1}) \end{pmatrix}, \quad \underset{\vartheta_0}{\mathrm{Var}}(\hat{\boldsymbol{\vartheta}}) = \frac{2}{m-1}\,\mathbf{S}^{-1}_{\Sigma_0^{-1}},$$

and $\hat{\boldsymbol{\vartheta}}$ *and*

$$\mathbf{v} = \overline{\mathbf{Y}} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\overline{\mathbf{Y}} \sim N_n\Big[\mathbf{0}, \frac{1}{m}(\boldsymbol{\Sigma}_0 - \mathbf{X}\mathbf{C}_0^{-1}\mathbf{X}')\Big]$$

are stochastically independent.

P r o o f. Since $E_\vartheta(\mathbf{S}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, the estimator $\hat{\boldsymbol{\vartheta}}$ is obviously an unbiased estimator of $\boldsymbol{\vartheta}$. The expression for $\mathrm{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}})$ is implied by the equality

$$\mathrm{cov}_{\vartheta_0}[\mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}), \mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})] = \frac{2}{m-1}\,\mathrm{Tr}(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}).$$

The matrix $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \frac{1}{m-1}\sum_{i=1}^{m-1} \boldsymbol{\xi}_i\boldsymbol{\xi}_i',$$

where $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{m-1}$ are i.i.d. and $\boldsymbol{\xi}_i \sim N_n(\mathbf{0}, \boldsymbol{\Sigma}_0)$. Thus

$$\begin{aligned}
&\mathrm{cov}_{\vartheta_0}\,[\mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}), \mathrm{Tr}(\mathbf{S}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1})] \\
&\quad = \frac{1}{(m-1)^2}\,\mathrm{cov}_{\vartheta_0}\Big[\mathrm{Tr}\Big(\sum_{k=1}^{m-1}\boldsymbol{\xi}_k\boldsymbol{\xi}_k'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\Big), \mathrm{Tr}\Big(\sum_{k=1}^{m-1}\boldsymbol{\xi}_k\boldsymbol{\xi}_k'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\Big)\Big] \\
&\quad = \frac{1}{(m-1)^2}\sum_{k=1}^{m-1}\sum_{l=1}^{m-1}\mathrm{cov}_{\vartheta_0}(\boldsymbol{\xi}_k'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\xi}_k, \boldsymbol{\xi}_l'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\xi}_l) \\
&\quad = \frac{1}{(m-1)^2}\sum_{k+1}^{m-1}\mathrm{cov}_{\vartheta_0}(\boldsymbol{\xi}_k\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\xi}_k, \boldsymbol{\xi}_k'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\xi}_k) \\
&\quad = \frac{2}{m-1}\,\mathrm{Tr}(\boldsymbol{\Sigma}_0\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}_0\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}) \\
&\quad = \frac{2}{m-1}\,\mathrm{Tr}(\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}).
\end{aligned}$$

Here the relationships

$$\boldsymbol{\xi} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma}_0) \Rightarrow \forall\,\{\mathbf{A} = \mathbf{A}', \mathbf{B} = \mathbf{B}'\}\,\mathrm{cov}_{\vartheta_0}(\boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi}, \boldsymbol{\xi}'\mathbf{B}\boldsymbol{\xi}) = 2\,\mathrm{Tr}(\boldsymbol{\Sigma}_0\mathbf{A}\boldsymbol{\Sigma}_0\mathbf{B})$$

are utilized. The independence of the vectors $\mathbf{v}$ and $\hat{\boldsymbol{\vartheta}}$ is implied by the fact that $\mathbf{S}$ and $\overline{\mathbf{Y}}$ are independent. □

The expression for $\operatorname{Var}_{\vartheta_0}(\widehat{P\boldsymbol{\vartheta}}) = \frac{2}{m-1}\mathbf{S}^{-1}_{\Sigma_0^{-1}}$ shows that the values

$$\frac{\sqrt{\left\{\dfrac{2}{m-1}\,\mathbf{S}^{-1}_{\Sigma_0^{-1}}\right\}_{i,i}}}{\vartheta_i}, \quad i = 1, \dots, p,$$

can be made sufficiently small by sufficiently large number of replication.

Theorem 3.2. *Let* $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i)$ *and let the estimator* $\hat{\boldsymbol{\vartheta}}$ *of* $\boldsymbol{\vartheta}$ *be independent of* $\mathbf{Y}$*. Let* $\operatorname{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = \mathbf{W}$, $\{\mathbf{W}\}_{i,j} = w_{i,j}$, $i,j = 1,\dots,p$*. Then the covariance matrix of the plug-in estimator* $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})$ *is approximately given by the relationship*

$$\begin{aligned}
&\operatorname*{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})]\\
&\approx \mathbf{C}_0^{-1} + \sum_{i=1}^{p}\sum_{j=1}^{p} \delta\vartheta_i\delta\vartheta_j \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}\\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_{i,j}\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}.
\end{aligned}$$

If $\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) = (\mathbf{X}'\boldsymbol{\Sigma}_0^{-}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\overline{\mathbf{Y}}$ *and* $\hat{\boldsymbol{\vartheta}}$ *is given by Lemma* 3.1*, then*

$$\begin{aligned}
&\operatorname*{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})]\\
&\approx \frac{1}{m}\mathbf{C}_0^{-1} + \frac{1}{m}\sum_{i=1}^{p}\sum_{j=1}^{p} \delta\vartheta_i\delta\vartheta_j \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}\\
&\quad + \frac{2}{m(m-1)}\sum_{i=1}^{p}\sum_{j=1}^{p} s^{i,j}\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1},
\end{aligned}$$

where $\delta\vartheta_i = E_{\vartheta_0}(\widehat{\delta\vartheta})$ *and* $s^{i,j} = \{\mathbf{S}^{-1}_{\Sigma_0^{-1}}\}_{i,j}$, $i,j = 1,\dots,p$*.*

Proof. Since

$$\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}}) \approx \hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0) - \sum_{i=1}^{p} \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}\widehat{\delta\boldsymbol{\vartheta}}_i,$$

where $\mathbf{v} = \mathbf{Y} - \mathbf{X}\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{Y}$ and $\mathbf{v}$ and $\widehat{\delta\boldsymbol{\vartheta}}$ are stochastically independent, we can write

$$\begin{aligned}
\mathbf{C}_{i,j} &= \operatorname{cov}_{\vartheta_0}(\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}\widehat{\delta\vartheta}_i, \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}\widehat{\delta\vartheta}_j)\\
&= E_{\vartheta_0}(\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v}\mathbf{v}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1})E_{\vartheta_0}(\widehat{\delta\vartheta}_i\widehat{\delta\vartheta}_j)\\
&= \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_x)^{+}\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}(w_{i,j} + \delta\vartheta_i\vartheta_j).
\end{aligned}$$

In the case of a replicated model the matrices

$$\mathbf{C}_0^{-1},\ \ \boldsymbol{\Sigma}_0^{-1},\ \ \mathbf{V}_i,\ \ (\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+,\ \text{ and }\ \mathbf{V}_j,$$

must be substituted by

$$\frac{1}{m}\,\mathbf{C}_0^{-1},\quad m\boldsymbol{\Sigma}_0^{-1},\ \ \frac{1}{m}\,\mathbf{V}_i,\ \ m(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+,\ \text{ and }\ \frac{1}{m}\,\mathbf{V}_j,$$

respectively. The value $w_{i,j}$ must be substituted by $\frac{2}{m-1}\,s^{i,j}$. □

Let

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{1,1}, & \ldots, & \mathbf{U}_{1,p} \\ \vdots & \ddots & \vdots \\ \mathbf{U}_{p,1}, & \ldots, & \mathbf{U}_{p,p} \end{pmatrix},$$

where

$$\mathbf{U}_{i,j} = \mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1},\quad i,j=1,\ldots,p.$$

Then

$$\begin{aligned}\operatorname*{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})] \approx \mathbf{C}_0^{-1} &+ (\delta\boldsymbol{\vartheta}'\otimes\mathbf{I}_{k,k})\mathbf{U}(\delta\boldsymbol{\vartheta}\otimes\mathbf{I}_{k,k}) \\ &+ (\mathbf{1}_p'\otimes\mathbf{I}_{k,k})[(\mathbf{W}\otimes\mathbf{I}_{k,k})*\mathbf{U}](\mathbf{1}_p\otimes\mathbf{I}_{k,k}).\end{aligned}$$

In the case of a replicated model,

$$\begin{aligned}\operatorname*{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0+\widehat{\delta\boldsymbol{\vartheta}})] \approx \frac{1}{m}\,\mathbf{C}_0^{-1} &+ \frac{1}{m}\,(\delta\boldsymbol{\vartheta}'\otimes\mathbf{I}_{k,k})\mathbf{U}(\delta\boldsymbol{\vartheta}\otimes\mathbf{I}_{k,k}) \\ &+ \frac{2}{m(m-1)}(\mathbf{1}_p'\otimes\mathbf{I}_{k,k})\big[\big(\mathbf{S}_{\Sigma_0^{-1}}^{-1}\otimes\mathbf{I}_{k,k}\big)*\mathbf{U}\big](\mathbf{1}_p\otimes\mathbf{I}_{k,k}).\end{aligned}$$

R e m a r k 3.3. The term

$$(\delta\boldsymbol{\vartheta}'\otimes\mathbf{I}_{k,k})\mathbf{U}(\delta\boldsymbol{\vartheta}\otimes\mathbf{I}_{k,k})$$

can be neglected for a function $\mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathbb{R}^k$ if $\delta\boldsymbol{\vartheta}\in\mathcal{N}_h$, where $\mathcal{N}_h$ is the insensitivity region given as

$$\mathcal{N}_h = \{\delta\boldsymbol{\vartheta}\colon\ (\delta\boldsymbol{\vartheta})'\mathbf{W}_h\delta\boldsymbol{\vartheta} \leqslant 2\varepsilon\mathbf{h}'\mathbf{C}_0^{-1}\mathbf{h}\},$$

where

$$\{\mathbf{W}_h\}_{i,j} = \mathbf{h}'\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}\mathbf{h}.$$

If $\delta\boldsymbol{\vartheta}\in\mathcal{N}_h$, then $\operatorname{Var}_{\vartheta_0}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0+\delta\boldsymbol{\vartheta})] \leqslant (1+\varepsilon)^2\operatorname{Var}[\hat{\boldsymbol{\beta}}(\boldsymbol{\vartheta}_0)]$ (the term ε^2 is neglected). In more detail see [3], [4], [2]. The confidence region for $\delta\boldsymbol{\vartheta}$ must be used in order to be sure that $\boldsymbol{\vartheta}\in\mathcal{N}_h$.

4. Numerical example

Let

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\beta, \begin{pmatrix} \vartheta_1, & 0 \\ 0, & \vartheta_2 \end{pmatrix}\right],$$
$$\begin{pmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 1 \\ 16 \end{pmatrix}, \begin{pmatrix} 0.4, & 0 \\ 0, & 7 \end{pmatrix}\right]$$

($\sqrt{0.4}/1 = 0.632 \gg 0.1$, $\sqrt{7}/16 = 0.165 > 0.1$) and let $\mathbf{Y}$ and $\hat{\boldsymbol{\vartheta}}$ be stochastically independent.

The BLUE of β is

$$\hat{\beta} = \left[(1,1)\begin{pmatrix} 1/\vartheta_1, & 0 \\ 0, & 1/\vartheta_2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^{-1}(1,1)\begin{pmatrix} 1/\vartheta_1, & 0 \\ 0, & 1/\vartheta_2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{\vartheta_2 Y_1 + \vartheta_1 Y_2}{\vartheta_2 + \vartheta_1}.$$

The plug-in estimator is

$$\tilde{\beta} = \frac{\hat{\vartheta}_2 Y_1 + \hat{\vartheta}_1 Y_2}{\hat{\vartheta}_2 + \hat{\vartheta}_1}.$$

Regarding Theorem 3.2, where $\boldsymbol{\vartheta}_0 = \begin{pmatrix} 1 \\ 16 \end{pmatrix}$, we have that

$$(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+ = \frac{1}{1+16}\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix},$$
$$\mathbf{C}_0^{-1} = \left(\frac{1}{1} + \frac{1}{16}\right)^{-1} = \frac{16}{17},$$
$$\sum_{i=1}^{2}\sum_{j=1}^{2} E(\widehat{\delta\vartheta}_i)E(\widehat{\delta\vartheta}_j)\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}$$
$$= (\delta\vartheta_1)^2 0.0521 + (\delta\vartheta_2)^2 0.0002,$$
$$\sum_{i=1}^{2}\sum_{j=1}^{2} w_{i,j}\mathbf{C}_0^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j\boldsymbol{\Sigma}_0^{-1}\mathbf{X}\mathbf{C}_0^{-1}$$
$$= 0.4 \times \frac{256}{17^3} + 7 \times \frac{1}{17^3} = 0.0222.$$

Thus

$$\underset{\vartheta_0}{\mathrm{Var}}[\hat{\beta}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \approx 0.9412 + 0.0521(\delta\vartheta_1)^2 + 0.0002(\delta\vartheta_2)^2 + 0.0222 = 0.9634$$

(for $\delta\vartheta_1 = \delta\vartheta_2 = 0$).

The plug-in estimator was simulated 50000times and for $\delta\vartheta_1 = \delta\vartheta_2 = 0$ we obtained

$$\frac{1}{50000}\sum_{i=1}^{50000}(\tilde{\beta}^{(i)} - \overline{\tilde{\beta}})^2 = 0.9775.$$

The approximated value attains $98.6\,\% = 0.9634/0.9775 \times 100\,\%$ of the true value, even if the uncertainties of the estimators $\widehat{\delta\vartheta}_1$ and $\widehat{\delta\vartheta}_2$ are relatively large $(0.632; 0.165)$.

References

[1] *L. Kubáček*: Nonlinear error propagation law. Appl. Math. *41* (1996), 329–345.

[2] *L. Kubáček, L. Kubáčková, E. Tesaříková, J. Marek*: How the design of an experiment influences the nonsensitiveness regions in models with variance components. Appl. Math. *43* (1998), 439–460.

[3] *L. Kubáček, L. Kubáčková*: Nonsensitiveness regions in universal models. Math. Slovaca *50* (2000), 219–240.

[4] *L. Kubáček, L. Kubáčková*: Statistics and Metrology. Publishing House of Palacký University, Olomouc, 2000. (In Czech.)

[5] *C. R. Rao*: Linear Statistical Inference and Its Applications. John Wiley & Sons, New York-London-Sydney, 1965.

[6] *C. R. Rao, S. K. Mitra*: Generalized Inverse of Matrices and Its Applications. John Wiley & Sons, New York-London-Sydney-Toronto, 1971.

[7] *C. R. Rao, J. Kleffe*: Estimation of Variance Components and Applications. North Holland, Amsterdam-New York-Oxford-Tokyo, 1988.

Author's address: *L. Kubáček*, Dept. of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, Třída 17. listopadu 1192/12, CZ 771 46 Olomouc, Czech Republic, e-mail: `lubomir.kubacek@upol.cz`.