J. Schröder

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The two-parameter class of Schröder inversions

Joachim Schröder

In memoriam Gerhard Preuss, 1940–2011.

Abstract. Infinite lower triangular matrices of generalized Schröder numbers are used to construct a two-parameter class of invertible sequence transformations. Their inverses are given by triangular matrices of coordination numbers. The two-parameter class of Schröder transformations is merged into a one-parameter class of stretched Riordan arrays, the left-inverses of which consist of matrices of crystal ball numbers. Schröder and inverse Schröder transforms of important sequences are calculated.

Keywords: generalized Schröder numbers, coordination numbers, crystal ball numbers, stretched Riordan array, triangular matrix, sequence transformation, inversion, left-inverse

Classification: Primary 05A10; Secondary 05A15, 05A19

1. Introduction

Ernst Friederich Wilhelm Carl Schröder (1841–1902) [10] is the originator of the counting sequence which is now called large Schröder numbers $(\mathrm{Schr}\,(n,n,1))_{n\in\mathbb{N}}$. They count the number of paths with rectangular and diagonal steps weakly above the line $y=x$ from $(0,0)$ to (n,n), see J. Schröder [11] for more details.

Let $\mathbb{R}^{\mathbb{N}}$ be the direct product of the field $(\mathbb{R},+,\cdot)$ considered as an $\mathbb{R}$-vector space. A linear map $T:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}$ is called sequence transformation. If $\mathbf{a}=(a_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ is a sequence, then $T(\mathbf{a})$ is called T-transform of $\mathbf{a}$. T is called inversion, if it has a left-inverse. A left-inverse is not unique in general, see Tables 4 and 5. Its existence implies and is implied by injectivity of T. We are interested in inversions which can be described by countable infinite matrices M and the usual matrix-by-vector product $M\mathbf{a}$, $\mathbf{a}\in\mathbb{R}^{\mathbb{N}}$. The absence of a limit notion requires that each row of M contains a finite number of non-zero elements only. Note that M is not the matrix M_T usually associated with a linear map between linear spaces: M is not a $2^{\aleph_0}\times 2^{\aleph_0}$ matrix. Our matrices are sub-triangular matrices, i.e. non-zero entries may appear only on or below the diagonal, and they are allowed to be non-square matrices. Their entries will be generalized Schröder numbers and the entries of the inverses are coordination or crystal ball numbers. The author only knows of two other inversions with a similar important combinatorial interpretation: binomial and Stirling inversion.

There are numerous sequence transformations in the literature and established methods to obtain them. Riordan [8] and [9] is a good reference. Sloane [12], Bower [3] and Bernstein & Sloane [2] present a wealth of transformations, many with their combinatorial interpretations.

We are going to construct a two-parameter class of inversions, where parameters have a persuasive geometric meaning: the slope l of the restricting line and the number k of units which the path finishes above the line $y = lx$. There are other many-parameter classes, e.g. the Gould-class [6] and the Krattenthaler-class [7]. Throughout this paper we will use extensively the theory of (standard) Riordan arrays [14] and later on of stretched Riordan arrays, as outlined in Corsani, Merlini & Sprugnoli [5].

2. Prerequisites

Definition 1.
1. If p is a statement, then $[p] = 1$, if p is true, otherwise $[p] = 0$. $\mathbb{I}_n$ denotes the identity matrix of order n. $\mathbb{N} = \{0, 1, \ldots\}$ is the set of natural numbers. $\gamma^{\leftarrow}$ denotes the (compositional) inverse of the map γ, whereas γ^{-1} is the (left) inverse, if existent, w.r.t. a given multiplication. $\circ$ denotes composition of maps and $\cdot$ multiplication. $\mathbb{R}[[z]]$ is the ring of formal power series equipped with the usual addition and (Cauchy-) product.

$\lfloor x \rfloor$ is the largest integer smaller or equal to the real number x.

2. Let $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. Define $\mathrm{GF} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}[[z]]$ by $\mathrm{GF}(\mathbf{a}) := \sum_{\nu\geq 0} a_\nu z^\nu$. If $f(z) = \sum a_\nu z^\nu$, then $[z^n]f(z) := a_n$, i.e., $[z^n]$ can be identified with the projection $p_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ onto the n-th factor: $[z^n] \circ \mathrm{GF} = p_n$. Further $\mathrm{ord}(f) := \min\{n \mid [z^n]f(z) \neq 0\}$ and $\mathbb{R}_s[[z]] := \{f \in \mathbb{R}[[z]] \mid \mathrm{ord}(f) = s\}$.

Let $p(z^{1/r}) \in \mathbb{R}[[z^{1/r}]]$ be a Puiseux series. Define $\mathrm{GF}^{\leftarrow} : \mathbb{R}[[z^{1/r}]] \to \mathbb{R}^{\mathbb{N}}$ by

$$p_n(\mathrm{GF}^{\leftarrow}(p(z^{1/r}))) := \begin{cases} [z^{n/r}]p(z^{1/r}) & \text{if } \ r \mid n \\ 0 & \text{if } \ r \nmid n \end{cases}$$

i.e., we are taking only coefficients of terms with integral exponents. For instance $\mathrm{GF}^{\leftarrow}(\frac{1}{1-z^{1/3}}) = (1, 0, 0, 1, 0, 0, 1, \ldots)$.

3. Let $M = (m_{\mu,\nu})_{\mu,\nu\geq 0}$ be an infinite lower sub-diagonal matrix and $\mathbf{a} = (a_\nu)_{\nu\geq 0} \in \mathbb{R}^{\mathbb{N}}$. The M-transform of $\mathbf{a}$ is $M\mathbf{a} = (b_\mu)_{\mu\geq 0} = (\sum_{\nu\geq 0} m_{\mu,\nu} a_\nu)_{\mu\geq 0}$.

4. An infinite lower sub-triangular matrix R is called (general) Riordan array, if the GF of the ν-th column is $g(z)(z^s f(z))^\nu$ for some $g, f \in \mathbb{R}_0[[z]]$, where $s \geq 1$ is fixed, see [14]. Observe the composition rule $(z^s f)(a(z)) = (a(z))^s f(a(z)) = ((z^s f) \circ a)(z)$. If $s = 1$, we speak of a standard Riordan array, if $s > 1$ of a (vertically) stretched Riordan array, see [5]. The pair $(g, z^s f)$ is called Riordan pair, $R(g, z^s f)$ is the corresponding Riordan array. The set of standard Riordan pairs $\{(g, zf)\}$ form a group with identity $(1, z)$, multiplication

$$(d, zh) * (g, zf) := (d \cdot (g \circ zh), zh \cdot (f \circ zh)) \tag{1}$$

and inverse

$$(g, zf)^{-1} := ((g \circ (zf)^{\leftarrow})^{-1}, (zf)^{\leftarrow}). \tag{2}$$

The map $R : (g, zf) \mapsto R(g, zf)$ is a group isomorphism between the Riordan group and the group of Riordan arrays with matrix multiplication. Setting $f = 0$ in Equation 1 immediately yields the generating function of the transformed sequence:

Lemma 2. *Let* $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ *and* $A(z) = GF(\mathbf{a})$ *Then*

$$GF(R(d, zh)\mathbf{a}) = d(z) \cdot A(zh(z)). \tag{3}$$

□

The theory of Riordan arrays skillfully separates the general from the specific, at least for convolution arrays.

5. Let $\mathrm{Schr}\,(n, m, l) = \mathrm{Schr}\,(lm + k, m, l)$ denote the number of paths on the rectangular grid from $(0, 0)$ to $(m, n) = (m, lm + k)$, $n \geq lm$, weakly above the line $y = lx$ with step set $\{(1, 0), (1, 1), (0, 1)\}$. Let $A_l := \mathrm{GF}((\mathrm{Schr}\,(lm, m, l))_{m\in\mathbb{N}})$. It is known that

$$A_l = 1 + z(A_l^l + A_l^{l+1}) \tag{4}$$

and

$$A_l^{k+1} = \mathrm{GF}((\mathrm{Schr}\,(lm + k, m, l))_{m\in\mathbb{N}}), \tag{5}$$

where

$$\mathrm{Schr}\,(lm + k, m, l) = \frac{k + 1}{lm + k + 1} \sum_{v=0}^{m} \binom{lm + k + 1}{m - v}\binom{lm + k + v}{v}, \tag{6}$$

see [11], [16].

6. Let $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, $d \in \mathbb{N}$. The ℓ^1-*norm* $|\mathbf{x}|_1$ of $\mathbf{x}$ is defined by $|\mathbf{x}|_1 := \sum |x_i|$. $\mathcal{S}_d(n) := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^d \text{ and } |\mathbf{x}|_1 = n\}$ is called $(d-1)$-*dimensional crystal sphere* of radius n. We set $S_d(n) := |\,\mathcal{S}_d(n)|$. The sequence $(S_d(n))_{n\in\mathbb{N}}$ is called *coordination-sequence* (or *-numbers*), cf. A035597 pp. and A035607 in the *Online Encyclopedia of Integer Sequences*, OEIS, [13]. It is known that

$$S_d(n) = \sum_{k=0}^{d} \binom{d}{k}\binom{n + d - k - 1}{d - 1} \quad \left(= \sum_{k=0}^{d} \binom{d}{k}\binom{n + k - 1}{d - 1}\right) \tag{7}$$

and

$$\mathrm{GF}((S_d(n))_{\mathbb{N}}) = \left(\frac{1+z}{1-z}\right)^d, \tag{8}$$

see [4, p. 9, Equation (16)].

7. The union $\bigcup_{\nu=0}^{n} \mathcal{S}_d(\nu) =: \mathcal{G}_d(n)$ is called *d-dimensional crystal ball* of radius n. We put $|\,\mathcal{G}_d(n)| =: G_d(n) = \sum_{\nu=0}^{n} S_d(\nu)$. It is known that

$$G_d(n) = \sum_{k=0}^{d} \binom{d}{k}\binom{n+d-k}{d} \quad \left(= \sum_{k=0}^{d} \binom{d}{k}\binom{n+k}{d}\right), \tag{9}$$

see [4, p. 9, Equation (17)], and obviously

$$\mathrm{GF}((G_d(n))_{\mathbb{N}}) = \left(\frac{1+z}{1-z}\right)^d \frac{1}{1-z}\,. \tag{10}$$

3. Schröder inversions and standard arrays

Theorem 3. *The inverse of the standard Riordan pair* $(g, zf) := (A_l^{k+1}, zA_l^l)$ *in the Riordan group is*

$$(g, zf)^{-1} = \left(\frac{1}{g\circ (zf)^{\leftarrow}}, (zf)^{\leftarrow}\right) = \left(\left(\frac{1-y}{1+y}\right)^{k+1}, y\left(\frac{1-y}{1+y}\right)^{l}\right). \tag{11}$$

Proof: Set $y = zA_l^l$. We obtain from Equation (4) $A_l = 1 + y + yA_l$,

$$A_l = \frac{1+y}{1-y} \qquad \text{and} \tag{12}$$

$$y = zA_l^l = z\left(\frac{1+y}{1-y}\right)^l, \tag{13}$$

$$z = y\left(\frac{1-y}{1+y}\right)^l = (zf)^{\leftarrow}(y). \qquad \text{Further} \tag{14}$$

$$\begin{aligned} \frac{1}{g\circ(zf)^{\leftarrow}} &= \frac{1}{A_l^{k+1}((zf)^{\leftarrow}(y))} = \frac{z^{\frac{k+1}{l}}}{\left(zA_l^l\circ(zf)^{\leftarrow}(y)\right)^{\frac{k+1}{l}}} \\ &= \frac{z^{\frac{k+1}{l}}}{y^{\frac{k+1}{l}}} = \frac{y^{\frac{k+1}{l}}\left(\frac{1-y}{1+y}\right)^{k+1}}{y^{\frac{k+1}{l}}} = \left(\frac{1-y}{1+y}\right)^{k+1}. \end{aligned} \tag{15}$$

□

Example 4.

$$\begin{pmatrix} 1 &&&&& \\ -2 & 1 &&&& \\ 2 & -6 & 1 &&& \\ -2 & 18 & -10 & 1 && \\ 2 & -38 & 50 & -14 & 1 & \\ -2 & 66 & -170 & 98 & -18 & 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ 2 & 1 &&&& \\ 10 & 6 & 1 &&& \\ 66 & 42 & 10 & 1 && \\ 498 & 326 & 90 & 14 & 1 & \\ 4066 & 2706 & 810 & 154 & 18 & 1 \end{pmatrix} = \mathbb{I}_6$$

TABLE 1. A matrix of Schröder numbers of slope $l = 2$, and $k = 0$ (right) and its inverse, a matrix of coordination numbers.

$$\begin{pmatrix} 1 &&&&& \\ -4 & 1 &&&& \\ 8 & -8 & 1 &&& \\ -12 & 32 & -12 & 1 && \\ 16 & -88 & 72 & -16 & 1 & \\ -20 & 192 & -292 & 128 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 &&&&& \\ 4 & 1 &&&& \\ 24 & 8 & 1 &&& \\ 172 & 64 & 12 & 1 && \\ 1360 & 536 & 120 & 16 & 1 & \\ 11444 & 4672 & 1156 & 192 & 20 & 1 \end{pmatrix} = \mathbb{I}_6$$

TABLE 2. A matrix of Schröder numbers of slope $l = 2$, and $k = 1$ (right) and its inverse, a matrix of coordination numbers. Note that merging this Schröder matrix with the Schröder matrix in Table 1 gives the Schröder matrix in Table 4.

Corollary 5. (a) *Let* $({}_{l,k}s_{\mu,\nu})_{\mu,\nu\geq 0} := R(A_l^{k+1}, zA_l^l)$ *denote the Schröder-matrix belonging to* l, k *(see Theorem* 3 *and Definition* 1, 4.*). Then*

$$ {}_{l,k}s_{\mu,\nu} = \mathit{Schr}\,(l\mu + k, \mu - \nu, l). \tag{16}$$

(b) *Let* $({}_{l,k}c_{n,\mu})_{n,\mu\geq 0} := R((\frac{1-y}{1+y})^{k+1}, y(\frac{1-y}{1+y})^l)$ *denote the Coordination-matrix belonging to* l, k *(see Theorem* 3 *and Definition* 1, 4.*). Then*

$$ {}_{l,k}c_{n,\mu} = (-1)^{n-\mu} S_{\mu l+k+1}(n - \mu). \tag{17}$$

□

Corollary 6. (a) *We have that*

$$\sum_{\tau=\nu}^{n} (-1)^{n-\tau} S_{l\tau+k+1}(n-\tau)\mathit{Schr}\,(l\tau + k, \tau - \nu, l) = [n = \nu], \tag{18}$$

(b) *we have that*

$$\sum_{\nu=t}^{s} Schr\,(ls+k, s-\nu, l)(-1)^{\nu-t} S_{lt+k+1}(\nu-t) = [s=t]. \tag{19}$$

Proof: (a) $({}_{l,k}c_{n,\mu})_{n,\mu\geq 0}({}_{l,k}s_{\mu,\nu})_{\mu,\nu\geq 0} = \mathbb{I}_\infty$ and
(b) $({}_{l,k}s_{\mu,\nu})_{\mu,\nu\geq 0}({}_{l,k}c_{n,\mu})_{n,\mu\geq 0} = \mathbb{I}_\infty$. □

Definition 7. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{\mathbb{N}}$.

(a) The (l,k)-Schröder transform of $\mathbf{a}$ is $R(A_l^{k+1}, zA_l^l)\mathbf{a} =: \mathrm{Schr}\,_{l,k}(\mathbf{a})$ (see Theorem 3 and Definition 1, 3. and 4.).

(b) The (l,k)-Coordination transform of $\mathbf{b}$ is $R((\frac{1-y}{1+y})^{k+1}, y(\frac{1-y}{1+y})^l)\mathbf{b} =:$ $\mathrm{Coord}\,_{l,k}(\mathbf{b})$ (see Theorem 3 and Definition 1, 3. and 4.).

Corollary 8. *If we have that*

$$A(z) = GF((a_i)_{i\in\mathbb{N}}), \tag{20}$$

then the GF of the (l,k)-Schröder transform is

$$A_l^{k+1} A(zA_l^l) \tag{21}$$

and the GF of the inverse Schröder transform (= (l,k)-Coordination transform) is

$$\left(\frac{1-y}{1+y}\right)^{k+1} A\left(y\left(\frac{1-y}{1+y}\right)^l\right). \tag{22}$$

Proof: See Lemma 2. □

Corollary 9. *Let* $\mathbf{a} \in \mathbb{R}^{\mathbb{N}}$. *Then the (l,k)-Schröder transform of* $\mathbf{a}$ *is*

$$\mathbf{b} = Schr\,_{l,k}(\mathbf{a}) = \left(\sum_{\nu=0}^{n} a_\nu Schr\,(ln+k, n-\nu, l)\right)_{n\in\mathbb{N}}. \tag{23}$$

The inverse Schröder transform, i.e., the (l,k)-Coordination transform, is

$$\mathbf{a} = Coord\,_{l,k}(\mathbf{b}) = \left(\sum_{\nu=0}^{n} b_\nu (-1)^{n-\nu} S_{l\nu+k+1}(n-\nu)\right)_{n\in\mathbb{N}}. \tag{24}$$

□

4. Schröder inversions and stretched arrays

The array of Schröder numbers $({}_l\widehat{s}_{\mu+k,\nu}) := ({}_{l,k}s_{\mu,\nu}) = (\mathrm{Schr}\,(l\mu+k, \mu-\nu, l))$, $0 \leq k \leq l-1$, $0 \leq \mu \leq M$, $0 \leq \nu \leq \mu$ is a $(M+1)\times l(M+1)$ array, see Table 4 and Table 6 for an example. We say that the array has shape $1 \times l$.

Lemma 10 ([5]). *Let $(d(t), t^s h(t))$, $d, h \in \mathbb{R}_0[[t]]$ represent a stretched Riordan array.*

(a) *Let $y = t^s h(t)$ and*

$$y^{1/s} = t \cdot (h(t))^{1/s} =: t \cdot u(t). \tag{25}$$

Let $(t \cdot u)^{\leftarrow} = u \cdot t(u)$. Then

$$(d(t), t \cdot u(t))^{-1} = ((d \circ (u \cdot t))^{-1}, u \cdot t) = (\frac{1}{d(y^{1/s}t(y^{1/s}))}, y^{1/s}t(y^{1/s})) \tag{26}$$

and the ν-th column of the left-inverse horizontally stretched Riordan array is

$$GF^{\leftarrow}\left((d(y^{1/s}t(y^{1/s})))^{-1}(y^{1/s}t(y^{1/s}))^{\nu}\right). \tag{27}$$

(b) *Let $\mathbf{a} \in \mathbb{R}^{\mathbb{N}}$ and $A(z) = GF(\mathbf{a})$. Then*

$$GF(R(d(t), t^s h(t))\mathbf{a}) = d \cdot A(t^s h(t)). \tag{28}$$

(The GF of the transform is $d \cdot A(t^s h(t))$).

(c) *Let $\mathbf{b} \in \mathbb{R}^{\mathbb{N}}$ and $B(z) = GF(\mathbf{b})$. Then*

$$GF(R((d(y^{1/s}t(y^{1/s})))^{-1}, y^{1/s}t(y^{1/s}))\mathbf{b}) = \frac{B(y^{1/s}t(y^{1/s}))}{d(y^{1/s}t(y^{1/s}))}, \tag{29}$$

with the understanding that the sequence of coefficients of terms with integral exponents coincides with the transform. □

Remark 11.

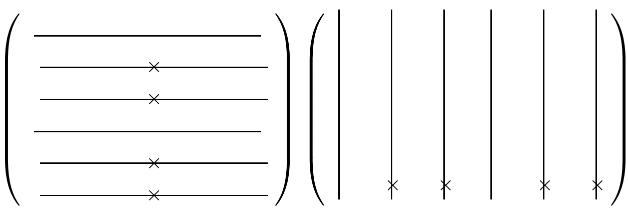

Table 3. $s = 3$

Formula (27) looks complicated, even more so if we would consider the fact that y has s different s-th roots (see [5]). But it is conceptually easy: We are presented with a vertically stretched Riordan array $R(d(t), t^s h(t))$. $R(d(t), t \cdot u(t))$ is the standard Riordan array obtained by filling in missing columns. From the inverse we are deleting all rows with index $\neq 0 \bmod s$, see Table 3.

In Theorem 12 and the subsequent corollaries we will go the opposite way and merge l standard Riordan arrays. Selection of rows is done by the commonly accepted method of employing powers of roots of unity (see Corollary 14 and Equation (36)).

In preparation for the next theorem we will join the Schröder matrices in Table 1 and Table 2. According to Theorem 3, the first one belongs to the Riordan pair $(A_2^1(z), zA_2^2(z))$, i.e., $k=0$, the latter to $(A_2^2(z), zA_2^2(z))$, i.e., $k=1$. We are stretching both vertically by the factor 2: $(A_2(z^2), z^2A_2^2(z^2))$ and $(A_2^2(z^2), z^2A_2^2(z^2))$, and join them: $(A_2(z^2)+zA_2^2(z^2), z^2A_2^2(z^2))$. We may call this method reverse multisectioning. In the sequel we require $l \geq 2$.

Theorem 12. *The Riordan pair $(g, z^l f)$ of the $1\times l$ Schröder array*

$$(_l s_{\sigma,\lambda})_{\sigma,\lambda\geq 0} = (\mathit{Schr}\,(\sigma, \lfloor\frac{\sigma}{l}\rfloor - \lambda, l))_{\sigma,\lambda\geq 0} \tag{30}$$

is

$$(g, z^l f) = \left(A_l(z^l)\frac{z^l A_l^l(z^l)-1}{zA_l(z^l)-1}, z^l A_l^l(z^l)\right). \tag{31}$$

The left-inverse Riordan pair is

$$(g, z^l f)^{-1} = \left(\frac{1-y^{1/l}}{1+y}, y^{1/l}\frac{1-y}{1+y}\right). \tag{32}$$

The ν-th column of the corresponding Riordan array with shape $l\times 1$ is

$$GF^{\leftarrow}\left(\left(\frac{1-y^{1/l}}{1+y}\right)\left(y^{1/l}\frac{1-y}{1+y}\right)^{\nu}\right). \tag{33}$$

Proof: **Equation** (31)**.** We are going to join l Riordan arrays. These l arrays are

$$(A_l(z), zA_l^l(z)), (A_l^2(z), zA_l^l(z)), (A_l^3(z), zA_l^l(z)), \ldots, (A_l^l(z), zA_l^l(z)).$$

We stretch them such that consecutive entries in a column are l lines apart:

$$(A_l(z^l), z^lA_l^l(z^l)), (A_l^2(z^l), z^lA_l^l(z^l)), (A_l^3(z^l), z^lA_l^l(z^l)), \ldots, (A_l^l(z^l), z^lA_l^l(z^l)).$$

In order to merge them, we have to shift the i-th Riordan array downwards by i lines. This is accomplished by multiplying by z^i:

$$(A_l(z^l) + zA_l^2(z^l) + z^2A_l^3(z^l) + \ldots + z^{l-1}A_l^l(z^l), z^lA_l^l(z^l)),$$

which is Equation (31).

Equation (32)**.** We will use Equation (26). From Equation 4, $A_l = 1 + z(A_l^l + A_l^{l+1})$, it follows $A_l(z^l) = 1 + z^l(A_l^l(z^l) + A_l^{l+1}(z^l))$. Set $w = zA_l(z^l), w^l = y$.

Hence $A_l(z^l) = 1 + w^l + A_l(z^l)w^l$ and

$$w = zA_l(z^l) = z + zw^l + w^{l+1},\ z = w\frac{1-w^l}{1+w^l} = y^{1/l}\frac{1-y}{1+y} = (z^l f)^{\leftarrow}.$$

Now

$$\begin{aligned} g &= \frac{zA_l(z^l)}{z}\frac{(zA_l(z^l))^l - 1}{zA_l(z^l)-1} = \frac{w}{z}\frac{w^l-1}{w-1} = \frac{w}{y^{1/l}\frac{1-y}{1+y}}\frac{w^l-1}{w-1} \\ &= \frac{y^{1/l}(1+y)}{y^{1/l}(1-y)}\frac{y-1}{y^{1/l}-1} = \frac{1+y}{1-y^{1/l}} = g\circ(zy)^{\leftarrow} \end{aligned}$$

and

$$(g, zf)^{-1} = \left(\frac{1-y^{1/l}}{1+y}, y^{1/l}\frac{1-y}{1+y}\right).$$

Equation (33)**.** See Formula (27). □

Corollary 13. *Let the crystal ball numbers G_σ be defined as in Definition* 1, 7. *and let*

$$({}_lg_{n,\sigma})_{n,\sigma\ge 0} := R\left(\left(\frac{1-y^{1/l}}{1+y}\right), y^{1/l}\left(\frac{1-y}{1+y}\right)\right) \tag{34}$$

denote the Coordination-matrix belonging to l. Then

$${}_lg_{n,\sigma} = \rho_l(\sigma)(-1)^{n+\lfloor\sigma/l\rfloor}G_\sigma\left(n - \lfloor\frac{\sigma+l-1}{l}\rfloor\right), \tag{35}$$

where ρ_l is a function satisfying

$$\rho_l(\sigma) = \begin{cases} 1, & \text{if } l \text{ divides } \sigma \text{ or } \ \sigma+1 \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

□

Corollary 14. *Let ω_l be a primitive l-th root of unity, $l > 1$. Let the crystal ball numbers G_ν be defined as in Definition* 1, 7. *and $\rho_l(\nu) := (1 + \omega_l^\nu + \omega_l^{2\nu} + \ldots + \omega_l^{(l-1)\nu} + 1 + \omega_l^{\nu+1} + \omega_l^{2(\nu+1)} + \ldots + \omega_l^{(l-1)(\nu+1)})/l$. Then*

$$\sum_{\nu\ge 0} Schr(\sigma, \lfloor\frac{\sigma}{l}\rfloor - \nu, l)(-1)^{\tau+\lfloor\nu/l\rfloor}\rho_l(\nu)G_\nu(\tau - \lfloor\frac{\nu+l-1}{l}\rfloor) = [\sigma=\tau]. \tag{37}$$

Proof: ρ_l satisfies the condition in Equation (36). $({}_lg_{n,\sigma})_{n,\sigma\ge 0}({}_ls_{\sigma,\lambda})_{\sigma,\lambda\ge 0} = \mathbb{I}_\infty$ by means of Equation (26) and Theorem 12. □

Example 15.

$$\begin{pmatrix} 1 \\ -1 & -1 & 1 \\ 1 & 3 & -5 & -1 & 1 \\ -1 & -5 & 13 & 7 & -9 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 & 1 \\ 4 & 1 \\ 10 & 6 & 1 \\ 24 & 8 & 1 \\ 66 & 42 & 10 & 1 \end{pmatrix} = \mathbb{I}_4$$

Table 4. A matrix of Schröder numbers of slope $l = 2$ (right) and its left inverse, a matrix of crystal ball numbers.

$$\frac{1}{20}\begin{pmatrix} 1 & 1 & -3 & -1 & -2 & 2 & 0 \\ -5 & -5 & 20 & 10 & 5 & -5 & 0 \\ 18 & 18 & -89 & -53 & 9 & 11 & 0 \\ -36 & -36 & 248 & 176 & -168 & -32 & 20 \end{pmatrix}$$

Table 5. The Moore-Penrose Pseudoinverse of the Schröder-matrix in Table 4.

Example 16.

$$\begin{pmatrix} 1 \\ -1 & 0 & -1 & 1 \\ 1 & 0 & 5 & -7 & 0 & -1 & 1 \\ -1 & 0 & -13 & 25 & 0 & 11 & -13 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \\ 14 & 8 & 1 \\ 32 & 10 & 1 \\ 54 & 12 & 1 \\ 134 & 80 & 14 & 1 \end{pmatrix} = \mathbb{I}_4$$

Table 6. A matrix of Schröder numbers of slope $l = 3$ (right) and its left inverse, a matrix of crystal ball numbers. Note the columns of zeros, caused by non-integer exponents in the Puiseux series.

Corollary 17. *Let $\mathbf{a} \in \mathbb{R}^{\mathbb{N}}$ and $l \geq 2$. Then the l-Schröder transform of $\mathbf{a}$ is*

$$\mathbf{b} = Schr_l(\mathbf{a}) = \left(\sum_{\nu=0}^{n/l} a_\nu Schr\,(n, \lfloor n/l \rfloor - \nu, l) \right)_{n\in\mathbb{N}}. \tag{38}$$

The left-inverse Schröder transform, i.e., the l-Coordination transform, is

$$\mathbf{b} = Coord_l(\mathbf{a}) = \left(\sum_{\nu=0}^{nl} b_\nu \rho_l(\nu)(-1)^{n+\lfloor \nu/l \rfloor} G_\nu(n - \lfloor (\nu + l - 1)/l \rfloor) \right)_{n\in\mathbb{N}} \tag{39}$$

□

Corollary 18. *Let*

$$A(z) = GF((a_i)_{i\in\mathbb{N}}), \tag{40}$$

then the GF of the l-Schröder transform is

$$A_l(z^l) \frac{z^l A_l^l(z^l) - 1}{z A_l(z^l) - 1} A\left(z^l A_l^l(z^l)\right) \tag{41}$$

and the GF of the l-Coordination transform is

$$\frac{1 - y^{1/l}}{1 + y} A\left(y^{1/l} \frac{1 - y}{1 + y} \right) \tag{42}$$

in the sense that the coefficients of terms with integer exponents in this fractional power series coincide with the transform.

Proof: See Lemma 10. □

Remark 19. The referee pointed out that the inverse (32) of a stretched Riordan array (31) is a linear combination of the inverses of the standard arrays of which (31) is composed. Indeed, if we look at Tables 1 and 2 then $1 - 2 + 2 - 2 = -1$ and $-(1 - 4 + 8) = -5$ gives the 4^{th} entry of column 1 and 2 in Table 4, etc. To turn this observation into a theorem we need:

Definition 20. Let $\mathbf{a} = (a_0, a_1, \ldots)$, $\mathbf{b} = (b_0, b_1, \ldots)$ and $\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \end{pmatrix}$ be two row and column vectors, respectively. The interlace of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ is defined by

$$\mathbf{a} \vee \mathbf{b} = (a_0, b_0, a_1, b_1, \ldots) \qquad \text{and} \qquad \mathbf{c} \vee \mathbf{d} = \begin{pmatrix} c_0 \\ d_0 \\ c_1 \\ d_1 \\ \vdots \end{pmatrix}.$$

Lemma 21. *We have that* $(\mathbf{a} \vee \mathbf{b}) \cdot (\mathbf{c} \vee \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$. □

Theorem 22. *Let* $\mathbf{X} = (\mathbf{x_0} \mid \mathbf{x_1} \mid \ldots \mid \mathbf{x_n})$ *and* $\mathbf{Y} = (\mathbf{y_0} \mid \mathbf{y_1} \mid \ldots \mid \mathbf{y_n})$ *be two matrices in column form. Let* $\mathbf{X^{-1}} = \begin{pmatrix} \mathbf{u_0} \\ \hline \mathbf{u_1} \\ \hline \vdots \\ \hline \mathbf{u_n} \end{pmatrix}$ *and* $\mathbf{Y^{-1}} = \begin{pmatrix} \mathbf{v_0} \\ \hline \mathbf{v_1} \\ \hline \vdots \\ \hline \mathbf{v_n} \end{pmatrix}$ *be their inverses written in row form. Then*

$$\begin{pmatrix} \mathbf{u_0} \\ \hline (\mathbf{u_0} + \mathbf{u_1}) \vee -\mathbf{v_0} \\ \hline (\mathbf{u_0} + \mathbf{u_1} + \mathbf{u_2}) \vee -(\mathbf{v_0} + \mathbf{v_1}) \\ \hline \vdots \\ \hline \sum_{\mu=0}^{n} \mathbf{u}_\mu \vee - \sum_{\mu=0}^{n-1} \mathbf{v}_\mu \end{pmatrix} \tag{43}$$

is a left-inverse of

$$(\mathbf{x_0} \vee \mathbf{y_0} \mid \mathbf{x_1} \vee \mathbf{y_1} \mid \ldots \mid \mathbf{x_n} \vee \mathbf{y_n}) \,.$$

Proof:

$$\left(\sum_{\mu=0}^{i} \mathbf{u}_\mu \vee - \sum_{\mu=0}^{i-1} \mathbf{v}_\mu \right) \cdot (\mathbf{x_j} \vee \mathbf{y_j}) = \sum_{\mu=0}^{i} \mathbf{u}_\mu \cdot \mathbf{x_j} - \sum_{\mu=0}^{i-1} \mathbf{v}_\mu \cdot \mathbf{y_j} = \begin{cases} 1-1=0, & \text{if } j<i \\ 1-0=1, & \text{if } j=i \\ 0-0=0, & \text{if } j>i \end{cases}$$

□

According to Tables 4 and 6, matrix (43) is a matrix of crystal ball numbers (and not merely a matrix of alternating sums of coordination numbers). This is indeed the case:

Lemma 23. *It holds that* $\sum_{\mu=0}^{\tau}(-1)^\mu S_{\sigma+1}(\mu) = (-1)^\tau G_\sigma(\tau)$.

Proof: We will use results contained in [11], Section 4.:

$$\mathrm{GF}((G_n(m))_{n,m\geq 0}) = \sum_{n,m\geq 0} G_n(m) x^n y^m = \frac{1}{1-x-y-xy}$$

and

$$\mathrm{GF}((S_{a+1}(b))_{a,b\geq 0}) = \sum_{a,b\geq 0} S_{a+1}(b) x^a y^b = \frac{1+y}{1-x-y-xy} \,.$$

Example 24.

$$\begin{pmatrix} 1 \\ -1 & -1 & 0 & 1 \\ 1 & 3 & 0 & -7 & -1 & 0 & 1 \\ -1 & -5 & 0 & 25 & 9 & 0 & -13 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -5 & 0 & -1 & 1 & 0 \\ 0 & -5 & 13 & 0 & 9 & -11 & 0 & -1 & 1 & 0 \end{pmatrix}$$

TABLE 7. Two more inverses of the Schröder matrix in Table 6, corresponding to $(l,k) = (3,0),(3,1)$ and $(l,k) = (3,1),(3,2)$, respectively.

Therefore

$$\sum_{n,m\geq 0}(-1)^m G_n(m)x^n y^m = \sum_{n,m\geq 0} G_n(m)x^n(-y)^m = \frac{1}{1-x+y+xy}$$
$$= \frac{1}{1-y}\frac{1-y}{1-x+y+xy} = \frac{1}{1-y}\sum_{a,b\geq 0} S_{a+1}(b)x^a(-y)^b$$
$$= \frac{1}{1-y}\sum_{a,b\geq 0}(-1)^b S_{a+1}(b)x^a y^b = \sum_{a,b\geq 0}\sum_{\mu=0}^{b}(-1)^\mu S_{a+1}(\mu)x^a y^b.$$

□

Theorem 22 indicates that we need inverses of two standard Riordan arrays only in order to obtain a left inverse of the stretched Riordan array. Remaining columns are filled with zeros. For instance, in Table 6 we used the inverses of the standard Riordan arrays corresponding to $(l,k) = (3,0)$ and $(l,k) = (3,2)$, as prescribed by the Riordan group. We could as well have used $(l,k) = (3,0)$, $(l,k) = (3,1)$ and $(l,k) = (3,1)$, $(l,k) = (3,2)$, see Table 7.

5. Examples

In the table below we apply the Schröder transformation and its inverse, the coordination transformation, to some important sequences. Transforms obtained were looked up in the *OEIS*. Results are based on numerical evidence only, a proof should make use of Corollary 8.

sequence	k,l	S-transform	C-transform
n	0,1	A065096 variant of Catalan	
$(-1)^n n$	0,1	A010683 variant of Schröder	A054459 related to $\sqrt{2}$
	-1,1		A119915 related to ternary words
	2,1	A065096 variant of Catalan	
n^2	-1,1	A065096 variant of Catalan	
2^n	-1,1	A109980 variant of Delannoy	
$(-2)^n$	0,1	A114710 variant of Schröder	A055099 related to $\sqrt{2}$
	-1,1	A104934 related to Fibonacci	
3^n	0,1	A134425 variant of Schröder	A087455 related to Gauß-Fibonacci
$(-3)^n$	-1,1		A122558 related to compositions
	0,1		A126473 number of certain strings
4^n	0,1		A078020 related to Lucas-Lehmer
$(-4)^n$	0,1		A126501 related to base 6 numbers
	-1,1		A122690 $a_n = 5a_{n-1} + 4a_{n-2}$
fibo	-1,1	A110122 variant of Delannoy	
$(-1)^n$fibo	0,1		A192704 |certain 4×3 matrices|
	-1,1	A102905 related to Fibonacci	
cata	-1,1		A168505 expansion of 1/(1-x/(1+x/...
$(-1)^n$cata	0,1		A110886 |certain Euler trees|

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Departement van Wiskunde, Universiteit van die Vrystaat, Posbus 339, Bloemfontein 9300, Suid Afrika

E-mail: schroderjd@qwa.ufs.ac.za

URL: `www16.brinkster.com/jodis`

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