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MODELLING FINANCIAL TIME SERIES USING REFLECTIONS OF COPULAS

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We have intensified studies of reflections of copulas (that we introduced recently in [6]) and found that their convex combinations exhibit potentially useful fitting properties for original copulas of the Normal, Frank, Clayton and Gumbel types. We show that these properties enable us to construct interesting models for the relations between investment in stocks and gold.

Keywords: copula, tail dependence, survival copula, reflections of copulas, stock index, returns of index investments, returns of gold investments

Classification: 93E12, 62A10

1. INTRODUCTION

It has been observed that during crisis periods of the international stock markets, the investments in commodities seemed to provide safer alternatives to investors. To investigate this phenomenon we have applied the reflections mainly of the Gumbel and Clayton copulas (that are known for non-zero values of certain tail dependencies coefficients) to derive suitable (alternative) models for relations between the returns of investments in the New York stock exchange index and gold.

The paper is organized as follows. The second section is devoted to a brief overview of the theory of copulas, the reflection copula and the tail dependence coefficients. The third section contains an application part of the paper. Finally, some conclusions are presented.

2. THEORETICAL CONCEPTS

Let (X, Y) be a 2–dimensional random vector with a joint distribution F_{XY} and marginal distribution functions F_X, F_Y. We will use the standard definition of a copula [5, 7] $C(u, v) : [0, 1]^2 \to [0, 1]$ satisfying

$$F_{XY}(x, y) = C(F_X(x), F_Y(y)) \tag{1}$$

and corresponding density function (if C is absolutely continuous)

$$c(u, v) = \frac{\partial^2}{\partial u\, \partial v} C(u, v). \tag{2}$$

In our subsequent investigations, we mainly utilize two families of one–parametric Archimedean copulas Gumbel and strict Clayton ([3, 5, 7]) .

For these families of copulas we apply the (left, right and composed) reflections that exhibit new interesting properties concerning additional coefficients of tail dependencies.

- Gumbel family
$$C_\theta^G(u,v) = \exp^{-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{\frac{1}{\theta}}} \tag{3}$$
where $\theta > 1$. Let us note that $C_1^G(u,v) = \Pi(u,v) = u \cdot v$.

- Strict Clayton family (Kimeldorf and Sampson)
$$C_\theta^{Cl}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}} \tag{4}$$
for $\theta > 0$, $C_0^{Cl}(u,v) = \Pi\,(u,v) = u \cdot v$.

To enrich the classes of considered models, we also consider the classes of Frank and Normal copulas.

- Frank family
$$C_\theta^F(u,v) = -\frac{1}{\theta} \log\left(1 - \frac{(1-e^{-\theta u})(1-e^{-\theta v})}{(1-e^{-\theta})}\right) \tag{5}$$
for $\theta > 0$, $C_0^F(u,v) = \Pi(u,v) = u \cdot v$.

- Normal family
$$C_\varrho^n(u,v) = \Phi_\varrho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) \tag{6}$$
for $|\varrho| \leq 1$, where Φ is the distribution function of the one-dimensional normalized normal distribution and Φ_ϱ is the 2-dimensional distribution function of normalized normal distribution with given Pearson's correlation coefficient ϱ.

A rich overview of Archimedean copulas is presented in Embrechts et. al [3], Genest and Favre [4], Joe [5] and Nelsen [7].

Let us recall that for a given copula $C(u,v)$ the *lower* (*left*) and *upper* (*right*) tail dependence coefficients are defined by

$$\begin{aligned} \lambda_L(C) &= \lim_{\delta \to 0} Pr\left(F_Y(y) \leq \delta \,|\, F_X(x) \leq \delta\right) = \lim_{\delta \to 0} \frac{C(\delta,\delta)}{\delta} \\ &= \lim_{\delta \to 0} Pr\left(F_X(x) \leq \delta \,|\, F_Y(y) \leq \delta\right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \lambda_R(C) &= \lim_{\delta \to 0} Pr\left(F_Y(y) \geq 1-\delta \mid F_X(x) \geq 1-\delta\right) = \lim_{\delta \to 0} \frac{2\delta - 1 + C(1-\delta, 1-\delta)}{\delta} \\ &= \lim_{\delta \to 0} Pr\left(F_X(x) \geq 1-\delta \mid F_Y(y) \geq 1-\delta\right). \end{aligned} \tag{8}$$

It is well known [5, 7] that the Gumbel copula C_θ^G and Clayton copula C_θ^{Cl} satisfy the relation
$$\lambda_R(C_\theta^G) = 2 - 2^{\frac{1}{\theta}}, \quad \lambda_L(C_\theta^G) = 0$$

and

$$\lambda_R(C_\theta^{Cl}) = 0, \quad \lambda_L(C_\theta^{Cl}) = 2^{-\frac{1}{\theta}}.$$

It is also well known (see [3]) that the values of λ_R and λ_L for normal and Frank copulas are equal to 0.

We follow the approach of Patton [10] and consider a so–called *survival copula* derived from a given copula $C(u,v)$ corresponding to the couple (X,Y) by

$$SC(u,v) = u + v - 1 + C(1-u, 1-v) \tag{9}$$

which is the copula corresponding to the couple $(-X,-Y)$ with the marginal distribution functions

$$F_{-X}(x) = 1 - F_X(-x^+) \text{ and } F_{-Y}(y) = 1 - F_Y(-y^+). \tag{10}$$

Obviously, the relations

$$\lambda_L(SC) = \lambda_R(C) \quad \text{and} \quad \lambda_R(SC) = \lambda_L(C)$$

hold.

Convex combinations of copulas and corresponding survival copulas has been successfully applied for modelling of exchange rates dependences (e.g. in Patton [10] and Ning [8, 9]).

We will attempt to use in our modelling procedures also copulas LC and RC corresponding to the couples $(-X,Y)$ and $(X,-Y)$.

They have the form

$$LC(u,v) = v - C(1-u, v)$$

and

$$RC(u,v) = u - C(u, 1-v).$$

We will call the copulas LC and RC the *left* and the *right reflections* of the copula C. Since the survival copula SC can be obtained in the form of the right reflection of the copula LC as well as the left reflection of the copula RC, we included SC also in the family of the reflections of the copula C. Observe that if C is an absolutely continuous copula with density function $c_C(u,v)$, then also all its reflections are absolutely continuous with the respective density functions

$$\begin{aligned} c_{LC}(u,v) &= c_C(1-u, v), \\ c_{RC}(u,v) &= c_C(u, 1-v) \end{aligned}$$

and

$$c_{SC}(u,v) = c_C(1-u, 1-v).$$

We recall the definitions of upper–lower and lower–upper tail dependencies for the copula $C(u,v)$ (c.f. [6]) by $\lambda_{RL}(C) = \lambda_R(LC)$ and $\lambda_{LR}(C) = \lambda_L(LC)$. It is obvious that for the Gumbel copula $C = C_\theta^G$ the equalities

$$\lambda_{RL}(LC) = \lambda_{LR}(RC) = \lambda_R(C) = 2 - 2^{\frac{1}{\theta}}$$

hold.

Similarly, for the Clayton copula $C = C_\theta^{Cl}$ we have

$$\lambda_{RL}(RC) = \lambda_{LR}(LC) = \lambda_L(C) = 2^{-\frac{1}{\theta}}.$$

Using the system Mathematica we easily get that the values of λ_{LR} and λ_{RL} are equal to 0 for all Gumbel, Clayton, Frank and normal copulas.

It is well known that for the convex sums of copulas, the corresponding density function is the convex sum (with the same weights) of incoming density functions. The same kind of mixing behaviour can be observed for the coefficients of tail dependencies λ_R, λ_L, λ_{LR} and λ_{RL}.

3. MODELLING APPLICATIONS

In the modelling part, we concentrate our attention to the relations between returns of the New York stock exchange and gold in the period 31. 8. 1999 – 3. 12. 2010 that is including two dramatic stock market crises (1. 9. 2000 – 8. 10. 2002 and 20. 7. 2007 – 4. 3. 2009). On the contrary, in those periods the prices of gold had not been dramatically suffering (which might have inspired the conjecture of shifts of investors' attention and money from volatility stocks to more safe commodities in the times of crises).

The following Figure 1 shows the parallel development of the values of the New York stock exchange and prices of gold in the considered time periods.

The next Figures 2 and 3 show development of returns of the New York stock index and gold in all the periods.

In order to avoid a possible violation of the i.i.d. property of the analysed data, we first filtered both investigated univariate time series (in all considered time periods) by ARMA–GARCH models (all computations have been performed using the package Time Series of software Mathematica). Subsequently we applied the fitting by copulas to the residuals of those filters.

On the basis of the above mentioned properties of the coefficients of tail dependencies for the individual families of copulas, we considered models from Normal, Frank, Clayton, Gumbel families and their pairwise convex combinations (following a suggestion of one of the reviewers) as well as the convex combinations of models from the last 2 above mentioned families with their left and right reflections and their survival copulas.

For selecting the optimal models we applied the Kolmogorov–Smirnov Anderson–Darling (KSAD, for which we use the abbreviation AD) [1] test statistic (for which we also constructed a GoF simulation based test), when comparing models with they submodels and different families of models. For testing of nested models we applied likelihood ratio (LR) tests [2].

We start with the models for the whole period (31. 8. 1999 – 3. 12. 2010). The minimum value of AD were achieved by the models

$$0.348 * C_{0.394}^{Cl} + 0.651 * LC_{0.236}^{Cl} \quad (A)$$

with $AD = 1.574$ and non-zero tail dependence coefficients $\lambda_L = 0.349 * 2^{-\frac{1}{0.394}} = 0.060$, $\lambda_{LR} = 0.651 * 2^{-\frac{1}{0.236}} = 0.034$ followed by

$$0.713 * RC_{1.125}^{G} + 0.287 * SC_{1.300}^{G} \quad (B)$$

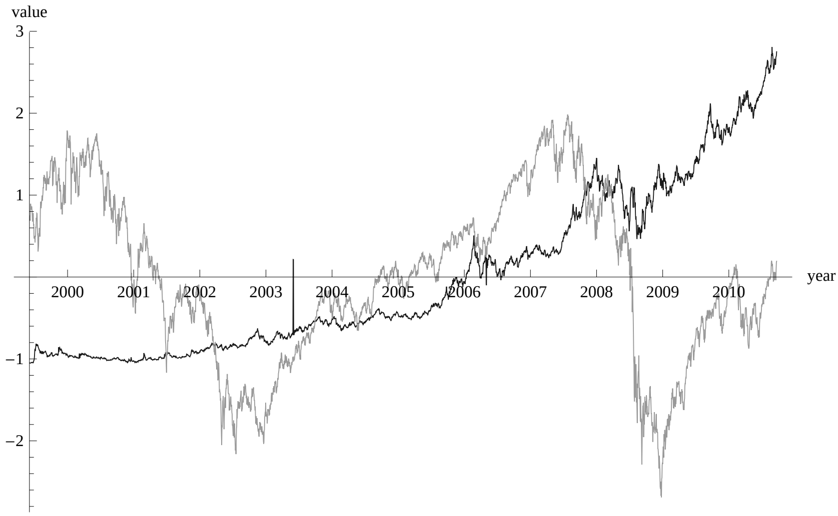

Fig. 1. Standardized values of N. Y. stock index (gray) and gold prices (black).

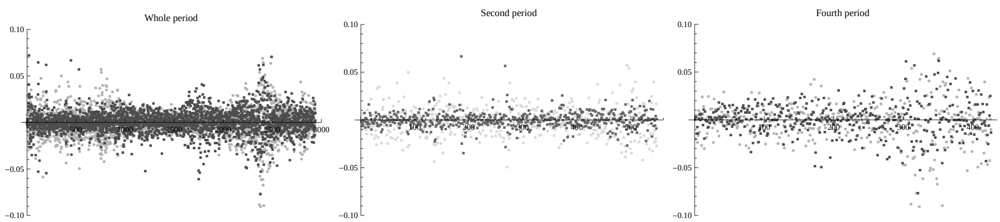

Fig. 2. Returns of gold investment (black) and N. Y. stock index (gray) in periods 31. 8. 1999 – 3. 12. 2010 (left), 1. 9. 2000 – 8. 10. 2002 (middle), 20. 7. 2007 – 4. 3. 2009 (right).

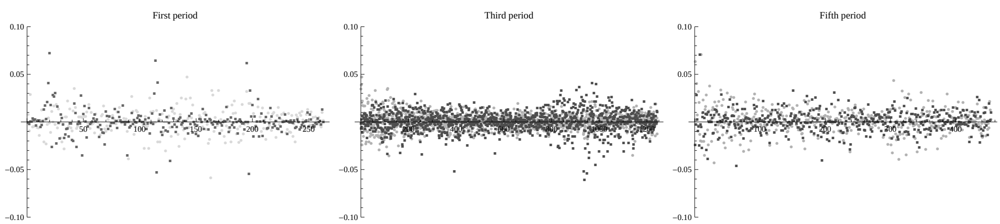

Fig. 3. Returns of gold investment (black) and N. Y. stock index (gray) in periods 31. 8. 1999 – 30. 8. 2000 (left), 9. 10. 2002 – 19. 7. 2007 (middle) and 5. 3. 2009 – 3. 12. 2010 (right).

with $AD = 1.606,\ \lambda_L = 0.287 * (2 - 2^{\frac{1}{1.300}}) = 0.085,\ \lambda_{LR} = 0.713 * (2 - 2^{\frac{1}{1.125}}) = 0.106$. (Note that $\lambda_{LR} > \lambda_L$). Next follow the models

$$0.364 * C^n_{-0.443} + 0.636 * C^{Cl}_{0.289} \quad (C)$$

with $AD = 1.954,\ \lambda_L = 0.636 * 2^{-\frac{1}{0.289}} = 0.058$ and

$$0.278 * LC^G_{1.380} + 0.722 * SC^G_{1.091} \quad (D)$$

with $AD = 2.203,\ \lambda_L = 0.722 * (2 - 2^{\frac{1}{1.091}}) = 0.081,\ \lambda_{LR} = 0.278 * (2 - 2^{\frac{1}{1.380}}) = 0.097$. For all above models, the simulation based GoF tests for AD yielded p–values > 0.1.

All one–dimensional nested submodels of the above two–dimensional models have substantially larger values of AD (between 2.98 and 9.6). All LR tests for comparisons the nested submodels yielded p–values < 0.001.

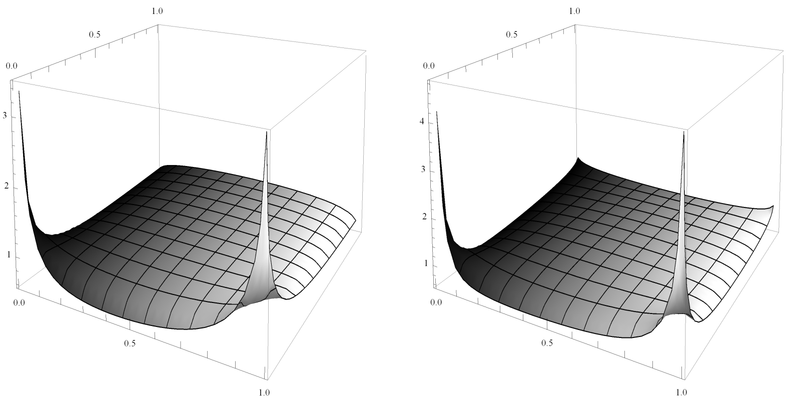

Fig. 4. 3D plot of the density functions for the optimal models (A – left, B – right) of the relation between the daily returns of N. Y. stock index and gold for the whole period (31. 8. 1999 – 3. 12. 2010).

Since the above types of models in the families ($C^{Cl} + LC^{Cl},\ RC^G + SC^G,\ C^n + C^{Cl},\ LC^G + SC^G$) will appear in the subsequent analyses, we will preserve for them the same notation *A, B, C, D* (in combination with the number of the considered time periods).

Next we concentrate at two crisis period. For the second period (1.9.2000 – 8.10.2002) the minimum value of AD was achieved for the model

$$0.497 * C^n_{-0.559} + 0.503 * C^{Cl}_{0.286} \qquad (2C)$$

with $AD = 1.505$ and $\lambda_L = 0.503 * 2^{-\frac{1}{0.286}} = 0.0446$.

A slightly greater values of AD has the competing models

$$0.602 * RC^G_{1.378} + 0.388 * SC^G_{1.136} \qquad (2B)$$

with $AD = 1.575,\ \lambda_L = 0.388 * (2 - 2^{\frac{1}{1.136}}) = 0.062,\ \lambda_{LR} = 0.602 * (2 - 2^{\frac{1}{1.378}}) = 0.208$ (again $\lambda_{LR} > \lambda_L$) and

$$0.359 * LC^G_{1.639} + 0.641 * SC^G_{1.046} \qquad (2D)$$

with $AD = 1.575,\ \lambda_L = 0.641 * (2 - 2^{\frac{1}{1.046}}) = 0.038,\ \lambda_{RL} = 0.359 * (2 - 2^{\frac{1}{1.639}}) = 0.170.$

Next follows the model

$$0.633 * C^{Cl}_{1.377} + 0.367 * LC^{Cl}_{0.560} \qquad (2A)$$

with $AD = 1.641,\ \lambda_L = 0.633 * 2^{-\frac{1}{1.377}} = 0.383,\ \lambda_{LR} = 0.367 * 2^{-\frac{1}{0.560}}) = 0.106.$

For all models (*2A*) – (*2D*), the simulation based GoF tests for AD yielded p–values > 0.1. Among all considered more dimensional models, the (clearly) lowest value of AD was reached by the optimal model in the family $C^{Cl} + LC^{Cl} + RC^{Cl} + SC^{Cl}$ with $AD = 1.669$. However, the LR test vs. the nested submodel (*2A*) yielded p–value > 0.1.

The attempt to unify the models (*2B*) and (*2D*) in $LC^G + RC^G + SC^G$ yielded a substantial increase of AD to 2.686.

All one–dimensional nested submodels of the models (*2A*) – (*2D*) the LR tests yielded p–values < 0.05 except for $RC^G_{1.146}$ vs. (*2B*) (p–value = 0.074). However, since the value of $AD = 4.717$ for this submodel is substantially greater than for (*2B*), we hesitate to consider that submodel more appropriate than (*2B*).

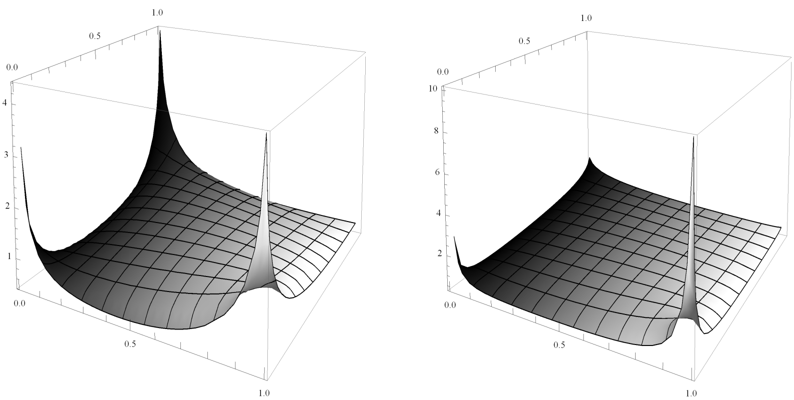

Fig. 5. 3D plot of the density functions for the optimal models (*2C* – left, *2B* – right) of the relation between the daily returns of N. Y. stock index and gold for the time period 1. 9. 2000 – 8. 10. 2002.

For the fourth period (20.7.2007 – 4.3.2009) the (practically identical) minimum values of $AD = 1.27$ were achieved by the 3 models

$$0.325 * C^{n}_{-0.741} + 0.675 * C^{Cl}_{0.474} \qquad (4C)$$

with $\lambda_L = 0.675 * 2^{-\frac{1}{0.474}} = 0.156;$

$$0.512 * RC^{G}_{1.453} + 0.488 * SC^{G}_{1.381} \qquad (4B)$$

with $\lambda_L = 0.488 * (2 - 2^{\frac{1}{1.381}}) = 0.170,\ \lambda_{LR} = 0.512 * (2 - 2^{\frac{1}{1.453}}) = 0.199;$

$$0.393 * LC^{G}_{1.819} + 0.607 * SC^{G}_{1.310} \qquad (4D)$$

with $\lambda_L = 0.607 * (2 - 2^{\frac{1}{1.310}}) = 0.184,\ \lambda_{RL} = 0.393 * (2 - 2^{\frac{1}{1.819}}) = 0.211.$

Slightly greater value of AD was achieved by the model

$$0.471 * C^{Cl}_{0.608} + 0.529 * LC^{Cl}_{0.651} \qquad (4A)$$

with $AD = 1.332,\ \lambda_L = 0.471 * 2^{-\frac{1}{0.608}} = 0.151,\ \lambda_{LR} = 0.529 * 2^{-\frac{1}{0.651}} = 0.182.$

Again the unified model for (*4B*) and (*4D*) in the family $LC^G + RC^G + SC^G$ yielded a substantially higher value of $AD = 1.723$ (higher than for some other 2–component models).

For all models (*4A*) – (*4D*), the simulation based GoF tests for AD yielded p–values > 0.1. Among all considered more dimensional models, the (clearly) lowest value of AD was reached by the optimal model in the family $C^G + LC^G + RC^G + SC^G$ with $AD = 1.732$. However, the LR test vs. the nested submodels (*4B*) and (*4D*) yielded p–values > 0.1.

All one–dimensional nested submodels of the models (*4A*) – (*4D*) the LR tests yielded p–values < 0.01.

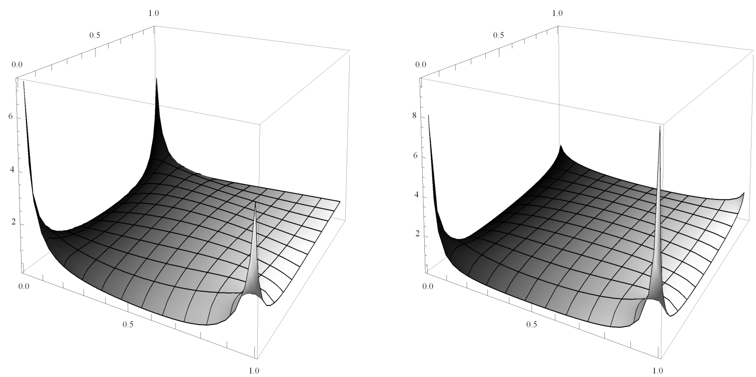

Fig. 6. 3D plot of the density functions for the optimal models (*4C* – left, *4B* – right) of the relation between the daily returns of N. Y. stock index and gold for the time period 20. 7. 2007 – 4. 3. 2009.

For the first period (31. 8. 1999 – 30. 8. 2000), the value of AD for the product copula Π was 1.44. The models with the minimum values of AD in all Archimedean copula classes have parameters that are extremely close to their limit values corresponding to the product copula Π. The models with minimum values of AD (among all considered models) were those with copula functions

$$0.94 * C^n_{-0.143} + 0.06 * C^G_{3.04} \quad \text{with } AD = 1.344 \qquad (1C)$$

and

$$0.867 * C^G_{1.023} + 0.133 * LC^G_{1.912} \quad \text{with } AD = 1.346. \qquad (1E)$$

The first model seems to be close to the Gauss family, where the optimal submodel $G^n_{-0.07}$ has $AD = 1.77$, which is substantially larger then the value $AD = 1.44$ for the product copula Π. Moreover, the LR test for the nested submodels $C_1^G = \Pi$ vs. $C^G + LC^G$ yielded p-value 0.37. Therefore, Π seems to be a satisfactory model for the investigated time series of residuals in the first period.

For the third period (9.10.2002 – 19.7.2007) the value of AD for Π was 3.39. The models with the minimum values of AD in all Archimedean copula classes have parameters that are extremely close to their limit values corresponding to the product copula Π. The lowest value of $AD = 2.104$ (p-value = 0.835) was (clearly) achieved for the model

$$0.358 * C^n_{-0.224} + 0.642 * C^{Cl}_{0.177} \qquad (3C)$$

with $\lambda_L = 0.642 * 2^{-\frac{1}{0.177}} = 0.013$.

Since both Π as well as the optimal submodel in the Gauss family $C^n_{-0.02}$ (with $AD = 4.9$) had substantially higher values of AD, we did not consider reductions of this model to 1–dimensional submodels.

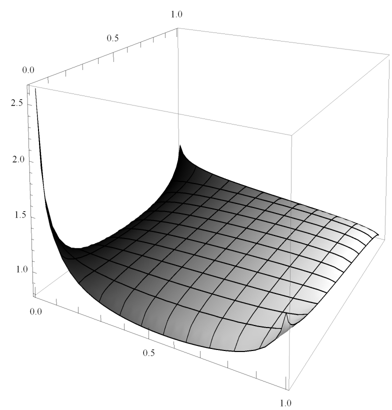

Fig. 7. 3D plot of the density function for the optimal model (*3C*) of the relation between the daily returns of N. Y. stock index and gold for the time period 9. 10. 2002 – 19. 7. 2007.

For the fifth period (5. 3. 2009 – 3. 12. 2010) the lowest value of AD was achieved by the one–component model

$$SC^G_{1.119} \qquad (5F)$$

with $AD = 1.243,\ \lambda_L = (2 - 2^{\frac{1}{1.119}}) = 0.142$ followed by

$$0.827 * C^n_{0.044} + 0.173 * C^{Cl}_{1.936} \qquad (5C)$$

with $AD = 1.286,\ \lambda_L = 0.173 * 2^{-\frac{1}{1.936}} = 0.121$,

$$C^{Cl}_{0.202} \qquad (5G)$$

with $AD = 1.334,\ \lambda_L = 2^{-\frac{1}{0.202}} = 0.032$ and by

$$0.695 * LC^G_{1.022} + 0.305 * SC^G_{1.654} \qquad (5D)$$

with $AD = 1.387$, $\lambda_L = 0.305 * (2 - 2^{\frac{1}{1.654}}) = 0.146$, $\lambda_{RL} = 0.695 * (2 - 2^{\frac{1}{1.022}}) = 0.021$.

For all above models, the simulation based GoF tests for AD yielded p–values > 0.1. The LR test for the nested submodels (*5F*) vs. (*5D*) and (*5G*) vs. (*5C*) yielded p–values > 0.05, thus we restrict our attention to (*5F*) and (*5G*).

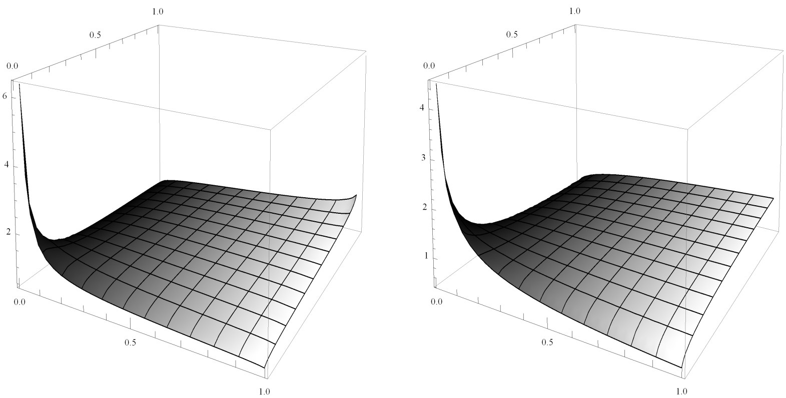

Fig. 8. 3D plot of the density functions for the optimal models (*5F* – left, *5G* – right) of the relation between the daily returns of N. Y. stock index and gold for the time period 5. 3. 2009 – 3. 12. 2010.

4. CONCLUSION

Our attempt to show that the models with negative deviations of stock returns accompanied by positive deviation of gold returns (i. e. with significantly positive values of the coefficient of lower–upper tail dependencies (that are typical for the models with RC^G or LC^{Cl} components) can be useful for explanation of the joint behaviour of the returns on investment in the N.Y. stock index and gold was partially supported by the outcomes of our analyses.

According the Kolmogorov–Smirnov Anderson–Darling criteria (AD) of GoF that we used for selection of the best models (in accordance with a suggestion of one reviewer) 2–component models were clearly favoured over their more dimensional alternatives. However, among 2–component models, several candidates of them scored very similar values of the above criteria.

For the whole time period (31. 8. 1999 – 3. 12. 2010), the minimum values of AD were really reached for the model families $C^{Cl} + LC^{Cl}$ and $RC^G + SC^G$. However, for the two crisis periods, some other models scored comparable values of AD, especially in the family Gauss + Clayton (that we included in our analysis following a suggestion of one reviewer). The optimum models in this family are interesting. Although they theoretically have only one non–zero coefficient of tail dependence (λ_L), the graphs of their densities show elevated values also in another corners of the unit square. This phenomenon might be an interesting subject of more detailed analyses.

We can observe that the quadruple of the best families of models (with respect to the AD criterion) for the whole period and for 2 crisis periods is the same and their ordering (with respect to AD) is identical for both crisis periods.

Concerning the considered non-crisis time periods, we concluded that the selected optimal models for them consist of one component only (for the first period, it was the product copula Π corresponding to the independence of the components).

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